Received: 9 June 2017 | Revised: 14 May 2018 | Accepted: 3 June 2018
DOI: 10.1002/zamm.201700157

**ORIGINAL PAPER**

ZAMM

# Three-dimensional problem of Rayleigh waves propagating in a half-space with restrained boundary

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In this paper we obtain the dispersion equation in three-dimensional wave propagation problem in elastic half-space with elastically restrained surface. It is shown that, in the case of plane strain, the elastic restriction of the surface leads to decrease of the degree of the surface wave localization.

**KEYWORDS**
elastic half-space, elastically restrained boundary, surface wave

**MSC (2010)**
74B05, 74J15

## 1 | INTRODUCTION

In the study of surface waves, the plane and antiplane deformation are generally considered. Existence of surface waves is first studied by Rayleigh,[1] where a plane problem for a half-space with stress free surface is examined. Later Rayleigh‘s problem is generalized and the solution of the three-dimensional problem is obtained by Knowles.[2] For more introduction and references on the subject we refer to [3]. See also [4] for another investigation on surface waves in three-dimensional bodies.

In [5] three-dimensional problem for elastic space waves propagation in isotropic half-space for two different boundary conditions is considered: free boundary and one of the shear displacement, tangential stresses and normal stress are equal to zero. In [6] 3D elastic waves propagation problems are reviewed. Study of 3DSW for various types of mixed boundary conditions is performed in [7]. It is shown, that the corresponding dispersion equation has a root for two types of boundary conditions: free surface and surface, where displacements in one tangential direction are prohibited. In [8–10] the asymptotic method is used to investigate propagation of cylindrical surface wave in the case of mixed boundary conditions.

Unlike the classical Rayleigh problem, two types of complex boundary conditions,instead of the condition of the free surface, are considered in [11]. It is assumed, that either normal stress is restricted in the perpendicular direction to the surface and shear stress is equal to zero, or the normal stress is equal to zero and the shear stress is restrained.Conditions, at which the surface wave cannot exist, are found.

The problem of periodic waves propagation in an elastic layer, when the normal and shear stresses are restrained on the surfaces, are investigated in [12,13]. The influence of restraint factor to the phase velocity of the symmetric and asymmetric oscillations is shown.

As a result of Radon integral transform, the space problems of the dynamic theory of elasticity are reduced to the plane problem with respect to Radon images of the characteristic quantities.[14] In [15] dynamic potentials are introduced for solving three-dimensional problems of the dynamic theory of elasticity, in which the antiplane displacement is not involved (for example, in the problem of the dynamics of the surface of an elastic half-space, where the contribution of the surface wave is dominant). Applying the Radon transform, the three-dimensional problem is reduced to a plane problem.

Development of asymptotic models of Rayleigh surface waves, Stoneley and Scholte-Gogoladze interface waves are studied in [16]. In [17] wave propagation problem in an elastic half-space is studied, when the restrained free boundary conditions are given. Using the Radon integral transform, the corresponding dispersion equation for determination of the velocity of the surface wave is obtained and studied numerically.

The study of free waves in a half-space (waves propagating with amplitudes determined by accuracy of arbitrary amplitude factor) for classical-traditional and auxetic-non-traditional materials is carried out in [18–25]. The propagation of Rayleigh waves in isotropic elastic semi-space (plane deformation) with impedance boundary conditions (on the semi-space boundary normal and tangential stresses are linear in corresponding displacement component multiplied by the frequency) are studied in [28–32]. The existence and uniqueness of the wave is proved and an analytical formula for the Rayleigh wave speed is obtained using the method of complex functions. From mathematical viewpoint, the derivation of the characteristic equation for the Rayleigh wave speed is a eigenvalue problem. In the case of impedance boundary conditions,[28–32] the eigenvalue also enters into the boundary conditions. Such eigenvalue problems are thoroughly studied in [33].

In this paper we obtain the dispersion equation of three-dimensional wave propagation problem in a half-space with elastically restrained border.Potential functions as in plane strain problems[3,5] are introduced.It is shown, that in plane strain the elastically restrained boundary leads to decrease of the surface wave localization degree. In the case of mixed boundary conditions, the propagation angle affects the phase velocity of the 3DSW.

