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Upper Bound for the Probability of Ruin in the Alternative Bonus-Malus System: Martingale Approach

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Abstract : This article presents the evaluation of the upper bound for the probability of ruin in a discrete-time insurance model with the help of martingale and supermartingale series. The Alternative Bonus-Malus System is a model where the next premium is the combination of the previous premium and the aggregate claim amount. The premium process forms a martingale series and the aggregate claims of the portfolio form a series of independent and identically distributed random variables.

Keywords: Probability of Ruin, Martingale, Insurance, Bonus-Malus System

MSC2010: 60G42; 62P05; 91B30; 91B70

I. INTRODUCTION

Consider a portfolio of insurance policies, where operates an Alternative bonus-malus system presented in [1] at the stability state. This means that starting from some time t the bonus and malus coefficients will not depend on time and we can consider the following premium model

$$P_k = (1-\alpha)P_{k-1} + \beta Y_k \tag{1}$$

where P_k-is an aggregate premium collected for k-th year of the portfolio, Y_k –is the aggregate claim loss for the given portfolio within $(k-1;k)$ time interval, α is the bonus factor and β is the malus factor. It is necessary to note that Y_k is independent of P_{k-1} for all k.

The surplus process of the portfolio is defined by

$$U_n = u + \sum_{k=1}^{n} P_k - \sum_{k=1}^{n} Y_k \tag{2}$$

where $u = U_0$ is the initial surplus, P_k is the aggregate premium defined by (1) and we assume that Y_k's are independent and identically distributed (i.i.d.) random variables with $VarY < \infty$.

Now recall some definitions from ruin theory (see for instance [2], [3]).

Definition 1: *The event that U ever falls below zero is called ruin:*

$$Ruin = \{U_n < 0 \; for \; some \; n\}.$$

Definition 2: *The time $\tau(u)$when the process falls below zero for the first time is called ruin time:*

$$\tau(u) = inf\{n > 0; U_n < 0\}.$$

The probability of ruin is then given by

$$\psi(u) = P\left(\bigcup_{n\geq 0}\{U_n < 0\}\,|U_0 = u\right) = P\left(\inf_{n\geq 0} U_n < 0\,|U_0 = u\right) = P(\tau(u) < \infty)$$

Evaluation with Martingales

Write

$$Z_k = Y_k - P_k = Y_k - (1-\alpha)P_{k-1} - \beta Y_k = (1-\beta)Y_k - (1-\alpha)P_{k-1} \qquad (3)$$

This variable shows the net loss of the portfolio at time k.

The total net loss of the portfolio up to time n is defined as

$$S_n = Z_1 + \cdots + Z_n, \quad n \geq 1, \quad S_0 = 0 \qquad (4)$$

So, for the probability of ruin we have the following equivalent expression:

$$\psi(u) = P\left(\inf_{n\geq 1}(-S_n) \leq -u\right) = P\left(\sup_{n\geq 1} S_n > u\right)$$

We know nothing about the independence of Z_k's, but we can calculate their variation. From (1) P_k can be expressed as:

$$P_k = (1-\alpha)^k P_0 + (1-\alpha)^{k-1}\beta Y_1 + \cdots + (1-\alpha)\beta Y_{k-1} + \beta Y_k \qquad (5)$$

So, using the independence of Y_k's we get

$$Var(P_k) = Var((1-\alpha)^k P_0 + (1-\alpha)^{k-1}\beta Y_1 + \cdots + (1-\alpha)\beta Y_{k-1} + \beta Y_k)$$
$$= Var(Y)\beta^2 \frac{1-(1-\alpha)^{2k}}{1-(1-\alpha)^2}$$

Using the relationship (3) and independence of Y_k and P_{k-1}we get:

$$Var(Z_k) = (1-\beta)^2 Var(Y) + (1-\alpha)^2 Var(P_{k-1})$$
$$= Var(Y)\left((1-\beta)^2 + (1-\alpha)^2\beta^2 \frac{1-(1-\alpha)^{2(k-1)}}{1-(1-\alpha)^2}\right)$$

For $Var(S_n)$ we can write:

$$Var(S_n) = Var\left(\sum_{k=1}^{n} Z_k\right) = \sum_{k=1}^{n} Var(Z_k) + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} cov(Z_i Z_j)$$

From (3) and (5) we conclude that

$$Var(S_n) \sim A \cdot n \cdot Var(Y) + B \cdot (1-(1-\alpha)^{C\cdot n}) \qquad (6)$$

where $A, B, C < \infty$ are constants.

