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Uniqueness Theorem for the Eigenvalues' Function

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Abstract—We study the family of Sturm–Liouville operators, generated by fixed potential q and the family of separated boundary conditions. We prove that the union of the spectra of all these operators can be represented as the values of a real analytic function of two variables. We call this function "the eigenvalues' function" of the family of Sturm–Liouville operators (EVF). We show that the knowledge of some eigenvalues for an infinite set of different boundary conditions is sufficient to determine the EVF, which is equivalent to uniquely determine the unknown potential. Our assertion is the extention of McLaughlin–Rundell theorem.

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1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Let us denote by $L(q,\alpha,\beta)$ the Sturm–Liouville boundary-value problem

$$\ell y \equiv -y'' + q(x)y = \mu y, \quad x \in (0,\pi), \quad \mu \in \mathbb{C}, \tag{1}$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \quad \alpha \in (0,\pi], \tag{2}$$

$$y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \quad \beta \in [0,\pi), \tag{3}$$

where q is a real-valued function, summable on $[0,\pi]$ (we write $q \in L^1_{\mathbb{R}}[0,\pi]$). By $L(q,\alpha,\beta)$ we also denote the self-adjoint operator, generated by the problem (1)–(3) (see [1–3]). It is known, that under these conditions the spectrum of the operator $L(q,\alpha,\beta)$ is discrete and consists of real, simple eigenvalues (see [1–4]), which we denote by $\mu_n = \mu_n(q,\alpha,\beta) = \lambda_n^2(q,\alpha,\beta)$, $n = 0,1,2,\ldots$, emphasizing the dependence of μ_n on q, α and β. We assume that eigenvalues are enumerated in the increasing order, i.e.

$$\mu_0(q,\alpha,\beta) < \mu_1(q,\alpha,\beta) < \cdots < \mu_n(q,\alpha,\beta) < \ldots . \tag{4}$$

For eigenvalues μ_n we have proved (see [4]) that

$$\lim_{\alpha\to 0}\mu_0(q,\alpha,\beta) = \lim_{\beta\to\pi}\mu_0(q,\alpha,\beta) = -\infty$$

and also we obtained new asymptotic formulae (when $n \to \infty$)

$$\mu_n(q,\alpha,\beta) = [n + \delta_n(\alpha,\beta)]^2 + \frac{1}{\pi}\int_0^\pi q(t)\,dt + r_n(q,\alpha,\beta), \tag{5}$$

where $\delta_n(\alpha,\beta)$, $n = 2,3,\ldots$, are smooth and bounded ($-1 \le \delta_n(\alpha,\beta) \le 1$) functions, defined on $[0,\pi] \times [0,\pi]$ (for more details, see [5]) and $r_n(q,\alpha,\beta) \to 0$ (when $n \to \infty$), uniformly by $\alpha,\beta \in [0,\pi]$ and q is from the bounded subsets of $L^1_{\mathbb{R}}[0,\pi]$.

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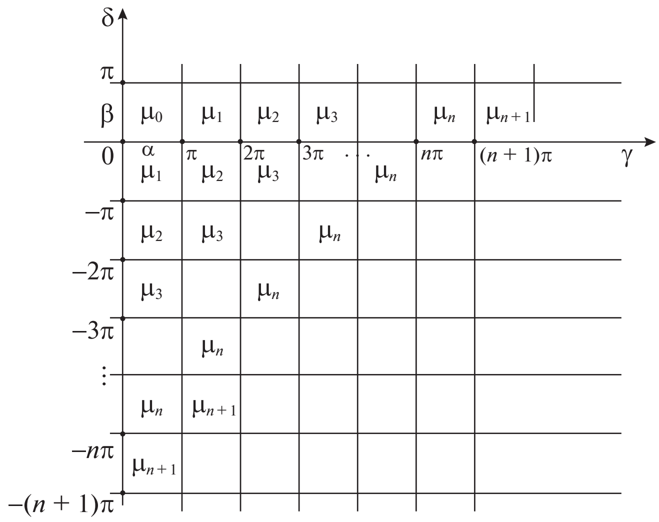

Fig. 1. The domain $(0, \infty) \times (-\infty, \pi)$.

