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M a t h e m a t i c s

ON CHARACTERISTIC FUNCTIONS OF CHECKING TESTS OF BINARY TABLES

Ed. V. YEGHIAZARYAN *

Chair of Discrete Mathematics and Theoretical Informatics YSU, Armenia

In the paper correspondence between the monotonic functions and the set of irredundant checking tests of binary tables is formed. The maximum number of irredundant checking tests of binary table is determined.

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Introduction. Let T be a binary table (consisting of 0 and 1) of n columns, rows that are pairwise distinct. A subset of the columns of this table with numbers $\{i_1, i_2 ..., i_k\} (1 \leq i_1 < i_2 < ... < i_k \leq n)$ is called a diagnostic test, or just a test of table T, if in the subtable, formed by these columns, all the rows are also pairwise distinct.

The test is called an irredundant test, if its any proper subset does not form a test.

Further, a subset of columns in a table is said to be checking test of the table, if in the subtable, formed by those columns, all the rows differ from the first.

The checking test is said to be an irredundant checking test, if its any proper subset does not form a checking test [1]. According to [2], the task of finding all the irredundant tests (irredundant checking tests) can be reduced to the task of deciphering a monotone Boolean function, depending upon the variables, which in some sense enumerates all the tests (checking test) of table.

This function is called to be the characteristic (monotone) function of the tests (checking tests) of the table. We recall some concepts related to the monotone functions [3].

Let $B = \{0, 1\}$, $B^n = \{\tilde{\alpha} / \tilde{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n), \alpha_i \in B, 1 \leq i \leq n\}$. We say that a vector $\tilde{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$ is preceded by another vector $\tilde{\beta} = (\beta_1, \beta_2, ..., \beta_n)$ and denote it by $\tilde{\alpha} \prec \tilde{\beta}$, if $\alpha_i \leq \beta_i$, $i = 1, 2, ..., n$. Two vectors are called incomparable, if they are not preceded by each other. A Boolean function $f(x_1, x_2, ..., x_n)$

* E-mail: eduardyeg@mail.ru

is called monotone, if for any vectors $\tilde{\alpha}, \tilde{\beta}$ with $\tilde{\alpha} \prec \tilde{\beta}$ we have $f(\alpha_1, \alpha_2, ..., \alpha_n) \leq f(\beta_1, \beta_2, ..., \beta_n)$.

A vector $\tilde{\beta} = (\beta_1, \beta_2, ..., \beta_n)$ is said to be the *bottom unit* of a monotone function f, if $f(\tilde{\beta}) = f(\beta_1, \beta_2, ..., \beta_n) = 1$, but for any (another) vector $\tilde{\alpha}$ with $\tilde{\alpha} \prec \tilde{\beta}$ we have $f(\tilde{\alpha}) = 0$. A vector $\tilde{\alpha}$ is called the *upper zero* of a monotone function f, if $f(\tilde{\alpha}) = 0$, and any (another) vector $\tilde{\beta}$ $\tilde{\alpha} \prec \tilde{\beta}$ satisfies $f(\tilde{\beta}) = 1$.

Let V and U be the sets of all upper zeros and all bottom units of $f(x_1, x_2, ..., x_n)$ respectively. It is easy to see that the set V possesses the following properties:

1) any two vectors from V are incomparable;

2) if $\tilde{\alpha} \notin V$, then $f(\tilde{\alpha}) = 0 \Leftrightarrow \exists \tilde{\beta} \in V(\tilde{\alpha} \prec \tilde{\beta})$.

Conversely, suppose there is given an arbitrary set V of pairwise incomparable sets. Then exists a unique monotonic function $f(x_1, x_2, ..., x_n)$, for which V is the set of all its upper zeros, i.e.

$f(\tilde{\alpha}) = 0$, if $\tilde{\alpha} \in V$ or $\exists \tilde{\beta} \in V(\tilde{\alpha} \prec \tilde{\beta})$, and $f(\tilde{\alpha}) = 1$, otherwise.

Similarly, the monotonic function is uniquely defined by the set U of all its bottom units.

