Subvariety structures in certain product varieties of groups

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Abstract. We classify certain cases when the wreath products of distinct pairs of groups generate the same variety. This allows us to investigate the subvarieties of some nilpotent-by-abelian product varieties $UV$ with the help of wreath products of groups. In particular, using wreath products, we find such subvarieties in nilpotent-by-abelian $UV$, which have the same nilpotency class, the same length of solubility, and the same exponent, but which still are distinct subvarieties. The classification we obtain strengthens our recent work on varieties generated by wreath products.

1 Introduction

The objective of this note is to study subvarieties generated by wreath products in certain product varieties of groups, and to discover the cases when wreath products of distinct pairs of groups generate the same variety of groups. Equivalently, in more specific notation given below, we investigate the cases when equality (***) holds for the pairs of groups $A_1, B_1$ and $A_2, B_2$.

Wreath products are among the main tools to study products $UV$ of varieties of groups. The methods used in the literature typically consider groups $A$ and $B$ generating $U$ and $V$, respectively, and then find extra conditions, under which the wreath product $A \text{ Wr } B$ generates $UV$, i.e., conditions, under which the equality

$$\text{var}(A \text{ Wr } B) = \text{var}(A) \text{ var}(B)$$

holds for $A$ and $B$ (here the Cartesian wreath product is assumed, but all the results we give are true for direct wreath products also). For the chronological development of this approach and for background information on varieties of groups or on wreath products we refer to [2, 3, 5, 8, 19, 20, 24] and to the literature cited therein.

Generalizing some results in the cited literature, we in [13–18] were able to suggest criteria classifying all the cases when (**) holds for groups from certain

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classes of groups: abelian groups, $p$-groups, nilpotent groups of finite exponent, etc. (see, in particular, the very brief outline of results in [17, Section 5]).

Here we turn to a sharper problem of comparison of two varieties, both generated by wreath products. Namely, take $A_1, B_1$ and $A_2, B_2$ to be pairs of non-trivial groups such that $\text{var}(A_1) = \text{var}(A_2)$, $\text{var}(B_1) = \text{var}(B_2)$, and distinguish the cases when the equality

$$\text{var}(A_1 \operatorname{Wr} B_1) = \text{var}(A_2 \operatorname{Wr} B_2) \quad (**)$$

holds. The main classification criterion given in Theorem 2.3 covers the cases of nilpotent $A_1, A_2$ and abelian $B_1, B_2$, with some restrictions on exponents.

Besides getting a generalization of $(*)$, our study of equality $(**)$ is motivated by some applications one of which we would like to outline here. The classification of subvariety structures of $\mathfrak{U} \mathfrak{V}$ is incomplete even when $\mathfrak{U}$ and $\mathfrak{V}$ are such “small” varieties as the abelian varieties $\mathfrak{A}_m$ and $\mathfrak{A}_n$, respectively. Here are some of the results in this direction: $\mathfrak{A}_p^2$ (for prime numbers $p$) are the simplest non-trivial varieties, as they consist of the Cartesian powers of the cycle $C_p$ only. Kovács and Newman in [10] fully described the subvariety structure in the product $\mathfrak{A}_p^2 = \mathfrak{A}_p \mathfrak{A}_p$ for $p > 2$. Later they continued this classification for the varieties $\mathfrak{A}_p \mathfrak{A}_p$. Their research was unpublished for many years, and it appeared in 1994 only [11] (parts of their proof are present in [4]). Another direction is the description of subvarieties in the product $\mathfrak{A}_m \mathfrak{A}_n$, where $m$ and $n$ are coprime. This is done by Houghton (mentioned by Hanna Neumann in [20, 54.42]), by Cossey (Ph.D. thesis [6], mentioned by Bryce in [4]). A more general result of Bryce classifies the subvarieties of $\mathfrak{A}_m \mathfrak{A}_n$, where $m$ and $n$ are nearly prime in the sense that, if a prime $p$ divides $m$, then $p^2$ does not divide $n$ (see [4]). In 1967 Hanna Neumann wrote that a classification of subvarieties of $\mathfrak{A}_m \mathfrak{A}_n$ for arbitrary $m$ and $n$ “seems within reach” [20]. Bryce in 1970 mentioned that “classifying all metabelian varieties is at present slight” [4]. However, nearly half a century later this task is not yet accomplished: Bakhturin and Olshanskii remarked in the survey [1] of 1988 and of 1991 that a “classification of all nilpotent metabelian group varieties has not been completed yet”.

As this brief summary shows, one of the cases, when the subvariety structure of $\mathfrak{U} \mathfrak{V}$ is less known, is the case when $\mathfrak{U}$ and $\mathfrak{V}$ have non-coprime exponents divisible by high powers $p^u$ for many prime numbers $p$. Thus, even if we cannot classify all the subvarieties in some product varieties $\mathfrak{U} \mathfrak{V}$, it may be interesting to find those subvarieties in $\mathfrak{U} \mathfrak{V}$, which are generated by wreath products. We, surely, can take any groups $A \in \mathfrak{U}$ and $B \in \mathfrak{V}$, and then $A \operatorname{Wr} B$ will generate some subvariety in $\mathfrak{U} \mathfrak{V}$. But in order to make this approach reasonable, we still have to detect whether or not two wreath products of that type generate the same subvariety, i.e., whether or not equality $(**)$ holds for the given pairs of groups.
Yet another outcome of this research may be stressed. In the literature the different subvarieties are often distinguished by their different nilpotency classes, different lengths of solubility, or different exponents (see, for example, the classification of subvarieties of $\mathfrak{W}_2^p$ in [10]). Using the wreath products technique, we construct below such subvarieties of $\mathfrak{W}$, which have the same nilpotency class, the same length of solubility, the same exponent, but which are distinct subvarieties (see the examples in Section 4 below).

Since our study of (*) has already been given in detail, including some very recent publications, we do not want to directly or indirectly repeat here any proof fragment which might have been presented in our earlier publications. Instead, we just refer to facts in the respective articles, and give the links to our ArXiv files in the References.

2 Equivalence of the $p$-primary components and the main theorem

Before we turn to the main focus, on groups of finite exponent only, let us briefly discuss equality (**) for some other cases also.

If $B_1$ and $B_2$ are both abelian groups that do not have finite exponent, then they are discriminating groups [13, 19, 20] and, thus, for any $A_1$ and $A_2$ equality (*) holds for the pairs $A_1, B_1$ and $A_2, B_2$ (see [19, 20]). If, in addition, we have the conditions $\text{var}(A_1) = \text{var}(A_2)$ and $\text{var}(B_1) = \text{var}(B_2)$, then clearly

$$\text{var}(A_1) \text{ var}(B_1) = \text{var}(A_2) \text{ var}(B_2),$$

and so (**) also holds for these groups $A_1, B_1, A_2, B_2$.