## 2 | STATEMENT OF THE PROBLEM

Consider harmonic vibrations of isotropic elastic half-space $\{-\infty < x < \infty, -\infty < z < \infty, 0 \leq y < \infty\}$. Then, in the half-space:[3]

$$(\lambda + \mu)\,\mathrm{graddiv}\,\vec{u} + \mu\Delta\vec{u} = \rho\ddot{\vec{u}}, \tag{2.1}$$

where $\vec{u}$ is the displacement vector, $\lambda, \mu$ are the Lame constants, $\rho$ is the density of the half-space.

Suppose, that on the boundary of the half-space $y = 0$ the following boundary conditions are given:[8]

$$\sigma_{yx} = \alpha_* u, \quad \sigma_{yy} = \beta_* v, \quad \sigma_{yz} = \gamma_* w \quad \left(\alpha_*, \beta_*, \gamma_* > 0\right) \tag{2.2}$$

These conditions are first proposed by Mindlin[27] for studying the elastic wave reflections from the boundary of the half-space. In particular, when $\alpha_* = \beta_* = \gamma_* = 0$, we obtain the known conditions of free boundary. In [11] the conditions for existence of Rayleigh waves in the case of elastically restrained boundary (plane strain) are considered. Periodic wave propagation in an elastic layer is studied in [12,13].

As in problems of plane strain,[5] the potentials $\varphi(x, y, z, t)$ and $\psi(x, y, z, t)$ are introduced according to

$$u = \frac{\partial\varphi}{\partial x} - \frac{\partial\psi}{\partial z}, \quad w = \frac{\partial\varphi}{\partial z} + \frac{\partial\psi}{\partial x}. \tag{2.3}$$

Substituting (2.3) into the projections of (2.1) onto the axes $Ox$ and $Oz$, we respectively obtain

$$\begin{aligned} &\frac{\partial}{\partial x}\left[c_l^2\Delta_2\varphi + c_t^2\frac{\partial^2\varphi}{\partial y^2} - \frac{\partial^2\varphi}{\partial t^2} + \left(c_l^2 - c_t^2\right)\frac{\partial v}{\partial y}\right] - \frac{\partial}{\partial z}\left(c_t^2\Delta\psi - \frac{\partial^2\psi}{\partial t^2}\right) = 0, \\ &\frac{\partial}{\partial z}\left[c_l^2\Delta_2\varphi + c_t^2\frac{\partial^2\varphi}{\partial y^2} - \frac{\partial^2\varphi}{\partial t^2} + \left(c_l^2 - c_t^2\right)\frac{\partial v}{\partial y}\right] + \frac{\partial}{\partial x}\left(c_t^2\Delta\psi - \frac{\partial^2\psi}{\partial t^2}\right) = 0, \end{aligned} \tag{2.4}$$

where

$$\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$$

is the two-dimensional Laplacian.

System (2.4), as in the case of plane deformation, are satisfied if the following equations hold:

$$c_l^2\Delta_2\varphi + c_t^2\frac{\partial^2\varphi}{\partial y^2} - \frac{\partial^2\varphi}{\partial t^2} + \left(c_l^2 - c_t^2\right)\frac{\partial v}{\partial y} = 0, \tag{2.5}$$

$$c_t^2\Delta\psi - \frac{\partial^2\psi}{\partial t^2} = 0, \tag{2.6}$$

Obviously, from (2.5) we derive

$$\frac{\partial v}{\partial y} = -\frac{1}{1-\theta}\left(c_l^2\Delta_2\varphi + \theta\frac{\partial^2\varphi}{\partial y^2} - \frac{1}{c_l^2}\frac{\partial^2\varphi}{\partial t^2}\right), \quad \theta = \frac{c_t^2}{c_l^2}. \tag{2.7}$$

Differentiating the projection of (2.1) onto the axis $Oy$ with respect to $y$, we derive

$$c_t^2\Delta\frac{\partial v}{\partial y} + \left(c_l^2 - c_t^2\right)\frac{\partial^2}{\partial y^2}\Delta_2\varphi + \left(c_l^2 - c_t^2\right)\frac{\partial^3 v}{\partial y^3} = \frac{\partial^2}{\partial t^2}\frac{\partial v}{\partial y}. \tag{2.8}$$

