PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2011, № 1, p. 23–27

Mechanics

ON ONE NONLINEAR DIFFERENTIAL SEVERAL PERSON GAME IN CASE OF MANY AIM SETS

A. A. STEPANYAN*

Chair of Mechanics, YSU

The differential several person game in case of aim sets is considered, the dynamics of which is described by a nonlinear differential equation. A balance set of strategies is constructed by means of the method of extreme aiming at the corresponding set.

***Keywords*:** differential several person games, many aim sets, balance set of strategies.

1. A differential game is considered in the following statement. The dynamics of system is described by the differential equation

$$\dot{x} = f(t, x, u_1, ..., u_k), \quad u_i \in P_i \subset R^{n_i}, \quad i = 1, 2, ..., k. \tag{1.1}$$

Here $x \in R^n$ is a phase vector, u_i is an operating influence of the i–th player, P_i is a compact set in R^{n_i} space, $f : [t_0, \infty) \times R^n \times R^{n_1} \times ... \times R^{n_k} \to R^n$ is the vector-function that is continuous on the set of all arguments at $t_0 \le t \le \theta$ (t_0 and θ are the given instants of time).

Let us introduce the following notations:

$$P = P_1 \times \ldots \times P_k, \; P^{(i)} = P_1 \times \ldots \times P_{i-1} \times P_{i+1} \times \ldots P_k,$$
$$u = (u_1, ..., u_k), \; u^{(i)} = (u_1, ..., u_{i-1}, u_{i+1}, ..., u_k),$$
$$K = \{1, 2, ..., k\}, \; K(i) = \{1, ..., i-1, i+1, ..., k\}.$$

It is assumed that function $f(\cdot)$ satisfies the condition of infinite continuation of solutions, the condition of Lipschitz with respect to x and there is a saddle point for the "small game" [1].

Let ϑ_j ($j = 1, \ldots, m$) be intermediate instants of time on $[t_0, \theta]$ interval such that $t_0 = \vartheta_0 < \vartheta_1 < \ldots < \vartheta_m = \theta$, and the compact sets $M_1^{(j)}, ..., M_k^{(j)}$ ($j = 1, ..., m$) satisfy the following conditions: $M_i^{(j)} \cap G(\vartheta_j, t_0, x_0) \ne \varnothing$ ($i = 1, 2, ..., k$, $j = 1, ..., m$).

Here the area of approachability of system (1.1) from the position $\{t_0, x_0\}$ at an instant ϑ_j is denoted as $G(\vartheta_j, t_0, x_0)$. The set $M_i^{(j)}$ is aim set for the i-th player at the instant ϑ_j.

* E-mail: Mexanikus2006@yahoo.com

Now consider the differential several person game for many aim sets, in which each player tries to reduce the total motion distance from the aim sets at the instants ϑ_j, i.e. the aim of the i-th player is to approach the sets $M_i^{(j)}$ at instants ϑ_j ($j=1,\ldots,m$).

In this work the questions of existence of balance set of strategies concerning to the initial position is considered.

Let $\{t_0,x_0\}$ be an initial position of system (1.1), where $x_0=x(t_0)$, Δ_r is the splitting of half-interval $t_0\le t<\infty$, $\tau_1^{(r)},\tau_2^{(r)},\ldots$ are splitting nodes, then the diameter of splitting will be $\delta_r=\sup\limits_s(\tau_{s+1}^{(r)}-\tau_s^{(r)})$. It is supposed that at any splitting Δ_r the instants ϑ_j ($j=1,\ldots,m$) are splitting nodes, i.e.

$$\tau_{s_0}^{(r)}=t_0=\vartheta_0,\ \tau_{s_1}^{(r)}=\vartheta_1,\ \ldots,\ \tau_{s_m}^{(r)}=\vartheta_m=\theta. \tag{1.2}$$