If one of $B_1$ and $B_2$ is of finite exponent and the other is not, then we have $\text{var}(B_1) \neq \text{var}(B_2)$, and we do not need to consider (**) in this case, at all.

These examples show why the finite exponent case is the most interesting one, and from here on we consider only that case.

By Prüper’s theorem any abelian group $B$ of finite exponent is a direct product of its finite cyclic subgroups. Recall that the $p$-primary component $B(p)$ of $B$ is the subgroup of $B$ consisting of all elements whose orders are powers of $p$. Since $B$ clearly is a direct product of its $p$-primary components $B(p)$, the orders of the aforementioned cyclic subgroups can be supposed to be powers of primes. If the cardinality of the cyclic factors $C_{p^u}$ of order $p^u$ in this decomposition is $\mu_{p^u}$, we can write their direct product as $C_{p^u}^{\mu_{p^u}}$. Then $B(p)$ is a product of factors of that type:

$$B(p) = C_{p^{u_1}}^{\mu_{p^{u_1}}} \times \cdots \times C_{p^{u_r}}^{\mu_{p^{u_r}}},$$

where we may suppose $u_1 \geq \cdots \geq u_r$. The cardinal numbers $\mu_{p^{u_1}}, \ldots, \mu_{p^{u_r}}$ are invariants of $B(p)$ in the sense that they characterize $B(p)$ uniquely (see Fuchs’
textbook [7, Section 35], from where we also adopted the symbols \( \mu_p^u \) and the above notation. If \( B(p) \) is finite, then all the cardinals \( \mu_p^u1, \ldots, \mu_p^ur \) clearly are finite. Otherwise, at least one of them will be infinite, and we can always choose the first one of such infinite invariants \( \mu_p^u1 \).

**Example 2.1.** Consider the group

\[ B = C_3^6 \times C_{3^3}^{N_0} \times C_3^5 \times C_3^4 \times C_5^2. \]

For the 3-component \( B(3) \) of \( B \) we have \( u_1 = 5, \mu_3u_1 = 6; u_2 = 3, \mu_3u_2 = \infty_0; u_3 = 2, \mu_3u_3 = 5; u_4 = 1, \mu_3u_4 = \infty. \) So, the first infinite factor of \( B(3) \) is \( C_3^{N_0} \), although the factor \( C_3^N \) is of higher cardinality. And for the 5-component \( B(5) \) we have \( u_1 = 3, \mu_5u_1 = 4; u_2 = 2, \mu_5u_2 = 1, \) so \( B(5) \) has no infinite factor.

Let \( B_1 \) and \( B_2 \) be abelian groups of finite exponent, and for a given prime \( p \), let their \( p \)-primary components \( B_1(p) \) and \( B_2(p) \) have direct decompositions of the above type:

\[
B_1(p) = C_{\mu_1}^{\mu_p^u1} \times \cdots \times C_{\mu_r}^{\mu_p^ur}, \quad (2.1)
\]

\[
B_2(p) = C_{\nu_1}^{\nu_p^v1} \times \cdots \times C_{\nu_s}^{\nu_p^vs}. \quad (2.2)
\]

Define a specific equivalence relation \( \equiv \) between such \( B_1(p) \) and \( B_2(p) \). Namely:

1. If \( B_1(p), B_2(p) \) are both finite, then \( B_1(p) \equiv B_2(p) \) if and only if \( B_1(p) \) and \( B_2(p) \) are isomorphic, i.e., \( r = s \) and \( u_i = v_i, \mu_p^u_i = \mu_p^v_i \) for each \( i = 1, \ldots, r \).
2. If \( B_1(p), B_2(p) \) are both infinite, then \( B_1(p) \equiv B_2(p) \) if and only if there is a \( k \) such that:
   
   (i) \( u_i = v_i \) and \( \mu_p^u_i = \mu_p^v_i \) for each \( i = 1, \ldots, k - 1 \),
   
   (ii) \( C_{\mu_p^u_k}^{\mu_p^u_k} \) is the first infinite factor in decomposition (2.1), \( C_{\mu_p^v_k}^{\mu_p^v_k} \) is the first infinite factor in decomposition (2.2), and \( u_k = v_k \).
3. \( B_1(p), B_2(p) \) are not equivalent for all other cases.

Notice that in point (ii) above we do not require that \( C_{\mu_p^u_k}^{\mu_p^u_k} \) and \( C_{\mu_p^v_k}^{\mu_p^v_k} \) be isomorphic. We require them both to be direct products of infinitely many copies of the same cyclic group \( C_{\mu^u_k} \). In particular, \( \mu_p^u_k \) and \( \mu_p^v_k \) can be distinct infinite cardinal numbers.

**Example 2.2.** In order to get a group equivalent to the group \( B \) of Example 2.1, we can replace in \( B \) the factors \( C_3^{3^2} \) and \( C_3^N \) by arbitrary direct product of copies of the cycles \( C_3^2 \) and \( C_3 \). Also, we can replace \( C_3^{N_0} \) by, say, \( C_3^N \). However, we cannot alter any of the remaining factors \( C_3^{5^6}, C_5^4, C_5^2 \).
In these terms our main theorem is:

**Theorem 2.3.** Let $A_1, A_2$ be non-trivial nilpotent groups of exponent $m$ generating the same variety, and let $B_1, B_2$ be non-trivial abelian groups of exponent $n$ generating the same variety, where any prime divisor $p$ of $n$ also divides $m$. Then equality (***) holds for $A_1, A_2, B_1, B_2$ if and only if $B_1(p) \equiv B_2(p)$ for each $p$.

Notice how the roles of the passive and active groups of these wreath products are different: for $A_1, A_2$ we just require that $\text{var}(A_1) = \text{var}(A_2)$, whereas for $B_1, B_2$ we put extra conditions on the structures of their decompositions.

**Corollary 2.4.** In the notation of Theorem 2.3:

(a) Equality (***) holds for finite groups $B_1, B_2$ if and only if $B_1$ and $B_2$ are isomorphic.

(b) Equality (***) never holds if one of the groups $B_1, B_2$ is finite, and the other is infinite.

### 3 The proof of Theorem 2.3

Since we are going to intensively use nilpotency classes of wreath products, let us proceed to introduce Shield’s formula [22, 23]. By the well-known theorem of Baumslag, a Cartesian wreath product $A \text{Wr} B$ of non-trivial groups $A$ and $B$ is nilpotent if and only if $A$ is a nilpotent $p$-group of finite exponent, and $B$ is a finite $p$-group for a prime number $p$ [2]. The analog of this also holds for direct wreath products. Liebeck calculated the nilpotency class of $A \text{Wr} B$ for the particular case when the groups $A$ and $B$ of Baumslag’s theorem are abelian [12]. The complete formula for any nilpotent $p$-group $A$ of finite exponent and any finite $p$-group $B$ was found by Shield, and to present it we need to briefly introduce the $K_p$-series. See also [15] where we explain the construction in more detail, with illustrative examples.