Further, substituting (2.7) into (2.8), after some transformations, we arrive at

$$\Delta^2\varphi - \left(\frac{1}{c_l^2} + \frac{1}{c_t^2}\right)\frac{\partial^2}{\partial t^2}\Delta\varphi + \frac{1}{c_l^2c_t^2}\frac{\partial^4\varphi}{\partial t^4} = 0$$

or

$$\left(\Delta - \frac{1}{c_t^2}\frac{\partial^2}{\partial t^2}\right)\left(\Delta\varphi - \frac{1}{c_l^2}\frac{\partial^2\varphi}{\partial t^2}\right) = 0.$$

Taking into account the decay conditions

$$\lim_{y\to\infty}\vec{u} = 0, \quad \lim_{y\to\infty}\varphi = 0, \quad \lim_{y\to\infty}\psi = 0,$$

for the functions $u, v, w, \varphi$ and $\psi$ we obtain the following representations:[5]

$$\begin{gathered} u(x,y,z,t) = -i\left[Ak\cos\gamma e^{-\upsilon_1 ky} + (Bk\cos\gamma + Ck\sin\gamma)e^{-\upsilon_2 ky}\right]\exp i(\omega t - xk\cos\gamma - zk\sin\gamma), \\ v(x,y,z,t) = -k\left[A\upsilon_1 e^{-\upsilon_1 ky} + B\upsilon^{-1}{}_2 e^{-\upsilon_2 ky}\right]\exp i(\omega t - xk\cos\gamma - zk\sin\gamma), \\ w(x,y,z,t) = -i\left[Ak\sin\gamma e^{-\upsilon_1 ky} + (Bk\sin\gamma - Ck\cos\gamma)e^{-\upsilon_2 ky}\right]\exp i(\omega t - xk\cos\gamma - zk\sin\gamma), \\ \varphi(x,y,z,t) = \left[Ae^{-\upsilon_1 ky} + Be^{-\upsilon_2 ky}\right]\exp i(\omega t - xk\cos\gamma - zk\sin\gamma), \\ \psi(x,y,z,t) = Ce^{-\upsilon_2 ky}\exp i(\omega t - xk\cos\gamma - zk\sin\gamma), \end{gathered} \tag{2.9}$$

where $k$ is the wave number, $\nu_1 = \sqrt{1-\theta\eta}$, $\nu_2 = \sqrt{1-\eta}$, $\theta = \frac{c_t^2}{c_l^2} < 1$, $\eta = \frac{\omega^2}{k^2c_t^2} < 1$ is the dimensionless phase velocity of the three-dimensional surface wave, $\gamma$ is the sharp angle of the wave propagation in the plane $Oxz$, $A, B$ and $C$ are integration constants.

Using Hooke's law, the boundary conditions (2.2) take the form

$$\begin{gathered} \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \alpha_* u = 0, \\ \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) - \gamma_* w = 0, \\ \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial v}{\partial y} - \beta_* v = 0. \end{gathered} \tag{2.10}$$

## 3 | DISPERSION EQUATIONS AND NUMERICAL RESULTS

Substitution of the solution (2.9) into the boundary conditions (2.10) leads to the following system of linear homogeneous algebraic equations with respect to the coefficients $A$, $B$ and $C$:

$$\begin{cases} A\left(\beta_0\nu_1 + \nu_1^2\theta^{-1} - \theta^{-1}(1-2\theta)\right) + B\left(2 + \beta_0\nu_2^{-1}\right) = 0 \\ A\left(2\nu_1 + \alpha_0\right) + B\left(\nu_2^{-1}\left(1+\nu_2^2\right) + \alpha_0\right) + C\tan\gamma\left(\nu_2 + \alpha_0\right) = 0 \\ A\tan\gamma\left(2\nu_1 + \gamma_0\right) + B\tan\gamma\left(\nu_2^{-1}(2-\eta) + \gamma_0\right) - C\left(\nu_2 + \gamma_0\right) = 0. \end{cases}$$