With the aim to investigate "the movement" of the eigenvalues dependending on α and β we introduced the concept of "the eigenvalues function (EVF)" of the family of operators (see [4, 6, 7]) in the following way.

First, we note that arbitrary $\gamma \in (0, \infty)$ can be represented in the form $\gamma = \alpha + \pi n$, where $\alpha \in (0, \pi]$ and $n = 0, 1, 2, \ldots$; and arbitrary $\delta \in (-\infty, \pi)$ can be represented in the form $\delta = \beta - \pi n$, where $\beta \in [0, \pi)$ and $n = 0, 1, 2, \ldots$. In what follows, we assume q is fixed, so when we say "the function $\mu(q, \gamma, \delta)$ of two arguments" we understand that arguments are γ and δ.

Definition. *The function $\mu(q, \gamma, \delta)$ of two arguments, defined on $(0, \infty) \times (-\infty, \pi)$ by the formula*

$$\mu(q, \gamma, \delta) = \mu(q, \alpha + \pi n, \beta - \pi m) := \mu_{n+m}(q, \alpha, \beta), \tag{6}$$

where $\mu_k(q, \alpha, \beta)$, $k = 0, 1, 2, \ldots,$ are the eigenvalues of $L(q, \alpha, \beta)$, enumerated in the increasing order (see (4)), we shall call the eigenvalues' function of the family of operators $\{L(q, \alpha, \beta), \alpha \in (0, \pi], \beta \in [0, \pi)\}$.

The meaning of this definition is easy to understand from Figure 1.

The values of EVF is the union of all spectra $(\{\mu_n(q, \alpha, \beta)\}_{n=0}^{\infty})$ of operators $L(q, \alpha, \beta)$, when (α, β) changes on $(0, \pi] \times [0, \pi)$. For the case $q(x) \equiv 0$ we constract the part of the graph of EVF $\mu(0, \gamma, \delta)$, which contains the part of μ_0 (see Figure 2).

In [4] we have proved, that EVF is a real analitic function of two arguments. This property gives us hope, that the following assertion may be true.

Theorem 1. *Let the EVF $\mu(q_1, \cdot, \cdot)$ and $\mu(q_2, \cdot, \cdot)$ be such that*

$$\mu(q_1, \gamma_k, \beta_1) = \mu(q_2, \gamma_k, \beta_2), \quad k = 1, 2, \ldots, \tag{7}$$

where $\beta_1, \beta_2 \in [0, \pi)$ and $\{\gamma_k\}_{k=1}^{\infty}$ is the sequence with distinct positive elements γ_k, that converges to some $\gamma_0 > 0$, i.e. $\lim_{k\to\infty} \gamma_k = \gamma_0$. Then $q_1(x) = q_2(x)$ a.e. on $[0, \pi]$ and $\beta_1 = \beta_2$.

Similarly, if

$$\mu(q_1, \alpha_1, \delta_k) = \mu(q_2, \alpha_2, \delta_k), \quad k = 1, 2, \ldots, \tag{8}$$

where $\alpha_1, \alpha_2 \in (0, \pi]$ and $\{\delta_k\}_{k=1}^{\infty}$ is the sequence with distinct elements $\delta_k \in (-\infty, \pi)$, that converges to some $\delta_0 \in (-\infty, \pi)$, i.e. $\lim_{k\to\infty} \delta_k = \delta_0$, then $q_1(x) = q_2(x)$ a.e. on $[0, \pi]$ and $\alpha_1 = \alpha_2$.

The paper is organized as follows: in Section 2 we give some preliminaries, in Section 3 we prove Theorem 1 and in Section 4 we consider the connection of our theorem and the theorem of McLaughlin–Rundell.

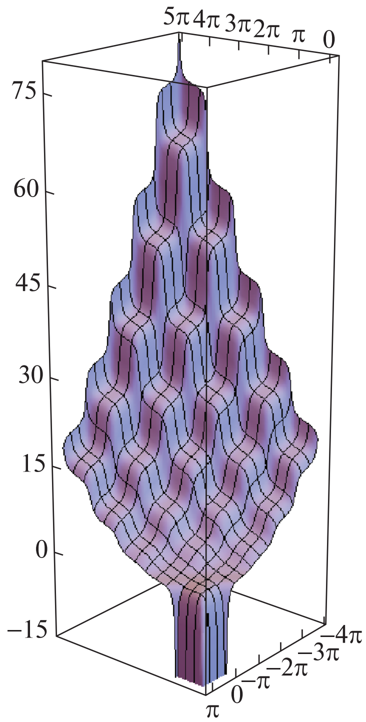

Fig. 2. The graph of the function $\mu(0, \gamma, \delta)$.