The sectionally positioning control $u_i^{(s)}[\tau_s^{(r)},x[\tau_s^{(r)}]]$ and strategies of players U_i, the Euler's broken lines

$$x_\Delta^{(s)}\left[\cdot,\tau_s^{(r)},x_\Delta^{(s)}[\tau_s^{(r)}],u_{i_1}[\tau_s^{(r)},x_\Delta^{(s)}[\tau_s^{(r)}]],\ldots,u_{i_l}[\tau_s^{(r)},x_\Delta^{(s)}[\tau_s^{(r)}]],u_{i_{l+1}}[t],\ldots,u_{i_k}[t]\right],$$

generated by the controls of l players ($1\le l\le k$), are defined on $\left[\tau_s^{(r)},\tau_{s+1}^{(r)}\right)$ $(s=0,1,\ldots)$ intervals. On the whole interval $[t_0,\theta]$ the motion of system is defined as an absolutely continuous function, for which one can find the Euler's broken lines that uniformly converge to it. By $X[t_0,x_0,U_{i_1},\ldots,U_{i_l}]$ we denote the totality of all movements leaving the position $\{t_0,x_0\}$ and generated by the strategies $U_{i_1},\ldots,U_{i_l}$ of l players ($1\le l\le k$). It is called the bunch of movements.

2. Let the players work for the stability of situation: the choice of strategy for each player is based on some principle, the deviation from which can result in an increase of the gain of other players. Such a principle is the case at the choice of balance set of strategies defined as follows:

Definition 1. The set of strategies $\{U_1^o,\ldots,U_k^o\}$ is referred to as balanced with respect to the initial position $\{t_0,x_0\}$, if for each number $i\in K$ any movement $x[\cdot]$ from the bunch $X[t_0,x_0,U_1^o,\ldots,U_{i-1}^o,U_{i+1}^o,\ldots,U_k^o]$ avoids meeting the set $M_i^{(j)}$ up to the instant ϑ_j for all $j=1,\ldots,m$.

The balance set of strategies of players will be constructed using the method of aiming at the corresponding set [1].

Let the set W be given in space $[t_0,\theta]\times R^n$, in which for all $t\in[t_0,\theta]$ $W(t)=\{x\mid \{t,x\}\in W\}\ne\varnothing$ and which satisfies the following two conditions:

Condition 1. The set W is closed, and for all $i\in K$ and $j=1,\ldots,m$

$$W(\vartheta_j)\cap M_i^{(j)}=\varnothing.$$

Condition 2. Irrespective of the position $\{t_*,x_*\}\in W$, the number $i\in K$, the vector $u_i^*(\cdot)\in P_i$ and instant of time $t^*\in[t_*,\theta]$, there are admissible strategies $u_\alpha(\cdot)\in P_\alpha$, $\alpha\in K(i)$, such that for the solution $x(\cdot)$ of differential equation

$$\dot{x} = f(t,x,u_1,...,u_{i-1},u_i^*,u_{i+1},...,u_k), \quad x(t_*) = x_*, \tag{2.1}$$

it holds that $\{t^*, x(t^*)\} \in W$ (or, what is the same, there is a movement $x(\cdot)$ from a bunch of movements $X[t_*, x_*, U_i^*]$ such that $\{t^*, x(t^*)\} \in W$).

Now define the set of strategies $U_1^e,...,U_k^e$ extreme to set W, that satisfies the following two conditions [1]:

Condition 3. If $\{t,x\} \notin W$ and $W(t) = \{x / \{t,x\} \in W\} \neq \varnothing$, then for all $i \in K$ the strategies $u_i^e = U_i^e(t,x)$ are found from equality

$$\begin{aligned} &\max_{u_\alpha \in P_\alpha(\alpha \in K(i))} (x-w)' f\left(t,x,u_1,\ldots,u_{i-1},u_i^e,u_{i+1},\ldots,u_k\right) = \\ &= \min_{u_i \in P_i} \max_{u_\alpha \in P_\alpha(\alpha \in K(i))} (x-w)' f\left(t,x,u_1,\ldots,u_k\right). \end{aligned} \tag{2.2}$$

Here w is watever vector (the same for all i) satisfying the condition

$$\|x-w\| = \min_{w \in W(t)} \|x-w\|.$$

Condition 4. If $\{t,x\} \notin W$, but $W(t) = \varnothing$ or $\{t,x\} \in W$, then $u_i^e(t,x)$ is an arbitrary vector from P_i for all $i \in K$.