Equating the determinant of the system to zero, provides the following dispersion equation:

$$\begin{aligned} &4\nu_1\nu_2 - (2-\eta)^2 + \nu_2\left[\eta\left(\alpha_0 + \beta_0\nu_1\nu_2^{-1}\right) + \alpha_0\beta_0\left(\nu_2^{-1} - \nu_1\right)\right] \\ &\quad + \gamma_0\left[4\nu_1 + \eta\left(\alpha_0 + \beta_0\nu_1\nu_2^{-1}\right) + \alpha_0\beta_0\left(\nu_2^{-1} - \nu_1\right) - (2-\eta)^2\nu_2^{-1}\right] \\ &\quad + tg^2\gamma\left[4\nu_1\nu_2 - (2-\eta)^2 + \nu_2\eta\left(\gamma_0 + \beta_0\nu_1\nu_2^{-1}\right) + \nu_2\gamma_0\beta_0\left(\nu_2^{-1} - \nu_1\right)\right. \\ &\quad \left. + \alpha_0\left(4\nu_1 - (2-\eta)^2\nu_2^{-1} + \eta\left(\gamma_0 + \beta_0\nu_1\nu_2^{-1}\right) + \gamma_0\beta_0\left(\nu_2^{-1} - \nu_1\right)\right)\right] = 0, \end{aligned} \tag{3.1}$$

where

$$\alpha_0 = \frac{\alpha_*}{\mu k}, \quad \beta_0 = \frac{\beta_*}{\mu k}, \quad \gamma_0 = \frac{\gamma_*}{\mu k}.$$

### 3.1 | Particular cases

In the case of plane strain ($\gamma = \gamma_0 = 0$), the dispersion Equation 3.1 takes to the following form:

$$(2-\eta)^2 - 4\nu_1\nu_2 - \alpha_0\eta\nu_2 - \beta_0\eta\nu_1 - \alpha_0\beta_0\left(1 - \nu_1\nu_2\right) = 0. \tag{3.2}$$

Moreover, in the case of $\alpha_0 = \beta_0 = 0$, (3.2) coincides with Rayleigh‘s classical equation. Compared with the Rayleighs Equation 3.2 is dispersive, since its solution depends on$\alpha_0$, $\beta_0$. Equation 3.2 when $\alpha_0 = 0$ or $\beta_0 = 0$ is obtained in [11], where the conditions of existence of surface waves are derived, depending on the coefficient characterizing the elastic restraint and wave length.

Equation 3.2 admits trivial solution, corresponding to $\eta = 0$. Following to [26], eliminating the root corresponding to $\eta = 0$, (3.2) is reduced to

$$D(\eta) \equiv \eta - (1-\theta)\nu_2\left(\nu_1 + \nu_2\right)^{-1}\left(4 - \alpha_0\beta_0\right) - \alpha_0\nu_2 - \beta_0\nu_1 - \alpha_0\beta_0 = 0. \tag{3.3}$$

Obviously,

$$D(0) = -0.5(1-\theta)\left(4 - \alpha_0\beta_0\right) - \alpha_0 - \beta_0 - \alpha_0\beta_0,$$

$$D(1) = 1 - \beta_0\sqrt{1-\theta} - \alpha_0\beta_0.$$

Equation 3.3 has a solution in the interval $\eta \in (0, 1)$, if $D(0) < 0$, $D(1) > 0$, and this solution is unique if $\frac{dD}{d\eta} > 0$. Choosing $\alpha_0$ and $\beta_0$ appropriately, we can compute the phase velocity of the surface wave, depending on the degree of restraint surface of the half-space.

In Table 1 $\eta$ is computed according to (3.3) depending on $\alpha_0$ and $\beta_0$ when $\theta = 0.33$. It is evident, that restriction of the boundaries in the direction of either unit normal or unit tangent leads to increase in the dimensionless phase velocity of the surface wave. When the boundary is restrained in both directions simultaneously, the phase velocity of the surface wave first

**TABLE 1** Dependence of dimensionless phase velocity of the surface wave on parameters of the surface restraint when $\theta = 0.33$

| $\alpha_0$ | $\beta_0$ | $\eta$ |
|---|---|---|
| 0 | 0 | 0.8464 |
| 0 | 0.2 | 0.9005 |
| 0 | 0.4 | 0.9405 |
| 0 | 0.6 | 0.9686 |
| 0 | 0.8 | 0.9867 |
| 0 | 1 | 0.9966 |
| 0.2 | 0 | 0.8712 |
| 0.4 | 0 | 0.8903 |
| 0.6 | 0 | 0.9052 |
| 0.8 | 0 | 0.9172 |
| 1 | 0 | 0.9269 |
| 0.2 | 0.2 | 0.9045 |
| 0.4 | 0.4 | 0.9238 |
| 0.6 | 0.6 | 0.9216 |
| 0.8 | 0.8 | 0.9001 |
| 1 | 1 | 0.8540 |

increases, reaching its maximal value, but then decreases. Thus, the restriction of the boundary of the elastic half-space reduces the degree of localization of the surface wave (slow decay of the amplitude).