2. SOME PRELIMINARIES

Let us denote by $\varphi(x, \mu, \alpha, q)$ (sometimes, for brevity, we will write $\varphi(x, \mu)$) the solution of (1), that satisfies the initial conditions

$$\varphi(0, \mu, \alpha, q) = \sin \alpha, \quad \varphi'(0, \mu, \alpha, q) = -\cos \alpha, \tag{9}$$

and by $\psi(x, \mu, \beta, q)$ (for brevity $\psi(x, \mu)$) the solution of (1), that satisfies the initial conditions

$$\psi(\pi, \mu, \beta, q) = \sin \beta, \quad \psi'(\pi, \mu, \beta, q) = -\cos \beta. \tag{10}$$

It follows from (9) that $\varphi(x, \mu, \alpha, q)$ satisfies the boundary condition (2) and from (10) that $\psi(x, \mu, \beta, q)$ satisfies the boundary condition (3). Therefore, μ^* will be the eigenvalue of $L(q, \alpha, \beta)$ if $\psi(x, \mu^*, \beta, q)$ satisfies (2), or if $\varphi(x, \mu^*, \alpha, q)$ satisfies (3). Thus, the following statement is true.

Lemma 1. *μ^* is an eigenvalue of $L(q, \alpha, \beta)$* if and only if

$$\psi(0, \mu^*, \beta, q) \cos \alpha + \psi'(0, \mu^*, \beta, q) \sin \alpha = 0, \tag{11}$$

or

$$\varphi(\pi, \mu^*, \alpha, q) \cos \beta + \varphi'(\pi, \mu^*, \alpha, q) \sin \beta = 0.$$

Thus, $\varphi(x, \mu_n(q, \alpha, \beta), \alpha, q)$ and $\psi(x, \mu_n(q, \alpha, \beta), \beta, q)$ are eigenfunctions of $L(q, \alpha, \beta)$, corresponding to the eigenvalue $\mu_n(q, \alpha, \beta), n = 0, 1, 2, \ldots$.

It is well-known (see [1, 8]) that $\varphi(x, \mu)$, $\varphi'(x, \mu)$, $\psi(x, \mu)$, $\psi'(x, \mu)$, for a fixed $x[0, \pi]$, are entire functions of μ.

Let us define the functions f_0 and f_π by formulae

$$f_0(\mu) := \psi(0, \mu, \beta_1, q_1)\psi'(0, \mu, \beta_2, q_2) - \psi'(0, \mu, \beta_1, q_1)\psi(0, \mu, \beta_2, q_2),$$
$$f_\pi(\mu) := \varphi(\pi, \mu, \alpha_1, q_1)\varphi'(\pi, \mu, \alpha_2, q_2) - \varphi'(\pi, \mu, \alpha_1, q_1)\varphi(\pi, \mu, \alpha_2, q_2).$$

It is easy to see that f_0 and f_π are also entire functions of μ.

Lemma 2. *If both $\psi(x, \mu^*, \beta_1, q_1)$ and $\psi(x, \mu^*, \beta_2, q_2)$ satisfy (11), then $f_0(\mu^*) = 0$.*

Lemma 3. *If $\psi(x, \mu^*, \beta_1, q_1)$ satisfies (11) and $f_0(\mu^*) = 0$, then $\psi(x, \mu^*, \beta_2, q_2)$ satisfies the (11).*

The proofs of these lemmas are carried out by direct verification.

Similar lemmas are true for the functions $\varphi(x, \mu^*, \alpha, q)$.