For an (arbitrary) group $B$ and prime number $p$ the $K_p$-series $K_{i,p}(B)$ of $B$ is defined for $i = 1, 2, \ldots$ as the product

$$K_{i,p}(B) = \prod \{\gamma_r(B)^{p^j} : \text{for all } r, j \text{ such that } rp^j \geq i\},$$

where $\gamma_r(B)$ is the $r$-th term of the lower central series of the group $B$, and $\gamma_r(B)^{p^j} = \langle g^{p^j} : g \in \gamma_r(B) \rangle$ is the $p^j$-th power of $\gamma_r(B)$. Clearly, $K_{1,p}(B) = B$ holds for any $B$. If $B$ is abelian, then $\gamma_2(B) = [B, B] = \{1\}$, and so in the product above only the powers $\gamma_1(B)^{p^j} = B^{p^j}$ of the initial term $\gamma_1(B) = B$ need to be considered.
When $B$ is a finite $p$-group, the following additional parameters are introduced: let $d$ be the maximal integer such that $K_{d,p}(B) \neq \{1\}$. For each $s = 1, \ldots, d$ define $e(s)$ by the equality $p^{e(s)} = |K_{s,p}/K_{s+1,p}|$, and set $a$ and $b$ by the rules:

$$a = 1 + (p - 1) \sum_{s=1}^{d} (s \cdot e(s)), \quad b = (p - 1)d.$$ 

Finally, let $s(h)$ be defined as follows: $p^{s(h)}$ is the exponent of the $h$-th term $\gamma_{h}(A)$ of the lower central series of the nilpotent $p$-group $A$ of finite exponent. Then Shield’s formula [23] states that the nilpotency class of the wreath product $A \text{Wr} B$ is equal to the maximum

$$\max_{h=1,\ldots,c} \{a \cdot h + (s(h) - 1)b\}. \quad (3.1)$$

After these preparations we turn back to the proof of Theorem 2.3. The first and very simple step to start with is to reduce equality (**) to its particular case when $A_1 = A_2 = A$:

$$\text{var}(A \text{Wr} B_1) = \text{var}(A \text{Wr} B_2). \quad (3.2)$$

Recall that for any given class $\mathfrak{X}$ of groups, $Q\mathfrak{X}$, $S\mathfrak{X}$, and $C\mathfrak{X}$ denote the classes of all homomorphic images, subgroups, and Cartesian products of groups of $\mathfrak{X}$, respectively. By Birkhoff’s theorem [20, 15.23], for any class of groups $\mathfrak{X}$ the equality $\text{var}(\mathfrak{X}) = QSC \mathfrak{X}$ holds.

**Lemma 3.1.** For any groups $A_1, A_2$ generating the same variety the equality

$$\text{var}(A_1 \text{Wr} B) = \text{var}(A_2 \text{Wr} B)$$

holds for any group $B$.

**Proof.** If $A_1 \in \text{var}(A_2)$, then we have $A_1 \in QSC \{A_2\}$. Then by [20, 22.11] and [13, Lemma 1.1] we have $A_1 \text{Wr} B \in \text{var}(A_2 \text{Wr} B)$ for any $B$. The inverse inclusion $A_2 \text{Wr} B \in \text{var}(A_1 \text{Wr} B)$ is proved analogously. \hfill \Box

Returning to the notation of Theorem 2.3, we always have the equality

$$\text{var}(A_1 \text{Wr} B_1) = \text{var}(A_2 \text{Wr} B_1)$$

for the groups $A_1, A_2, B_1$. So, we can just assume that $A_1 = A_2$ and write $A$ for the group $A_1 = A_2$, and reduce our study of equality (**) to the study of (3.2), which is going to be our main objective for the sequel. We are going to achieve it in the following steps: first we consider the case of $p$-groups only, and find a few necessary conditions for equality (3.2) for some specific cases of $p$-groups in Lemma 3.2, Lemma 3.5. Then Lemma 3.6 shows that the combination of these
necessary conditions is also sufficient. In Lemma 3.8 and in the final proof we will
deduce the general case from the cases obtained for $p$-groups.

Suppose $A, B_1, B_2$ are non-trivial nilpotent $p$-groups of finite exponent, and
$B_1, B_2$ have the decompositions (2.1) and (2.2), respectively (in which we clearly
have $B_1(p) = B_1$, $B_2(p) = B_2$, since we deal with $p$-groups).

For any groups $X$ and $Y$ of finite exponent we have

$$\exp(X \text{ Wr } Y) = \exp(X) \cdot \exp(Y).$$

Thus, the first easy observation is that, if (3.2) holds for $A, B_1, B_2$, then the
exponents of $B_1$ and $B_2$ are equal, i.e., $u_1 = v_1$. Otherwise, the wreath pro-
ducts $A \text{ Wr } B_1$ and $A \text{ Wr } B_2$ would also have distinct exponents, and they would
generate distinct varieties.

If $m_{p^{u_1}}$ and $m_{p^{v_1}}$ are finite and equal, then the first factors $C_{p^{u_1}}^{m_{p^{u_1}}}$ and $C_{p^{v_1}}^{m_{p^{v_1}}}$
in (2.1) and (2.2) are coinciding finite groups. For the sequel let $t$ denote the index
for which $C_{p^{u_1}}^{m_{p^{u_1}}}$ and $C_{p^{v_1}}^{m_{p^{v_1}}}$ in (2.1) and (2.2) are coinciding finite factors for each
$i = 1, \ldots, t - 1$, but not for $i = t$. That is, the $t$-th factors $C_{p^{u_t}}^{m_{p^{u_t}}}$ and $C_{p^{v_t}}^{m_{p^{v_t}}}$ are
either non-isomorphic finite groups, or at least one of them is infinite. Clearly,
if $B_1 \neq B_2$, then such a $t$ exists, and the case $t = 1$ is not ruled out.

**Lemma 3.2.** In the above circumstances, if the $t$-th factors $C_{p^{u_t}}^{m_{p^{u_t}}}$ and $C_{p^{v_t}}^{m_{p^{v_t}}}$ are
non-isomorphic finite groups, then equality (3.2) does not hold for $A, B_1, B_2$.