In the case of free boundary ($\alpha_0 = \beta_0 = \gamma_0 = 0$), (3.1) in 3D problem is reduced to

$$\left(1 + tg^2\gamma\right)\left[(2-\eta)^2 - 4\nu_1\nu_2\right] = 0. \tag{3.4}$$

In the case of different mixed boundary conditions,[7] from dispersion Equation 3.1 we obtain:

**a)** displacement is prohibited in one of the tangential directions: $\sigma_{yy} = 0$, $\sigma_{yz} = 0$, $u = 0$,

$$(2-\eta)^2 - 4\nu_1\nu_2 - \eta\nu_2^2 ctg^2\gamma = 0, \tag{3.5}$$

**b)** displacement is prohibited in one of the tangential directions: $\sigma_{yy} = 0$, $\sigma_{yx} = 0$, $w = 0$,

$$(2-\eta)^2 - 4\nu_1\nu_2 - \eta\nu_2^2 tg^2\gamma = 0, \tag{3.6}$$

**c)** displacement is prohibited in both tangential directions: $\sigma_{yy} = 0$, $u = 0$, $w = 0$,

$$\eta\left(1 + tg^2\gamma\right) = 0,$$

**d)** displacement is prohibited in the direction of the outer normal: $v = 0$, $\sigma_{yx} = 0$, $\sigma_{yz} = 0$,

$$\eta\nu_1\left(1 + tg^2\gamma\right) = 0,$$

**e)** displacement is prohibited in one of the tangential directions and in the direction of the outer normal: $v = 0$, $w = 0$, $\sigma_{yx} = 0$,

$$\eta\nu_1 + \nu_2 tg^2\gamma\left(1 - \nu_1\nu_2\right) = 0,$$

**f)** displacement is prohibited in one of the tangential directions and in the direction of the outer normal: $v = 0$, $u = 0$, $\sigma_{yz} = 0$,

$$\eta\nu_1 + \nu_2 ctg^2\gamma\left(1 - \nu_1\nu_2\right) = 0,$$

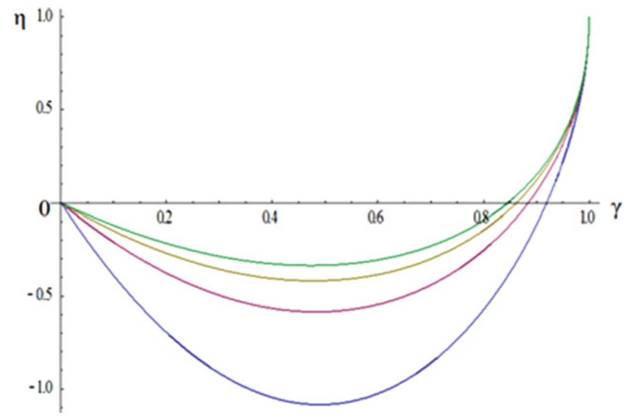


**FIGURE 1** Dependence of the phase velocity of the 3DSW on the propagation angle

**TABLE 2** Dependence of dimensionless phase velocity of the 3DSW on propagation angle when $\theta = 0.33$

| $\gamma$ | **0** | $\pi/6$ | $\pi/4$ | $\pi/3$ | $\pi/2$ |
|---|---|---|---|---|---|
| $\eta$ (3.5) | - | 0.9198 | 0.8850 | 0.8624 | 0.8464 |
| $\eta$ (3.6) | 0.8464 | 0.8624 | 0.8850 | 0.9198 | - |

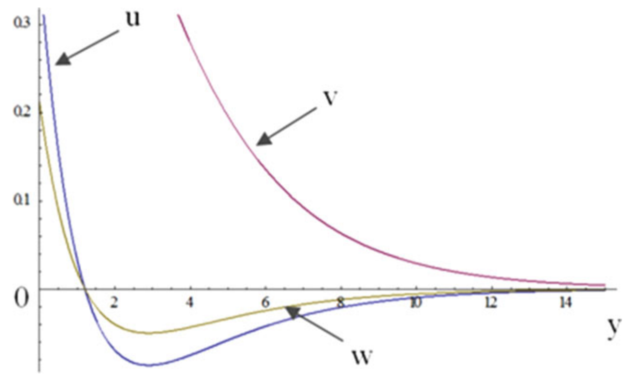


**FIGURE 2** $u, v, w$ displacements dependencies on propagation angle $\gamma = \frac{\pi}{6}$ in the case of free boundary