Denote (see (7))

$$\nu_k := \mu(q_1, \gamma_k, \beta_1) = \mu(q_2, \gamma_k, \beta_2), \quad k = 1, 2, \dots .$$

Since, by the condition of Theorem 1, the sequence $\{\gamma_k\}_{k=1}^{\infty}$ converges ($\lim_{k\to\infty} \gamma_k = \gamma_0$), then, according to the smoothness of EVF, the sequence $\{\nu_k\}_{k=1}^{\infty}$ also converges:

$$\lim_{k\to\infty} \nu_k = \lim_{k\to\infty} \mu(q_{1,2}, \gamma_k, \beta_{1,2}) = \mu(q_{1,2}, \gamma_0, \beta_{1,2}) = \nu_0.$$

Since the positive γ_k we can represent in the form $\gamma_k = \alpha_k + \pi m$, where $\alpha_k \in (0, \pi]$ and some $m = 0, 1, 2, \dots$, and

$$\nu_k = \mu(q_{1,2}, \alpha_k + \pi m, \beta_{1,2}) = \mu_m(q_1, \alpha_k, \beta_1) = \mu_m(q_2, \alpha_k, \beta_2),$$

are eigenvalues of the operator $L(q_1, \alpha_k, \beta_1)$, then the functions $\psi(x, \nu_k, \beta_1, q_1)$ and $\psi(x, \nu_k, \beta_2, q_2)$ are eigenfunctions of this operator and, therefore, they satisfy the conditions

$$\psi(0, \nu_k, \beta_1, q_1) \cos \alpha_k + \psi'(0, \nu_k, \beta_1, q_1) \sin \alpha_k = 0,$$
$$\psi(0, \nu_k, \beta_2, q_2) \cos \alpha_k + \psi'(0, \nu_k, \beta_2, q_2) \sin \alpha_k = 0,$$

for all $k = 1, 2, \dots$. By Lemma 2 it follows that $f_0(\nu_k) = 0$. Since f_0 is entire function and $\{\nu_k\}_{k=1}^{\infty}$ converges, it follows that $f_0(\mu) \equiv 0, \mu \in \mathbb{C}$. Thus, the following assertion is true.

Lemma 4. *If the condition (7) is satisfied, then $f_0(\mu) \equiv 0, \mu \in \mathbb{C}$. Similarly, if the condition (8) is satisfied, then $f_\pi(\mu) \equiv 0, \mu \in \mathbb{C}$.*

3. THE PROOF OF THEOREM 1

Let now $\mu_n^{(1)} := \mu(q_1, \alpha_1 + \pi n, \beta_1) = \mu_n(q_1, \alpha_1, \beta_1)$, $n = 0, 1, 2, \dots$, i.e. $\{\mu_n^{(1)}\}_{n=0}^{\infty}$ is the spectrum of $L(q_1, \alpha_1, \beta_1)$ for some $\alpha_1 \in (0, \pi]$. In particular, this means that the equality

$$\psi(0, \mu_n^{(1)}, \beta_1, q_1) \cos \alpha_1 + \psi'(0, \mu_n^{(1)}, \beta_1, q_1) \sin \alpha_1 = 0$$

holds for all $n = 0, 1, 2, \dots$. Since $f_0(\mu) \equiv 0$, then $f_0(\mu_n^{(1)}) = 0$ for all $n = 0, 1, 2, \dots$, and from Lemma 3 it follows that the equality

$$\psi(0, \mu_n^{(1)}, \beta_2, q_2) \cos \alpha_1 + \psi'(0, \mu_n^{(1)}, \beta_2, q_2) \sin \alpha_1 = 0$$

holds for all $n = 0, 1, 2, \dots$. This means that all $\mu_n^{(1)}$ are eigenvalues of the operator $L(q_2, \alpha_1, \beta_2)$, or, which is the same, $\{\mu_n(q_1, \alpha_1, \beta_1)\}_{n=0}^{\infty} \subset \{\mu_n(q_2, \alpha_1, \beta_2)\}_{n=0}^{\infty}$. This implies that each $\mu_n(q_1, \alpha_1, \beta_1) = \mu_m(q_2, \alpha_1, \beta_2)$ for some m. But from the asymptotics of eigenvalues (see (5)) it follows that $m = n$, i.e.

$$\mu_n(q_1, \alpha_1, \beta_1) = \mu_n(q_2, \alpha_1, \beta_2), \quad n = 0, 1, 2, \dots . \tag{12}$$