Theorem 1. Let the set W satisfies Conditions 1 and 2 and $\{t,x\} \in W$. Then the set of strategies defined by the Conditions 3 and 4 is balanced (in the sense of Definition 1) with respect to the initial position $\{t,x\}$.

Proof. Let the i-th player select an admissible strategy U_i. It will be shown that $x(t^*,t,x,U_1^e,...,U_{i-1}^e,U_i,U_{i+1}^e,...,U_k^e) \in W(t^*)$ at $t \le t^* \le \theta$.

Now denote as $\left(K(i),\ i,\ W(t),\ M_i^{(j)}, \{\vartheta_j\}, j=1,...,m\right)$ the differential game of two persons, in which the set of players $K(i)$ seeks to hold out the movement in set W by instants ϑ_j, and the i-th player seeks to deviate the movement of system from sets $W(\vartheta_j)$, $j=1,...,m$.

The strategies defined by Conditions 3 and 4 keep the system state in the set W for any movement from $X\left[t_0,x_0,U_1^e,\ldots,U_{i-1}^e,U_{i+1}^e,...,U_k^e\right]$ commenced in it up to the instants ϑ_j ($j=1,...,m$). Therefore, the extreme strategies form barriers around the set W that impede the exit of movements $x[t]$ from W up to the instants ϑ_j. Hence, according to [1] (Lemma 15.1), the strategies $U_1^e,...,U_{i-1}^e,U_{i+1}^e,...,U_k^e$ would hold the system movement on the set W till the instants of time ϑ_j, $j=1,...,m$, for any strategy U_i, i.e. the set of strategies $U_1^e,...,U_{i-1}^e,U_{i+1}^e,...,U_k^e$ is balanced initial positions from W.

For construction of set W for the following reasoning will be used.

Let the sets $N^{(j)}(j=1,...,m)$ are the convex compacts in R^n that satisfy the conditions $N^{(j)} \cap M_i^{(j)} = \varnothing$, $\quad i \in K$, $j=1,...,m$.

Let the system reach the position $\{t,x\}$ ($\vartheta_{r-1} \le t < \vartheta_r$). Now define

$$\varepsilon_i^0(t,x) = \max_{\|l_\beta^{(i)}\|=1, \beta=r,\dots,m} \sum_{j=r}^{m} (< l_j^{(i)}, x > + \min_{-q\in N^{(j)}} < l_j^{(i)}, q > + \\ + \int_t^\theta \max_{u_i\in P_i} \min_{u_\alpha\in P_\alpha (\alpha\in K(i))} < l_j^{(i)}, f^j(\tau, x, u_1, \dots, u_k) > d\tau), \tag{2.3}$$

if the right hand side is more than zero and $\varepsilon_i^0(t,x)=0$ otherwise. Here

$$f^j(\tau,x,u_1,\dots,u_k) = \begin{cases} f(\tau,x,u_1,\dots,u_k) & \text{at } \tau \le \vartheta_j, \\ 0 & \text{at } \tau > \vartheta_j. \end{cases}$$

Let

$$\varepsilon^0(t,x) = \max_{i\in K} \varepsilon_i^0(t,x), \tag{2.4}$$

$\varepsilon_i^0(t,x)$, $\varepsilon^0(t,x)$ being continuous functions of their arguments. Now introduce the following notations:

- denote as $I^{(0)}(t,x)$ the set of maximizing indexes in (2.4) for position $\{t,x\}$, where $\varepsilon^0(t,x)>0$ and $I^{(0)}(t,x)=K$, if $\varepsilon^0(t,x)=0$;
- denote as $L_i^0(t,x)$ the totality of sets of vectors $l_j^{(i)}$ $(j=r,\dots,m)$ maximizing the functions $\varepsilon_i^0(t,x)$ (2.3), if $\varepsilon_i^0(t,x)>0$ and $L_i^0(t,x)$ completely coincides with the unit sphere in position $\{t,x\}$, where $\varepsilon_i^0(t,x)=0$;

$$L^0(t,x) = \bigcup_{i\in I^0(t,x)} L_i^0(t,x);$$

$$S_i^0(t,x) = \left\{ s_i = \sum_{j=r}^{m} l_j^{(i)}, l_j^{(i)} \in L_i^0(t,x) \right\}; \quad S^0(t,x) = \bigcup_{i\in I^0(t,x)} S_i^0(t,x).$$

We assume that the following conditions are satisfied:

Condition 5. For each number $i\in K$ and for any position $\{t,x\}$, where $\varepsilon_i^0(t,x)>0$, in (2.3) maxima are reached on a unique set $l_j^{(i)^0}$ $(j=r,\dots,m)$.

Condition 6. In each position $\{t,x\}$, where $\varepsilon^0(t,x)>0$, for any numbers $i\in I^0\{t,x\}$, vector $s_i\in S_i^0$ and index $\alpha\in K$ there takes place

$$\max_{u_i\in P_i} \min_{u_\alpha\in P_\alpha} < s_i, f(t,x,u_1,\dots,u_k) > \ \ge \ \max_{u_\alpha\in P_\alpha} \min_{u_i\in P_i} < s_i, f(t,x,u_1,\dots,u_k) >. \tag{2.5}$$

Condition 7. In each position $\{t,x\}$, where $\varepsilon^0(t,x)>0$, for any number $i\in K$ there is a vector $u_i^0\in P_i$, such that for all $s_i\in S_i^0$ there takes place an equality

$$\min_{u_i\in P_i} \max_{u_\alpha\in P_\alpha (\alpha\in K(i))} < s, f(t,x,u_1,\dots,u_k) > = \max_{u_\alpha\in P_\alpha (\alpha\in K(i))} < s, f(t,x,u_1,\dots,u_{i-1},u_i^0 u_{i+1},\dots,u_k) >.$$

Theorem 2. Under Conditions 5, 6, 7 the set $W=\{\{t,x\}, \varepsilon^0(t,x)\le 0\}$ will satisfy Conditions 1, 2.

Proof. As from the definition of $\varepsilon^0(t,x)$, i.e. from (2.3) and (2.4), there follows the closure of set W, hence, the Condition 1 is satisfied. The fulfillment of Condition 2 will be shown by contradiction assuming that there is a position

$\{t_*, x_*\} \in W$, a number $i \in K$, a vector $u_i^0 \in P_i$ and an instant $t^* \in (t_*, \theta)$ such that the solution of differential equation

$$\dot{x} = f(t, x, u_1, ..., u_{i-1}, u_i^0, u_{i+1}, ..., u_k), \quad x(t_*) = x_*, \tag{2.6}$$

for any u_α $(\alpha \in K(i))$, leaves the set W at the instant t^*. Now choose vectors u_α^*, $\alpha \in K(i)$, satisfying the Condition 7:

$$\max_{u_\beta \in P_\beta (\beta \in K(\alpha))} < s, f(t, x, u_1, ..., u_{\alpha-1}, u_\alpha^*, u_{\alpha+1}, ..., u_k) > =$$
$$= \min_{u_\alpha \in P_\alpha} \max_{u_\beta \in P_\beta (\beta \in K(\alpha))} < s, f(t, x, u_1, ..., u_{\alpha-1}, u_\alpha, u_{\alpha+1}, ..., u_k) >, \quad s \in S^0(t, x). \tag{2.7}$$

Here $x(\cdot)$ is a solution of equation (2.6) for controls from (2.7). Then according to above assumptions there is an interval $[t_1, t_2] \subset [t_*, t^*]$ such that $\varepsilon^0(t, x) > 0$ almost for all $t \in [t_1, t_2]$ and $\varepsilon^0(t_1, x(t_1)) < \varepsilon^0(t_2, x(t_2))$.