**Proof.** We have two options:

(a) $u_t = v_t$, and then $m_{p^{u_t}} \neq m_{p^{v_t}}$, and so we may suppose $m_{p^{u_t}} > m_{p^{v_t}}$,

(b) $u_t \neq v_t$, and we may suppose $u_t > v_t$ (the values of $m_{p^{u_t}}$ and $m_{p^{v_t}}$ are
immaterial in this case).

Let us give the proof for the first option; the second can be proved by slight adap-
tation, by adding to $B_2$ the “missing” factor $C_{p^{v_t}}^{m_{p^{u_t}}}$ with $v_t = u_t$ and $m_{p^{v_t}} = 0$.

We suppose equation (3.2) holds, and proceed to derive a contradiction. Denote
$u_t = v_t$ by $w$, and set

$$P_1 = (A \text{ Wr } B_1)^{p^{w-1}} \quad \text{and} \quad P_2 = (A \text{ Wr } B_2)^{p^{w-1}}.$$

Since the wreath product $A \text{ Wr } B_2$ is an extension of its base subgroup $A^{B_2}$ by the
active group $B_2$, we have

$$P_2/(A^{B_2} \cap P_2) \cong (P_2 A^{B_2})/A^{B_2} \leq (A \text{ Wr } B_2)/A^{B_2} \cong B_2,$$

i.e., the subgroup $P_2$ of $A \text{ Wr } B_2$ is an extension of some subgroup $A^{B_2} \cap P_2$ of
$A^{B_2}$ by some subgroup of $B_2$ and, in fact, of $B_2^{p^{w-1}}$ taking into account the multi-
plication rule in wreath products. By the Kaloujnine–Krasner theorem [20, 22.21],

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$P_2$ can be embedded in the wreath product $A_{B_2} B_2^{p_{u-1}}$, and we can find an upper bound for the nilpotency class of $P_2$ by applying Shield’s formula to this wreath product. Clearly,

$$B_2^{p_{u-1}} \cong C_{p^{v_{1-w}+1}}^{m} \times \cdots \times C_{p^{v_{t-1-w}+1}}^{m} \times C_{p}^{m},$$

and it is a simple routine computation to find the $K_p$-series for (3.3):

$$K_{1,p}(B_2^{p_{u-1}}) = B_2^{p_{u-1}}, \quad K_{2,p}(B_2^{p_{u-1}}) = \cdots = K_{p,p}(B_2^{p_{u-1}}) = (B_2^{p_{u-1}})^p,$$

because for the indices $i = 2, \ldots, p$ the least power $p^j$ for which $p^j \geq i$ is given by $j = 1$. Clearly,

$$(B_2^{p_{u-1}})^p \cong C_{p^{v_{1-w}}}^{m} \times \cdots \times C_{p^{v_{t-1-w}}}^{m},$$

(notice how the factor $C_{p}^{m}$ has disappeared as it is of exponent $p$).

$$K_{p+1,p}(B_2^{p_{u-1}}) = \cdots = K_{p^2,p}(B_2^{p_{u-1}}) = (B_2^{p_{u-1}})^{p^2},$$

because for the indices $i = p + 1, \ldots, p^2$ the least power $p^j$ for which $p^j \geq i$ is given by $j = 2$. So,

$$(B_2^{p_{u-1}})^{p^2} \cong C_{p^{v_{1-w-1}}}^{m} \times \cdots \times C_{p^{v_{t-1-w-1}}}^{m}.$$

As the process continues so more and more factors will be lost. In the last steps we obtain, for some $r$,

$$K_{p^r+1,p}(B_2^{p_{u-1}}) = \cdots = K_{p^{r+1},p}(B_2^{p_{u-1}}) = (B_2^{p_{u-1}})^{p^{r+1}} = C_{p}^{m},$$

and, finally,

$$K_{p^{r+1}+1,p}(B_2^{p_{u-1}}) = K_{p^{r+1}+2,p}(B_2^{p_{u-1}}) = \cdots = \{1\}.$$

In this notation $d = p^{r+1}$. The values of $e(s), s = 1, \ldots, d$ are easy to obtain as they are non-zero in the following cases only:

$$|K_{1,p}(B_2^{p_{u-1}})/K_{2,p}(B_2^{p_{u-1}})| = p^{m} = p^{e(1)},$$

$$|K_{p,p}(B_2^{p_{u-1}})/K_{p+1,p}(B_2^{p_{u-1}})| = p^{e(p)},$$

$$|K_{p^2,p}(B_2^{p_{u-1}})/K_{p^2+1,p}(B_2^{p_{u-1}})| = p^{e(p^2)},$$

$$\vdots$$

$$|K_{p^{r+1},p}(B_2^{p_{u-1}})/K_{p^{r+1}+1,p}(B_2^{p_{u-1}})| = |K_{d,p}(B_d^{p_{u-1}})/K_{d+1}(B_d^{p_{u-1}})| = p^{e(d)}$$

(by our assumptions all these $e(s), s = 1, \ldots, d$, are finite).
It remains to compute the values of \(a, b,\) and \(s(h),\) and to obtain the nilpotency class of \(A^{B_2} \text{Wr} B_2^{p w - 1}\) by formula (3.1). We do not need to write down that class because the information relevant to this proof is already given above.

Next we construct a specific subgroup \(P_1^*\) in \(P_1.\) Let \(\{a_l : l \in \mathcal{L}\}\) be any generating set of \(A,\) and choose a set of generators for the first \(t\) factors of \(B_1\) in (2.1) as follows. For each \(i = 1, \ldots, t\) let \(c_{ij}\) be a generator of the \(j\)-th copy of the cycle \(C_{p^{u_i}}\) in \(C_{p^{m_{u_i}}}^{m_{u_i}}\) for \(j = 1, \ldots, m_{u_i},\) i.e.,

\[
C_{p^{u_i}}^{m_{u_i}} = \prod_{j=1}^{m_{u_i}} \langle c_{ij} \rangle.
\]

We have \(m = m_{p^{u_1}} + \cdots + m_{p^{u_t}}\) generators \(c_{ij}\) in total, and each element \(b\) in the product of the first \(t\) factors of \(B_1\) can be written as

\[
b = c_1^{e_1} \cdots c_t^{e_m} c_{p^{w_1}}^{m_{p^{w_1}}},
\]

for some non-negative integer exponents \(e_1, \ldots, e_m.\) For each generator \(a_l\) define an element \(\pi_l\) in the base subgroup \(A^{B_1}\) of \(A\text{ Wr} B_1\) as follows. Set \(\pi_l (b) = a_l,\) if in presentation (3.4) of \(b\) the first exponent \(e_1\) is zero, and all the remaining exponents \(e_2, \ldots, e_m\) are less than \(p^{w_1} - 1.\) Set \(\pi_l (b) = 1\) for all other values of \(b \in B_1.\)