In the case of (6) we can use Markov's theorem on the applicability of the law of large numbers with any type of dependence, so we have

$$\frac{Var(S_n)}{n^2} \sim \frac{A \cdot Var(Y)}{n} + \frac{B \cdot (1-(1-\alpha)^{Cn})}{n^2} \xrightarrow[n\to\infty]{} 0$$

S_n satisfies to the law of large numbers, that is $\frac{S_n}{n} \to EZ$, which in particular implies, that $S_n \to +\infty$ or $S_n \to -\infty$ according to the sign of EZ. Hence, if $EZ \geq 0$, ruin is unavoidable in the case of any starting capital u.

Proposition: *If EY is finite and the condition*

$$EZ = (1-\beta)EY - (1-\alpha)P_0 \geq 0$$

holds then, for any fixed $u > 0$, ruin occurs with probability 1.

The insurance company should choose the bonus and malus coefficients in such a way, that $EZ < 0$. In this case the company avoids ruin occurring with probability 1 and may hope to have $\psi(u) < 1$.

Definition 3: *(Net Profit Condition): The process Z satisfies to the net profit condition (NPC), if*

$$EZ = (1-\beta)EY - (1-\alpha)P_0 < 0 \qquad (7)$$

Taking the expectations in (5), recalling $P_0 = EP_k$ martingale property and using the i.i.d. property of Y_k's we have:

$$P_0 = (1-\alpha)^k P_0 + \frac{1-(1-\alpha)^k}{\alpha}\beta EY,$$

which gives:

$$P_0 = \frac{\beta EY}{\alpha}$$

By inputting it in the condition (7), we get the following NPC

$$\alpha < \beta.$$

This result is in accordance with the interpretation of $\frac{\alpha}{\beta}$ as the loss ratio of the portfolio.

Using (5) for S_n we can write:

$$S_n = \sum_{k=1}^{n} Y_k - \sum_{k=1}^{n} P_k$$
$$= (Y_1 + Y_2 + \cdots + Y_n)$$
$$- \left(P_0 \sum_{k=1}^{n} (1-\alpha)^k + \beta Y_1 \sum_{k=1}^{n} (1-\alpha)^{k-1} + \beta Y_2 \sum_{k=2}^{n} (1-\alpha)^{k-2} + \cdots + \beta Y_n \right)$$
$$= -P_0 \frac{(1-\alpha)(1-(1-\alpha)^n)}{\alpha} - \left(\frac{\beta}{\alpha} - 1\right) Y_1 - \cdots \left(\frac{\beta}{\alpha} - 1\right) Y_n$$
$$+ \frac{\beta}{\alpha} ((1-\alpha)^n Y_1 + \cdots + (1-\alpha) Y_n)$$

Denote

$$C_n = P_0 \frac{(1-\alpha)(1-(1-\alpha)^n)}{\alpha}$$

$$M_n = Y_1 + Y_2 + \cdots + Y_n$$

and

$$G_n = (1-\alpha)^n Y_1 + \cdots + (1-\alpha) Y_n$$

So, we have

$$S_n = -C_n - \left(\frac{\beta}{\alpha} - 1\right) M_n + \frac{\beta}{\alpha} G_n$$

Denote $\varphi_X(t) = Ee^{tX}$ the moment generating function (m.g.f.) of the random variable X.

Lemma 1: *For any $\gamma > 0$, and for any $\alpha, \beta \in (0,1)$ satisfying the NPC, the sequence*

$$M_n^{'} = \frac{e^{-\gamma\left(\frac{\beta}{\alpha}-1\right)M_n}}{\varphi_Y^n\left(-\gamma\left(\frac{\beta}{\alpha}-1\right)\right)}$$

is a martingale.