From [1, 2] it follows that $\varepsilon^0 : t \to \varepsilon^0(t, x(t))$ is a differentiable function of t almost everywhere on $[t_1, t_2]$ and $\exists\, p \in I^0(t, x(t))$

$$\frac{d\varepsilon^0(t, x(t))}{dt} = \frac{d\varepsilon_p(t, x(t))}{dt} = \sum_{j=r}^{m} < l_j^{(p)}, f(t, x, u_1^*, ..., u_{i-1}^*, u_i^0, u_{i-1}^*, ..., u_k^*) > -$$
$$- \sum_{j=r}^{m} \min_{u_p \in P_p} \max_{u_\alpha \in P_\alpha (\alpha \in K(p))} < l_j^{(p)} f(t, x, u_1, ..., u_{p-1}, u_p, u_{p+1}, \ldots, u_k) >. \tag{2.8}$$

After transformations with due regard for (2.7) from (2.8) we obtain

$$\frac{d\varepsilon(t, x(t))}{dt} \le \max_{u_i \in P_i} \min_{u_\rho \in P_\rho} \min_{\substack{u_\alpha \in P_\alpha, \\ \alpha \in K, \alpha \ne p, i}} < s^\rho, f(t, x, u_1, ..., u_k) > -$$
$$- \max_{u_p \in P_p} \min_{u_i \in P_i} \min_{\substack{u_\alpha \in P_\alpha, \\ \alpha \in K, \alpha \ne p, i}} < s^\rho, f(t, x, u_1, ..., u_k) > \le 0.$$

The validity of the last inequality follows from Condition 6. It turns out that almost everywhere on the interval $[t_1, t_2]$ $\frac{d\varepsilon_p}{dt} \le 0$, but $\varepsilon^0(t_1, x(t_1)) \ge \varepsilon^0(t_2, x(t_2))$, what contradicts the assumption. Hence, the Condition 2 holds.

Thus, it is shown that for differential several person games in case of many aim sets the strategies, extreme to a corresponding stable set, make a balance set with respect to initial positions.

Received 24.12.2010

REFERENCES

1. **Krassovski N.N., Subbotin A.I.** Positional Differential Games. M.: Nauka, 1974, 456 p. (in Russian).
2. **Lutmanov S.V.** Vestnik of Perm. Gos. Univ. Matematica, Mechanica, Informatica. Issue 2, № 2, p. 46–54 (in Russian).
3. **Vaisbord E.M., Zhukovsky V.I.** Introduction to the Several Side Differential Games and their Appendix. M., 1980, 304 p. (in Russian).

Ա. Ա. Ստեփանյան

Մի քանի խաղացողներով ոչ գծային դիֆերենցիալ խաղի մասին շատ նպատակային բազմությունների դեպքում

Դիտարկվում է մի քանի խաղացողների դիֆերենցիալ խաղ շատ նպատակային բազմությունների դեպքում, որի դինամիկան բնութագրվում է ոչ գծային դիֆերենցիալ հավասարումներով: Համապատասխան բազմության վրա էքստրեմալ նշանառության մեթոդով կառուցված են հավասարակշռված ստրատեգիաները:

А. А. Степанян

Об одной нелинейной дифференцильной игре нескольких лиц при многих целевых множествах

Рассматривается дифференциальная игра нескольких лиц при многих целевых множествах, динамика которой описывается нелинейным дифференциальным уравнением. Методом экстремального прицеливания на соответствующее множество сконструирован равновесный набор стратегий.