Using standard arguments (see, for example, [21, Section 5]), it is easy to verify that

\[
(c_1^{p^{w_1}})^{p^{w_1} - 1} = (c_1^{p^{w_1}})^{p^{w_1} - 1}\theta_l,
\]

where \(\theta_l (b) = a_l,\) if in presentation (3.4) all the exponents \(e_1, \ldots, e_m\) are less than \(p^{w_1} - 1,\) and \(\theta_l (b) = 1\) for other values \(b \in B_1.\) Since

\[
\theta_l = (c_1^{p^{w_1}})^{-1}(c_1^{p^{w_1}})^{p^{w_1} - 1},
\]

we have

\[
\theta_l \in P_1 = (A\text{ Wr} B_1)^{p^{w_1} - 1},
\]

and the elements \(\theta_l\) together with all the powers \(c_{ij}^{p^{w_1}}\) generate a subgroup \(P_1^*\)

\(P_1^*\)

in \(P_1.\)

Each element \(c_{ij}^{p^{w_1}}\) generates a cyclic subgroup \(\langle c_{ij}^{p^{w_1}} \rangle\) of order \(p^{u_i - w_1 + 1}\) inside the respective cyclic subgroup

\[
\langle c_{ij} \rangle \cong C_{p^{u_i}}.
\]

Taking into account the “\(p^{w_1} - 1\) step shifting effect” of the elements \(c_{ij}^{p^{w_1}}\) on the base subgroup for any indices \(i', j',\) it is easy to see that \(P_1^*\) is isomorphic to the wreath product

\[
A\text{ Wr} B_1^{p^{w_1} - 1} \cong A\text{ Wr}(C_{p^{u_1 - w_1 + 1}}^{m_{p^{u_1}} - 1} \times \cdots \times C_{p^{u_t - w_1 + 1}}^{m_{p^{u_t}} - 1} \times C_{p^{u_t}}^{m_{p^{u_t}}})
\]

\(= A\text{ Wr} B_1^*.
\)

(3.5)
If we now compute the nilpotency class of (3.5) by Shield’s formula, it will also be a lower bound for the nilpotency class of $P_1$.

The $K_p$-series for the active group $B_1^*$ is

$$K_1, p(B_1^*) = B_1^*,$$

$$K_{2, p}(B_1^*) = \cdots = K_{p, p}(B_1^*) = (B_1^*)^p \cong C_p^{\mu_1}_{u_1} \times \cdots \times C_p^{\mu_t}_{u_t}$$

(the factor $C_p^{\mu_t}$ has again disappeared as it is of exponent $p$) and

$$K_{p+1, p}(B_1^*) = \cdots = K_{p^2, p}(B_1^*) = (B_1^*)^{p^2} \cong C_p^{\mu_1}_{u_1} \times \cdots \times C_p^{\mu_t}_{u_t}.$$ 

In the last steps we obtain, for the same wreath products $A$ of $W$ and, finally,

$$K_{p^r+1, p}(B_1^*) = \cdots = K_{p^{r+1}, p}(B_1^*) = (B_1^*)^{p^{r+1}} = C_p^{\mu_1}_{u_1},$$

and, finally,

$$K_{p^r+1+1, p}(B_1^*) = K_{p^{r+1}+2, p}(B_1^*) = \cdots = \{1\}.$$ 

We again have the same $d = p^{r+1}$. The only non-zero values of $e(s)$ for $s = 1, \ldots, d$ are:

$$|K_{1, p}(B_1^*)| = p^{\mu_{u_1} + \cdots + \mu_{u_t}} = p^{e(1)},$$

$$|K_{p, p}(B_1^*)| = p^{e(p)},$$

$$|K_{p^2, p}(B_1^*)| = p^{e(2)},$$

$$\vdots$$

$$|K_{p^{r+1}, p}(B_1^*)| = p^{e(p^{r+1})} = p^{e(d)}.$$ 

Then we can compute the values of $a$, $b$, and $s(h)$, and get the nilpotency class of $W^*$ by formula (3.1).

Let us compare the parameters $e(s)$, $a$, $b$, $s(h)$ that we calculated above for the wreath products $A^{B_2, Wr} B_2^{P_{u-1}}$ and $A \wr B_1^*$, respectively. The parameter $e(1)$ is larger for the wreath product $A \wr B_1^*$ than for $A^{B_2, Wr} B_2^{P_{u-1}}$ because

$$\mu_{p^{u_t}} > \mu_{p^v},$$

and so the direct factors $C_p$ appear in $B_1^*$ strictly more times than in $B_2^{P_{u-1}}$.

The parameters $e(2), e(3), \ldots, e(d)$ will be the same for $A^{B_2, Wr} B_2^{P_{u-1}}$ and for $A \wr B_1^*$ because $u_i = v_i$ for all $i = 1, \ldots, t - 1$.

Since $a$ is calculated as $a = 1 + (p - 1) \sum_{s=1}^{d}(s \cdot e(s))$, we have that the value of the parameter $a$ is strictly larger for $A \wr B_1^*$ rather than for $A^{B_2, Wr} B_2^{P_{u-1}}$. Since $b = (p - 1)d$, this parameter is the same for both wreath products.
It remains to compare the parameters $s(h)$. For $A^{B_2} \text{Wr} B_1^{p^{w-1}}$ the value of $s(h)$ is set so that $p^{s(h)}$ is the exponent of the $h$-th term $\gamma_h(A^{B_2})$ of $A^{B_2}$. And for $A^{B_1} \text{Wr} B_2^{*}$ the value of $s(h)$ is set so that $p^{s(h)}$ is the exponent of $\gamma_h(A)$ of $A$. Since these exponents clearly are equal, the parameters $s(h)$ are also the same for both wreath products.

We get that the only value that is different in Shield’s formula applied to two wreath products is $e(1)$, and it is strictly larger for $A^{B_2} \text{Wr} B_1^{*}$. Thus, the nilpotency class of $A^{B_2} \text{Wr} B_1^{*}$ is strictly larger than that of $A^{B_2} \text{Wr} B_2^{p^{w-1}}$. In other words, a lower bound for the class of $P_1$ is larger than an upper bound for $P_2$.

If $c_2$ is the nilpotency classes of $P_2$, then $P_2 = (A \text{ Wr } B_2)^{p^{w-1}}$ is in nilpotent variety $\mathfrak{N}_{c_2}$, and thus, the group $A \text{ Wr } B_2$ (together with the variety it generates) is in the product $\mathfrak{N}_{c_2} \mathfrak{N}_{p^{w-1}}$ of $\mathfrak{N}_{c_2}$ and the Burnside variety $\mathfrak{B}_{p^{w-1}}$. As we saw, however, $A \text{ Wr } B_1$ does not belong to this product. Thus, also

$$\text{var}(A \text{ Wr } B_1) \neq \text{var}(A \text{ Wr } B_2).$$

This is a contradiction which completes the proof.