Similarly, if we take $\mu_n^{(2)} = \mu_n(q_1, \alpha_2, \beta_1)$, $n = 0, 1, 2, \dots$, where $\alpha_2 \in (0, \pi]$, $\alpha_1 \neq \alpha_2$, we will obtain, that

$$\mu_n(q_1, \alpha_2, \beta_1) = \mu_n(q_2, \alpha_2, \beta_2), \quad n = 0, 1, 2, \dots . \tag{13}$$

It follows from the equalities (12) and (13), that $q_1(x) = q_2(x)$ a.e. on $[0, \pi]$ and $\beta_1 = \beta_2$ according to the classical two spectrum version of the inverse Sturm–Liouville problem, studied by G. Borg (see [9–13]). Thus the first part of our Theorem 1 is proved. The proof of the second part is completely analogous.

It should be noted that the same proof shows, that the spectrum $\{\mu_n(q_1, \alpha, \beta)\}_{n=0}^{\infty}$ of the operator $L(q_1, \alpha, \beta)$ coincides with the spectrum $\{\mu_n(q_2, \alpha, \beta)\}_{n=0}^{\infty}$ of the operator $L(q_2, \alpha, \beta)$, for arbitrary $\alpha \in (0, \pi]$ and $\beta \in [0, \pi)$. It means that EVF $\mu(q_1, \gamma, \delta) \equiv \mu(q_2, \gamma, \delta)$ for all $\gamma \in (0, \infty)$ and $\delta \in (-\infty, \pi)$. For this reason we call our theorem the uniqueness theorem for the EVF.

4. THEOREM OF MCLAUGHLIN–RUNDELL

In [14] J. McLaughlin and W. Rundell considered the Sturm–Liouville problem

$$-y'' + q(x)y = \lambda y, \quad y(0) = 0, \quad y'(\pi) + \beta y(\pi) = 0,$$

for $q \in L^2_{\mathbb{R}}[0,\pi]$ and denoted by $\lambda_n(q,\beta)$ the eigenvalues of this problem, and proved the following theorem:

Theorem 2. *Let $q_1, q_2 \in L^2_{\mathbb{R}}[0,\pi]$. Fix j, a positive integer. Suppose β_k for $k = 1, 2, \ldots$ are distinct real numbers and $\lambda_j(q_1, \beta_k) = \lambda_j(q_2, \beta_k)$, $\quad k = 1, 2, \ldots$. Then $q_1(x) = q_2(x)$*, a.e.

It is clear that each real number β_k can be represented as $\beta_k = \cot\tilde{\beta}_k$, where $\tilde{\beta}_k \in (0,\pi)$. Since β_k are distinct, then $\tilde{\beta}_k$ are also distinct. The sequence $\{\tilde{\beta}_k\}_{k=1}^{\infty}$ forms a bounded set in $[0,\pi]$ and, consequently has at least one accumulation point $\tilde{\beta}^0$, i.e. there exist a subsequence $\tilde{\beta}_{k_l}$ s.t. $\lim_{k_l\to\infty}\tilde{\beta}_{k_l} = \tilde{\beta}^0$. If we define $\delta_{k_l} = \tilde{\beta}_{k_l} - \pi j$, then $\{\delta_{k_l}\}$ forms a real distinct sequence, which converges to $\tilde{\beta}^0 - \pi j$. Note, that $\lim_{k_l\to\infty}\cot\tilde{\beta}_{k_l} = \cot\tilde{\beta}^0 = \dfrac{\cos\tilde{\beta}^0}{\sin\tilde{\beta}^0}$. And so, the boundary condition $y'(\pi) + \beta y(\pi) = 0$ we can write in the form $y(\pi)\cos\tilde{\beta} + y'(\pi)\sin\tilde{\beta} = 0$ and the whole problem can be considered as the problem $L(q,\pi,\tilde{\beta})$ in our notations. Thus, McLaughlin–Rundell uniqueness theorem is the particular case of our, where the condition (8) is written in the form

$$\mu(q_1, \pi, \delta_m) = \mu(q_2, \pi, \delta_m), \quad m = 1, 2, \ldots,$$

where $\delta_m = \tilde{\beta}_m - \pi j$ converges to $\tilde{\beta}^0 - \pi j$.

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