Proof: Note that $M_{n+1} = M_n + Y_{n+1}$, Calculate

$$E\left(M_{n+1}^{'}\middle|Y_1,Y_2,\dots,Y_n\right)=E\left(\frac{e^{-\gamma\left(\frac{\beta}{\alpha}-1\right)M_{n+1}}}{\varphi_Y^{n+1}\left(-\gamma\left(\frac{\beta}{\alpha}-1\right)\right)}\middle|Y_1,Y_2,\dots,Y_n\right)$$

$$=E\left(\frac{e^{-\gamma\left(\frac{\beta}{\alpha}-1\right)(M_n+Y_{n+1})}}{\varphi_Y^{n+1}\left(-\gamma\left(\frac{\beta}{\alpha}-1\right)\right)}\middle|Y_1,Y_2,\dots,Y_n\right)=M_n^{'}\frac{Ee^{-\gamma\left(\frac{\beta}{\alpha}-1\right)Y_{n+1}}}{\varphi_Y\left(-\gamma\left(\frac{\beta}{\alpha}-1\right)\right)}=M_n^{'}$$

■

Lemma 2: *For any $\gamma>0$, such that $\varphi_Y(\gamma)<\infty$ and for any $\alpha,\beta\in(0,1)$ satisfying the NPC, the sequence*

$$G_n^{'}=\frac{e^{\gamma\frac{\beta}{\alpha}G_n}}{\varphi_Y^n\left(\gamma\frac{\beta}{\alpha}(1-\alpha)\right)}$$

is a supermartingale.

Proof: Note that

$$G_{n+1}=(1-\alpha)^{n+1}Y_1+\dots+(1-\alpha)^2Y_n+(1-\alpha)Y_{n+1}$$
$$=(1-\alpha)((1-\alpha)^nY_1+\dots+(1-\alpha)Y_n)+(1-\alpha)Y_{n+1}=(1-\alpha)(G_n+Y_{n+1})$$

Calculate

$$E\left(G_{n+1}^{'}\middle|Y_1,Y_2,\dots,Y_n\right)=E\left(\frac{e^{\gamma\frac{\beta}{\alpha}G_{n+1}}}{\varphi_Y^{n+1}\left(\gamma\frac{\beta}{\alpha}(1-\alpha)\right)}\middle|Y_1,Y_2,\dots,Y_n\right)=$$

$$=\frac{e^{\gamma\frac{\beta}{\alpha}(1-\alpha)G_n}}{\varphi_Y^n\left(\gamma\frac{\beta}{\alpha}(1-\alpha)\right)}E\left(\frac{e^{\gamma\frac{\beta}{\alpha}(1-\alpha)Y_{n+1}}}{\varphi_Y\left(\gamma\frac{\beta}{\alpha}(1-\alpha)\right)}\middle|Y_1,Y_2,\dots,Y_n\right)$$

$$=\frac{\left(e^{\gamma\frac{\beta}{\alpha}G_n}\right)^{(1-\alpha)}}{\varphi_Y^n\left(\gamma\frac{\beta}{\alpha}(1-\alpha)\right)}\cdot\frac{Ee^{\gamma\frac{\beta}{\alpha}(1-\alpha)Y_{n+1}}}{\varphi_Y\left(\gamma\frac{\beta}{\alpha}(1-\alpha)\right)}$$

The second multiplier is 1 according to the definition of m.g.f. The denominator of the first multiplier is the same as the denominator of the $G_n^{'}$ and we have that $(1-\alpha)\in(0,1)$, so, the nominator is less than the nominator of $G_n^{'}$, so we get:

$$E\left(G'_{n+1} \middle| Y_1, Y_2, \dots, Y_n\right) \le G'_n$$

which is the definition of the supermartingale. ■

Lemma 3: *For any* $\gamma > 0$, *such that* $\varphi_Y(\gamma) < \infty$ *and for any* $\alpha, \beta \in (0,1)$ *satisfying the NPC, the sequence*

$$S'_n = \frac{e^{\gamma S_n}}{\varphi_Y^n\left(-\gamma\left(\frac{\beta}{\alpha} - 1\right)\right)\varphi_Y^n\left(\gamma\frac{\beta}{\alpha}(1-\alpha)\right)} \quad (8)$$

is a supermartingale.