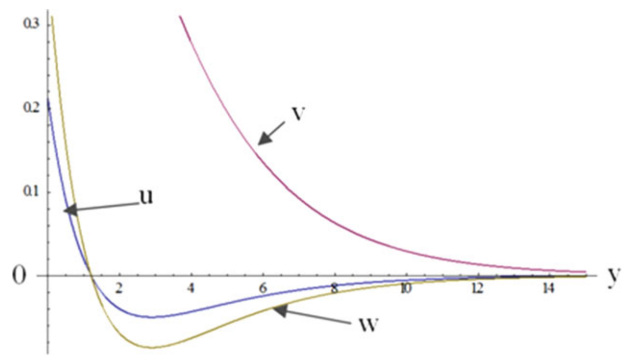


**FIGURE 3** $u, v, w$ displacements dependencies on propagation angle $\gamma = \frac{\pi}{3}$ in the case of free boundary

**g)** displacement is prohibited in all directions: $u = 0$, $v = 0$, $w = 0$,

$$\left(1 + tg^2\gamma\right)\left(1 - \nu_1\nu_2\right) = 0.$$

It turns out, that three-dimensional surface wave exists only for two kinds of boundary conditions. In the case, when the half-surface is stress-free, the known Rayleigh equation is obtained (cf. (3.4)). In the case of restrained free surface (displacement is prohibited in the tangential direction) dispersion equations are reduced to (3.5) or (3.6).

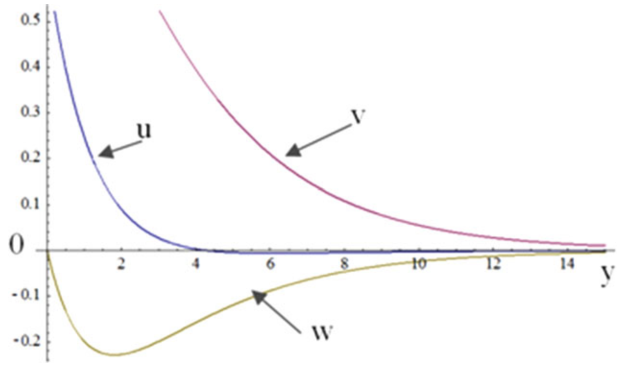


**FIGURE 4** $u, v, w$ displacements dependencies on propagation angle $\gamma = \frac{\pi}{4}$ in the case of elastically restrained free boundary

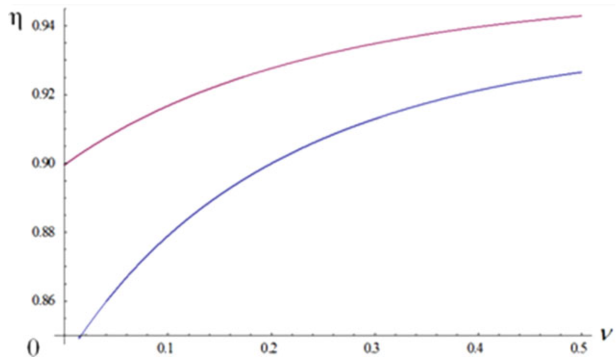


**FIGURE 5** Dependence of phase velocity on propagation angle $\gamma = \frac{\pi}{4}, \frac{\pi}{3}$ for different values of Poisson coefficient in the case of elastically restrained free boundary