Consistency between the set of tests (checking tests) of tables and characteristic monotone functions is established as follows. Let $\tilde{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n) \in B^n$ be a vector such that $\alpha_{i_1} = \alpha_{i_2} = ... = \alpha_{i_k} = 1$ and the other components are equal to 0. So, if the columns corresponding to the test (checking test) of the table T are numbered by $\{i_1, i_2 ..., i_k\}$, then we put $f(\tilde{\alpha}) = 1$, otherwise $f(\tilde{\alpha}) = 0$. It is easy to see that the function f is monotonic, and its bottom units correspond to the irredundant (checking irredundant) tests of the table T. The described characteristic function bears in itself all the information about the structure of the tests (checking tests) of the table T.

We have the following natural problem: can any monotonic function be a characteristic function of a tests (checking tests) of some table? In the case of tests this question has negative answer: there is no table of three columns [4], for which the monotonic function $f(x_1, x_2, x_3) = x_1 x_2 \vee x_3$ is the characteristic function of the tests.

In this paper we consider the case of checking tests, where the situation is different. We prove the following

Theorem. For any monotonic function $f(x_1, x_2, ..., x_n)$, which is not identically 0 or 1, there exists a table of n columns, for which $f(x_1, x_2, ..., x_n)$ is the characteristic function of its checking tests.

Proof. Let vectors $\tilde{\beta}_i = (\beta_{i1}, \beta_{i2}, ..., \beta_{in}) \in B^n$, $i = 1, 2, ..., m$, form a set $V = \{\tilde{\beta}_1, \tilde{\beta}_2, ..., \tilde{\beta}_m\}$ of upper zeros of a function $f(x_1, x_2, ..., x_n)$. We use them to construct the following table T of $m+1$ rows and n columns:

$$T = \begin{array}{l} 1\ 1 ....1 \\ \beta_{11}\beta_{12}.....\beta_{1n} \\ \beta_{21}\beta_{22}.....\beta_{2n} \\ .... \\ \beta_{i1}\beta_{i2}......\beta_{in} \\ .... \\ \beta_{m1}\beta_{m2}....\beta_{mn} \end{array}.$$

We assert, that the function $f(x_1,x_2,...,x_n)$ is the characteristic function of the checking tests of the table T. Indeed, let the characteristic function of the checking tests of the table T is the monotonic function $g(x_1,x_2,...,x_n)$. Let us find out, the vectors satisfying $g(x_1,x_2,...,x_n) = 0$. Let $\tilde{\delta} = (\delta_1,\delta_2,...,\delta_n)$ be a vector such, that $\delta_{i_1} = \delta_{i_2} = ... = \delta_{i_k} = 1$, the other components are equal to 0 and $g(\delta_1,\delta_2,...,\delta_n) = 0$. This means, that the subset of the columns of the table T with numbers $\{i_1,i_2,...,i_k\}$ is not its checking test. This will occur, if and only if a subset $\{i_1,i_2,...,i_k\}$ of the columns from a row is selected only from the identity components, i.e. $\beta_{ii_1} = \beta_{ii_2} = ... = \beta_{ii_k} = 1$. But this means that $\tilde{\delta} = (\delta_1,\delta_2,...,\delta_n) \prec \prec \tilde{\beta}_i = (\beta_{i1},\beta_{i2},...,\beta_{in})$. From this and from property 2) of the set of upper zeros it follows that the set $V = \{\tilde{\beta}_1,\tilde{\beta}_2,...,\tilde{\beta}_m\}$ is also the set of all upper zeros the function $g(x_1,x_2,...,x_n)$. Since the monotonic function is uniquely determined by the set of all its upper zeros, then $g(x_1,x_2,...,x_n) = f(x_1,x_2,...,x_n)$. □

In [5] it is shown that there is no table of n columns, having $C_n^{[\frac{n}{2}]}$ irredundant tests. However, for checking tests the following statement holds:

Corollary of the Theorem. The maximum number of irredundant checking tests from the table of n columns is not greater then $C_n^{[\frac{n}{2}]}$ and there exists a table, for which this value is achieved.

The proof follows from the Theorem and from the fact that the maximum number of the bottom units of a monotonic function equal to $C_n^{[\frac{n}{2}]}$ (this is the maximum number of pairwise incomparable vectors of length n [6]).

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