Proof: Note that

$$S_{n+1} = -C_{n+1} - \left(\frac{\beta}{\alpha} - 1\right)M_{n+1} + \frac{\beta}{\alpha}G_{n+1}$$
$$= -(C_n + (1-\alpha)^{n+1}) - \left(\frac{\beta}{\alpha} - 1\right)(M_n + Y_{n+1}) + \frac{\beta}{\alpha}(1-\alpha)(G_n + Y_{n+1})$$

Calculate

$$E\left(S'_{n+1} \middle| Y_1, Y_2, \dots, Y_n\right) = E\left(\frac{e^{\gamma S_{n+1}}}{\varphi_Y^{n+1}\left(-\gamma\left(\frac{\beta}{\alpha} - 1\right)\right)\varphi_Y^{n+1}\left(\gamma\frac{\beta}{\alpha}(1-\alpha)\right)} \middle| Y_1, Y_2, \dots, Y_n\right)$$

$$= E\left(\frac{e^{\gamma\left(-(C_n+(1-\alpha)^{n+1}) - \left(\frac{\beta}{\alpha}-1\right)(M_n+Y_{n+1}) + \frac{\beta}{\alpha}(1-\alpha)(G_n+Y_{n+1})\right)}}{\varphi_Y^{n+1}\left(-\gamma\left(\frac{\beta}{\alpha} - 1\right)\right)\varphi_Y^{n+1}\left(\gamma\frac{\beta}{\alpha}(1-\alpha)\right)} \middle| Y_1, Y_2, \dots, Y_n\right)$$

$$= \frac{e^{\gamma\left(-(C_n+(1-\alpha)^{n+1}) - \left(\frac{\beta}{\alpha}-1\right)M_n + \frac{\beta}{\alpha}(1-\alpha)G_n\right)}}{\varphi_Y^{n}\left(-\gamma\left(\frac{\beta}{\alpha} - 1\right)\right)\varphi_Y^{n}\left(\gamma\frac{\beta}{\alpha}(1-\alpha)\right)} E\left(\frac{e^{\gamma\left(-\left(\frac{\beta}{\alpha}-1\right)Y_{n+1} + \frac{\beta}{\alpha}(1-\alpha)Y_{n+1}\right)}}{\varphi_Y\left(-\gamma\left(\frac{\beta}{\alpha} - 1\right)\right)\varphi_Y\left(\gamma\frac{\beta}{\alpha}(1-\alpha)\right)} \middle| Y_1, Y_2, \dots, Y_n\right)$$

Here γ, $(1-\alpha)$ and $\left(\frac{\beta}{\alpha} - 1\right)$ are positive according to their definition and NPC, so it can be proved that $Cov\left(e^{-\gamma\left(\frac{\beta}{\alpha}-1\right)Y}, e^{\gamma\frac{\beta}{\alpha}(1-\alpha)Y}\right) \le 0$ (see for instance [4], [5]). For the second multiplier we write:

$$E\left(\frac{e^{\gamma\left(-\left(\frac{\beta}{\alpha}-1\right)Y_{n+1}+\frac{\beta}{\alpha}(1-\alpha)Y_{n+1}\right)}}{\varphi_Y\left(-\gamma\left(\frac{\beta}{\alpha}-1\right)\right)\varphi_Y\left(\gamma\frac{\beta}{\alpha}(1-\alpha)\right)}\middle|Y_1,Y_2,\dots,Y_n\right)=\frac{E\left(e^{-\gamma\left(\frac{\beta}{\alpha}-1\right)Y}\cdot e^{\gamma\frac{\beta}{\alpha}(1-\alpha)Y}\right)}{Ee^{-\gamma\left(\frac{\beta}{\alpha}-1\right)Y}Ee^{\gamma\frac{\beta}{\alpha}(1-\alpha)Y}}$$

$$=\frac{Cov\left(e^{-\gamma\left(\frac{\beta}{\alpha}-1\right)Y},e^{\gamma\frac{\beta}{\alpha}(1-\alpha)Y}\right)+Ee^{-\gamma\left(\frac{\beta}{\alpha}-1\right)Y}Ee^{\gamma\frac{\beta}{\alpha}(1-\alpha)Y}}{Ee^{-\gamma\left(\frac{\beta}{\alpha}-1\right)Y}Ee^{\gamma\frac{\beta}{\alpha}(1-\alpha)Y}}$$

$$=1+\frac{Cov\left(e^{-\gamma\left(\frac{\beta}{\alpha}-1\right)Y},e^{\gamma\frac{\beta}{\alpha}(1-\alpha)Y}\right)}{Ee^{-\gamma\left(\frac{\beta}{\alpha}-1\right)Y}Ee^{\gamma\frac{\beta}{\alpha}(1-\alpha)Y}}\leq 1$$