**Remark 3.3.** With further routine proofs we could show that the upper and lower bounds found above are, in fact, exactly the nilpotency classes of $A \text{ Wr } B_1$ and $A \text{ Wr } B_2$, respectively. We, however, refrain from doing that, as the proof above already accomplishes the task we need. Note also that even if $B_1$ and $B_2$, after their initial $t$ finite factors, contain some infinite factors $C_{p^{|C|}}$ or $C_{p^{*|C|}}$, $i > t$, these would play no role in Lemma 3.2, because all such factors would disappear in the $(p^{w-1})$-th powers $B_1^{p^{w-1}}$ and $B_2^{p^{w-1}}$.

We can already deduce:

**Proposition 3.4.** If $A$ is a non-trivial nilpotent $p$-group of finite exponent, and $B_1, B_2$ are non-trivial finite abelian $p$-groups, then equality (3.2) holds for $A, B_1, B_2$ if and only if $B_1 \cong B_2$.

**Proof.** Sufficiency of the condition is evident as $B_1 \cong B_2$ implies $A \text{ Wr } B_1 \cong A \text{ Wr } B_2$. Necessity follows from the lemma above.

It is time to allow one or both of the $t$-th factors in $B_1$ or $B_2$ to be infinite, preserving all other conditions and excluding the case covered by the previous lemma (when the $t$-th factors are finite non-isomorphic groups). We are not ruling out the option $t = 1$, i.e., the groups $B_1$ or $B_2$ may start with an infinite factor.

**Lemma 3.5.** In the above circumstances, if the $t$-th factors $C_{p^{|C|}}$ and $C_{p^{*|C|}}$ are not both infinite groups of the same exponent, then equality (3.2) does not hold for $A, B_1, B_2$. 

\[ \text{This is a contradiction which completes the proof.} \]
Proof. Without loss of generality we have the following cases of which the first three cover the situations when only one of the \( t \)-th factors is infinite:

**Case 1:** \( m_{p^u} \) is infinite, \( m_{p^v} \) is finite, and \( u_t = v_t = w \) are equal. This time we cannot apply Shield’s formula to \( P_1 = (A \text{ Wr } B_1)^{p^w-1} \) because \((B_1)^{p^w-1}\) is infinite. Consider a new group \( B'_1 \) which is obtained from \( B_1 \) by replacing its \( t \)-th factor \( C_{p^u} \) by \( C_{p^{u_t}}^{m_{p^u}} \), i.e., in its direct decomposition we replace the infinitely many copies of \( C_{p^u} \) by \( m_{p^u} + 1 \) copies (finitely many) of the same cycle. Then Shield’s formula can be applied to the group \((A \text{ Wr } B'_1)^{p^w-1}\), and by the proof of Lemma 3.2 we see that its nilpotency class is higher than the class \( c_2 \) of \( P_2 \). Thus, as in the previous proof, \( A \text{ Wr } B'_1 \) does not belong to the product variety \( \mathfrak{N}_{c_2} \mathfrak{V}_{p^w-1} \) which contains \( A \text{ Wr } B_2 \). Nevertheless, \( B'_1 \leq B_1 \) and so by \([20, 22.13]\) or by \([13, \text{Lemma 1.2}]\), \( A \text{ Wr } B'_1 \) is isomorphic to a subgroup of \( A \text{ Wr } B_1 \). Thus \( A \text{ Wr } B_1 \) is also not contained in \( \mathfrak{N}_{c_2} \mathfrak{V}_{p^w-1} \).

**Case 2:** \( m_{p^u} \) is infinite, \( m_{p^v} \) is finite, and \( u_t > v_t \). We can add to \( B_2 \) (immediately before the finite \( t \)-th factor) a new factor \( C_{p^{u_t}}^{m_{p^v}} \) with \( v_t = u_t \) and \( m_{p^v} = 0 \). We are then in a situation already covered by Case 1.

**Case 3:** \( m_{p^u} \) is infinite, \( m_{p^v} \) is finite, and \( u_t < v_t \). We can add to \( B_1 \) (immediately before the infinite factor) a new factor \( C_{p^{u_t}}^{m_{p^v}} \) with \( u_t = v_t \) and \( m_{p^u} = 0 \). We are then in a situation already covered by Lemma 3.2.

**Case 4:** Both \( m_{p^u} \) and \( m_{p^v} \) are infinite, and \( u_t \neq v_t \). We can reduce this to one of the previous cases by adding one more finite factor to one of the groups \( B_1 \) or \( B_2 \).

There only remains the case when both \( m_{p^u} \) and \( m_{p^v} \) are finite, and this is ruled out in the lemma above. \( \square \)

The series of necessary conditions restricted our consideration to the situation where \( B_1 \equiv B_2 \), i.e., \( B_1 \) and \( B_2 \) are of the same exponent; in their decompositions (2.1) and (2.2) the initial \( t - 1 \) finite factors are the same; \( B_1 \) and \( B_2 \) still may differ in their \( t \)-th factors, and in such a case both \( C_{p^u}^{m_{p^u}} \) and \( C_{p^v}^{m_{p^v}} \) are infinite and have the same exponent (these two factors need not be isomorphic, as \( m_{p^u} \) and \( m_{p^v} \) may be distinct infinite cardinal numbers). Two special cases are not ruled out: we may have \( t = 1 \) (i.e., the initial coinciding finite factors are absent in \( B_1 \) and \( B_2 \)); or \( t - 1 = r = s \), i.e., \( B_1 \) and \( B_2 \) are isomorphic finite groups.

**Lemma 3.6.** In the above circumstances, with equivalence \( B_1 \equiv B_2 \), equality (3.2) holds for \( A, B_1, B_2 \).