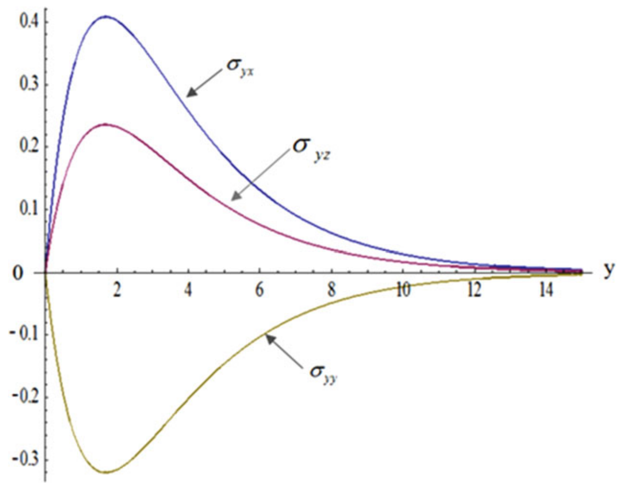


**FIGURE 6** Surface wave form in the case of free boundary for propagation angle $\gamma = \frac{\pi}{6}$

Figure 1 shows the dependence of the phase velocity of the 3DSW on the propagation angle. In Table 2 the values of $\eta$ are shown, which characterize the square of the phase velocity of the 3DSW depending on the propagation angle when $\theta = 0.33$.

Table 2 and plots show that in the case of mixed boundary conditions on the surface, the propagation angle affects the phase velocity of the 3DSW. By the displacement prohibition in one of the tangential directions, when the angle increases, the phase velocity of the 3DSW decreases. When $\gamma = \frac{\pi}{2}(0)$, the value of the phase velocity of the three-dimensional surface wave (3DSW) exactly coincides with the value of the phase velocity of the Rayleigh surface waves.

Figures 2–8 show the results of the 3DSW form computations for different values of propagation angle ($\theta = 0.33$). Plots show, that in the case of elastic restriction of the boundary of the half-space, the propagation angle acts on the damping velocity of the stress-strain condition far from the surface. When $\gamma = \frac{\pi}{2}(0)$, the 3DSW is a Rayleigh wave.

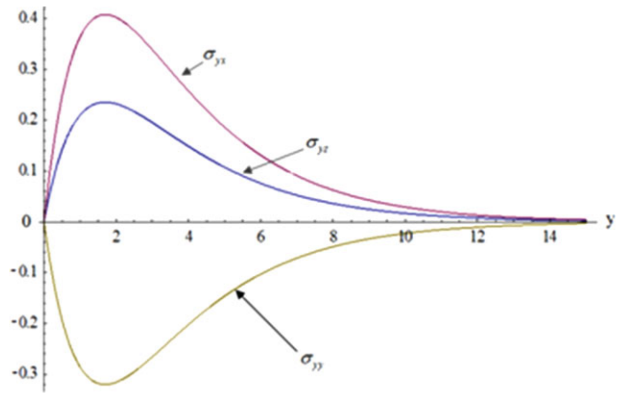


**FIGURE 7** Surface wave form in the case of free boundary for propagation angle $\gamma = \frac{\pi}{3}$

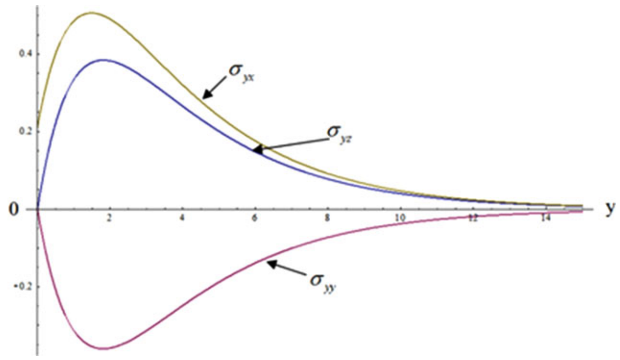


**FIGURE 8** Surface wave form in the case of elastically restrained free boundary for propagation angle $\gamma = \frac{\pi}{4}$

## CONCLUSION

Dispersion equations for three-dimensional problem of wave propagation in a half-space with an elastically restrained surface are obtained. The elastic restriction in plane strain leads to decrease of the localization degree of the surface wave (slow decay of the amplitude). 3DSW exists only for two kinds of boundary conditions: the surface is free from stresses or the surface is free and elastically restrained (displacement is prohibited in one of the tangential direction).

For mixed boundary conditions, the angle of propagation influences the phase velocity of the 3DSW. Increasing the propagation angle, the phase velocity of the 3DSW decreases, tending to the phase velocity of the Rayleigh surface wave.

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**How to cite this article:** Belubekyan MV, Sarkisyan SV. Three-dimensional problem of Rayleigh waves propagating in a half-space with restrained boundary. *Z Angew Math Mech*. 2018;98:1623–1631. https://doi.org/10.1002/zamm.201700157