Using Lemmas 1 and 2, for the first multiplier we have:

$$\frac{e^{\gamma\left(-(C_n+(1-\alpha)^{n+1})-\left(\frac{\beta}{\alpha}-1\right)M_n+\frac{\beta}{\alpha}(1-\alpha)G_n\right)}}{\varphi_Y^n\left(-\gamma\left(\frac{\beta}{\alpha}-1\right)\right)\varphi_Y^n\left(\gamma\frac{\beta}{\alpha}(1-\alpha)\right)}=\frac{e^{\gamma S_n}}{e^{\gamma((1-\alpha)^{n+1}+\alpha G_n)}\varphi_Y^n\left(-\gamma\left(\frac{\beta}{\alpha}-1\right)\right)\varphi_Y^n\left(\gamma\frac{\beta}{\alpha}(1-\alpha)\right)}$$

$$\leq\frac{e^{\gamma S_n}}{\varphi_Y^n\left(-\gamma\left(\frac{\beta}{\alpha}-1\right)\right)\varphi_Y^n\left(\gamma\frac{\beta}{\alpha}(1-\alpha)\right)}=S_n^{'}$$

The inequality holds as $\gamma((1-\alpha)^{n+1}+\alpha G_n)>0$.

So, we get

$$E\left(S_{n+1}^{'}\middle|Y_1,Y_2,\dots,Y_n\right)\leq S_n^{'} \qquad \blacksquare$$

Now we can find an upper bound for the probability of ruin in the model (2).

Theorem: *If for some $\gamma>0$, the process $S_n^{'}$ given by (8) is a supermartingale, where $S_n\to-\infty$ as $n\to\infty$, then*

$$\psi(u)\leq\frac{e^{-\gamma u}}{E\left(\frac{e^{-\gamma U_{\tau(u)}}}{\varphi_Y^{\tau(u)}\left(-\gamma\left(\frac{\beta}{\alpha}-1\right)\right)\varphi_Y^{\tau(u)}\left(\gamma\frac{\beta}{\alpha}(1-\alpha)\right)}\middle|\tau(u)<\infty\right)} \qquad (9)$$

Proof: Doob's Theorem on optional stopping time for the supermartingale $S_n^{'}$ at time $\tau(u)^\wedge T=\min(\tau(u),T)$ satisfies the inequality

$$ES_0^{'}\geq ES_{\tau(u)^\wedge T}^{'}$$

We cannot use the stopping time $\tau(u)$ directly because $P(\tau(u) = \infty) > 0$ and also because the conditions of the optional stopping theorem present a problem; however, using $\tau(u)^\wedge T$ invokes no problems because $\tau(u)^\wedge T$ is bounded by T.

Using the condition $S_0 = 0$, we get:

$$1 = ES_0' \geq ES_{\tau(u)^\wedge T}' = E\left(S_{\tau(u)}'; \tau(u) \leq T\right) + E\left(S_T'; \tau(u) > T\right) \qquad (10)$$

The second term in (10) converges to 0, as $T \to \infty$ due to the condition $S_n \to -\infty$ as $n \to \infty$.

Recall that $U_n = u - S_n$, and $\psi(u) = P(\tau(u) < \infty)$. For (10) write

$$1 \geq E\left(S_{\tau(u)}'; \tau(u) < \infty\right) = E\left(\frac{e^{\gamma S_{\tau(u)}}}{\varphi_Y^{\tau(u)}\left(-\gamma\left(\frac{\beta}{\alpha} - 1\right)\right)\varphi_Y^{\tau(u)}\left(\gamma\frac{\beta}{\alpha}(1-\alpha)\right)}; \tau(u) < \infty\right)$$

$$= e^{\gamma u} E\left(\frac{e^{-\gamma U_{\tau(u)}}}{\varphi_Y^{\tau(u)}\left(-\gamma\left(\frac{\beta}{\alpha} - 1\right)\right)\varphi_Y^{\tau(u)}\left(\gamma\frac{\beta}{\alpha}(1-\alpha)\right)}\middle| \tau(u) < T\right) P(\tau(u) < \infty)$$

So, we have the statement of the theorem. ■

II. CONCLUSION

The paper considered a surplus model, where the premiums form a martingale series. On the other hand there was no additional probabilistic assumption on claim process except i.i.d. The upper bound for the probability of ruin was found out with the help of martingales and supermartingales, which make the calculation processes much easier.

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