**Proof.** If \( B_1, B_2 \) are finite, then \( B_1 \equiv B_2 \) holds together with \( A \text{ Wr } B_1 \equiv A \text{ Wr } B_2 \), and we are left nothing to prove.
Suppose \( B_1, B_2 \) are infinite, their first coinciding \( t - 1 \) factors are finite, and their \( t \)-th factors are both infinite and of the same exponent. Let us first prove an auxiliary fact. Fix any infinite cardinal number \( s \), and denote by \( B_s \) the group \( C_m \times \cdots \times C_p \times C_s \), i.e., \( B_s \) can be obtained from \( B_1 \) (or from \( B_2 \)) by taking its first \( t - 1 \) finite factors and by adding one more infinite direct factor \( C_p \). In particular, when \( s = \aleph_0 \), we get the group \( B_{\aleph_0} \). The following equality holds:

\[
\text{var}(A \text{ Wr } B_{\aleph_0}) = \text{var}(A \text{ Wr } B_s). \tag{3.6}
\]

Indeed, since \( \aleph_0 \leq s \), it follows that \( B_{\aleph_0} \) is a subgroup of \( B_s \) and, by [20, 22.13] or [13, Lemma 1.2], \( A \text{ Wr } B_{\aleph_0} \) is a subgroup of \( A \text{ Wr } B_s \), and so

\[
\text{var}(A \text{ Wr } B_{\aleph_0}) \subseteq \text{var}(A \text{ Wr } B_s).
\]

If the inverse inclusion does not hold, then there is a word \( v(x_1, \ldots, x_k) \) in the absolutely free group \( F_k \), such that \( v(x_1, \ldots, x_k) \equiv 1 \) is an identity for the group \( A \text{ Wr } B_{\aleph_0} \) but not for the larger group \( A \text{ Wr } B_s \), that is, there are some elements \( g_1, \ldots, g_k \in A \text{ Wr } B_s \) such that \( v(g_1, \ldots, g_k) \neq 1 \).

Each \( g_j, j = 1, \ldots, k \), is of the form \( g_j = b_j \theta_j \), where \( b_j \in B_s \) and \( \theta_j \in A \). For any groups their Cartesian wreath product and direct wreath product always generate the same variety of groups [20, 22.31]. Thus, we may suppose \( g_1, \ldots, g_k \) are already in the direct wreath product, that is, the values \( \theta_j(d) \) are non-trivial for at most finitely many elements \( d_1, \ldots, d_l \in B_s \). Moreover, since the product

\[
B_s = C_m \times \cdots \times C_p \times C_s
\]

is also direct, the aforementioned elements \( d_1, \ldots, d_l \) and \( b_1, \ldots, b_k \) inside this direct product have at most finitely many non-trivial coordinates in respective copies of the cycles \( C_p, \ldots, C_p \) in \( B_s \). In particular, only finitely many coordinates are taken in the factor \( C_s \). Clearly, if we keep countably many copies of \( C_p \), and discard all the remaining copies of that cycle we will get the product \( C_{\aleph_0} \), and \( v(g_1, \ldots, g_k) \neq 1 \) will still hold in \( B_{\aleph_0} \). This is a contradiction.

Turning to the main proof, first notice that \( \aleph_0 \leq m \) and \( \aleph_0 \leq p \) and, thus, \( B_{\aleph_0} \) is a subgroup both in \( B_1 \) and \( B_2 \). So we have

\[
\text{var}(A \text{ Wr } B_{\aleph_0}) \subseteq \text{var}(A \text{ Wr } B_i), \quad i = 1, 2. \tag{3.7}
\]

Next denote by \( B_1 \) the group obtained from \( B_1 \) by replacing, in its decomposition (2.1), all the factors \( C_{p^j} \) by \( C_{p^i} \) for \( j = t + 1, \ldots, r \). We similarly
define the group $\overline{B}_2$. It is clear that
\[
C_{p^{u_1}}^{m_{p^{u_1}}} \times C_{p^{u_1}+1}^{m_{p^{u_1}}} \times \cdots \times C_{p^{u_r}}^{m_{p^{u_r}}} = C_{p^{u_1}}^{s_1}
\]
and
\[
C_{p^{v_1}}^{m_{p^{v_1}}} \times C_{p^{v_1}+1}^{m_{p^{v_1}}} \times \cdots \times C_{p^{v_s}}^{m_{p^{v_s}}} = C_{p^{v_1}}^{s_2} = C_{p^{v_1}}^{s_2},
\]
with
\[
s_1 = \max\{m_{p^{u_1}}, m_{p^{u_2}}, \ldots, m_{p^{u_r}}\}, \quad s_2 = \max\{m_{p^{v_1}}, m_{p^{v_2}}, \ldots, m_{p^{v_s}}\}
\]
(recall that $p^{u_1} = p^{v_1}$). We see that $\overline{B}_i$ is in fact $B_{s_i}, i = 1, 2$. On the other hand
\[
B_i \text{ is clearly a subgroup of } \overline{B}_i, \text{ and so}
\]
\[
\text{var}(A \text{ Wr } B_i) \subseteq \text{var}(A \text{ Wr } B_{s_i}), \quad i = 1, 2.
\]
(3.8)

Now let $s$ be the maximum of $s_1$ and $s_2$. By equalities (3.6), (3.7) and (3.8) we have
\[
\text{var}(A \text{ Wr } B_{s_0}) = \text{var}(A \text{ Wr } B_i) = \text{var}(A \text{ Wr } B_{s_1}) = \text{var}(A \text{ Wr } B_{s}), \quad i = 1, 2,
\]
as desired. \hfill \square

Lemmas 3.2–3.6 already prove the restricted version of Theorem 2.3 for $p$-groups:

**Proposition 3.7.** Let $A$ be a non-trivial nilpotent $p$-group of finite exponent, and let $B_1, B_2$ be non-trivial abelian $p$-groups of finite exponents. Then equality (3.2) holds for $A, B_1, B_2$ if and only if $B_1 \equiv B_2$.

From the assumptions made after Lemma 3.1 we have focussed on the case where $A, B_1$ and $B_2$ are $p$-groups only. From now on let $A$ be a non-trivial nilpotent group of class $c$ and of exponent $m$, and let $B_1, B_2$ be non-trivial abelian groups of exponent $n$ such that $n \mid m$. Denote var$(A)$ by $\mathbb{U}$, and assume $p_1, \ldots, p_l$ are all the prime divisors of $m$. Since $\mathbb{U}$ is nilpotent of class $c$, by [20, Corollary 35.12], $\mathbb{U}$ is generated by $F = F_c(\mathbb{U})$. Being a finite nilpotent group $F$ is a direct product of its Sylow $p_i$-subgroups, $i = 1, \ldots, l$,
\[
F = S_{p_1} \times \cdots \times S_{p_l},
\]
and all primes $p_i$ occur: none of the subgroups $S_{p_i}$ is trivial, as
\[
\exp(F) = \exp(\mathbb{U}) = \exp(A).
\]
Assume $p$ is a prime divisor of $n$ (and of $m$), i.e., it is one of the primes $p_i$. Denote by $p^u$ the highest power of $p$ dividing $n$ (in terms of (2.1) we could write $u = u_1$).

The following technical fact was proved in [17].
**Lemma 3.8** (see [17, proof of Lemma 3.3]). Any \( p \)-group in variety \( \text{var}(A \text{ Wr } B_1) \) belongs to \( \text{var}(S_p \text{ Wr } B_1(p)) \).

The analog of this lemma holds for \( p \)-groups in the variety \( \text{var}(A \text{ Wr } B_2) \) also.

In order to prove the necessity part of Theorem 2.3 (supposing by the assumptions made after Lemma 3.1 that \( A_1 = A_2 = A \)), assume equality (3.2) holds for \( A;B_1;B_2 \). Fix any \( p \) dividing \( n \), and let \( P \) be any group from the variety \( \text{var}(S_p \text{ Wr } B_1(p)) \). Since the group \( F \) and, thus, also \( S_p \) are in \( \mathcal{U} \), they can be obtained from \( A \) using the operations \( Q, S, C \). Thus, by [13, Lemma 1.1] and [13, Lemma 1.2] (see also [20, 22.11], [20, 22.13]), \( P \) belongs to \( \text{var}(A \text{ Wr } B_1) \). By assumption, the latter is equal to \( \text{var}(A \text{ Wr } B_2) \). Since \( P \) clearly is a \( p \)-group, and it is in \( \text{var}(A \text{ Wr } B_2) \), the group \( P \) also belongs to \( \text{var}(S_p \text{ Wr } B_2(p)) \) by Lemma 3.8. In the same way we can show that any group from \( \text{var}(S_p \text{ Wr } B_2(p)) \) is also in \( \text{var}(S_p \text{ Wr } B_1(p)) \), and so

\[
\text{var}(S_p \text{ Wr } B_2(p)) = \text{var}(S_p \text{ Wr } B_1(p)).
\]

Thus, according to Proposition 3.7, we get \( B_1(p) \equiv B_2(p) \) for any \( p \mid n \).

To prove the sufficiency part of Theorem 2.3, suppose \( B_1(p_1) \equiv B_2(p_i) \) for any of the prime divisors \( p_1, \ldots, p_h \) of \( n \), which means that the decompositions (2.1) and (2.2) for groups \( B_1(p_i) \) and \( B_2(p_i) \) both start with some coinciding finite factors

\[
C_{p_i}^{m_{u_{t_1}}} \times \cdots \times C_{p_i}^{m_{u_{t_i} - 1}} \quad \text{and} \quad C_{p_i}^{m_{v_{t_1}}} \times \cdots \times C_{p_i}^{m_{v_{t_i} - 1}}
\]

perhaps followed by infinite factors

\[
C_{p_i}^{m_{u_{t_i}}} \quad \text{and} \quad C_{p_i}^{m_{v_{t_i}}},
\]

respectively (in which the exponents \( p_i^{u_{t_i}} \) and \( p_i^{v_{t_i}} \) are equal, \( m_{u_{t_i}} \) and \( m_{v_{t_i}} \) may be any infinite cardinals, and the case \( t_i = 1 \) is not ruled out).

Now we are going to apply the idea and the notation from the proof of Lemma 3.6. Using the notation \( B_s \) introduced for any cardinal number \( s \) define

\[
B_s^* = \prod_{i=1}^{h} B_1(p_i)_s = \prod_{i=1}^{h} B_2(p_i)_s,
\]

i.e., for each \( p_i \) we take the first finite factors from the decomposition of \( B_1(p_i) \), and add one more factor \( C_{p_i}^{m_{u_{t_i}}} \); since \( B_1(p_i) \equiv B_2(p_i) \), we clearly have

\[
B_1(p_i)_s \equiv B_2(p_i)_s.
\]
It is clear that taking \( \mathfrak{s} = \mathfrak{S}_0 \), we get the group \( B^*_{\mathfrak{S}_0} \) which is contained as a subgroup in both \( B_1 \) and \( B_2 \) (since an infinite \( \mathfrak{m}_{p_i^{v_i}} \) is greater than or equal to \( \mathfrak{S}_0 \) for any \( i \)). On the other hand, taking \( \mathfrak{s} \) to be the maximum of the cardinals \( \mathfrak{m}_{p_i^{v_i+1}}, \mathfrak{m}_{p_i^{v_i+2}}, \ldots \) for all \( i \), we get the group \( B^*_\mathfrak{s} \) which contains isomorphic copies of both \( B_1 \) and \( B_2 \). By \([20, 22.13]\) or by \([13, \text{Lemma 1.2}]\), we have

\[
A \text{Wr} B^*_{\mathfrak{S}_0} \leq A \text{Wr} B_1 \leq A \text{Wr} B^*_\mathfrak{s}, \quad A \text{Wr} B^*_\mathfrak{s} \leq A \text{Wr} B_2 \leq A \text{Wr} B^*_{\mathfrak{S}_0}.
\]

Repeating some arguments about the identity \( v(x_1, \ldots, x_k) = 1 \) used in the proof of Lemma 3.6, we find that the wreath products \( A \text{Wr} B^*_{\mathfrak{S}_0} \) and \( A \text{Wr} B^*_\mathfrak{s} \) generate the same variety. Thus \( A \text{Wr} B_1 \) and \( A \text{Wr} B_2 \) also generate the same variety of groups.

The proof of Theorem 2.3 is thus completed.

4 Examples

**Example 4.1.** Take \( A_1 = A_2 = A = C_3 \), \( B_1 = C_3^2 \), and \( B_2 = C_3^2 \times C_3^4 \). For the group \( B_1 \) we have

\[
K_{1,3}(B_1) = B_1 = C_3^2,
K_{2,3}(B_1) = K_{3,3}(B_1) = B_1^{3^1} = C_3^2,
K_{4,3}(B_1) = K_{5,3}(B_1) = \cdots = \{1\}.
\]

This means \( d = 3 \); \( e(1) = 2 \), since \(|K_{1,3}(B_1)/K_{2,3}(B_1)| = 3^2 \), \( e(2) = 0 \) since \(|K_{2,3}(B_1)/K_{3,3}(B_1)| = 1 = 3^0 \), and \( e(3) = 2 \) since \(|K_{3,3}(B_1)/K_{4,3}(B_1)| = 3^2 \).

So we have \( a = 1 + 2(1 \cdot 2 + 2 \cdot 0 + 3 \cdot 2) = 17 \), \( b = (3 - 1)3 = 6 \). Also \( h = 1 \) (the nilpotency class of \( A \)) and the exponent of \( \gamma_1(A) = A \) is \( 3^1 \), and we have \( s(1) = 1 \). Thus the nilpotency class of \( A \text{Wr} B_1 \) is \( 17 \cdot 1 + (1 - 1)6 = 17 \).

Now do the same starting with \( B_2 \). We have

\[
K_{1,3}(B_2) = B_2 = C_3^2 \times C_3^4,
K_{2,3}(B_2) = K_{3,3}(B_2) = B_2^{3^1} = C_3,
K_{4,3}(B_2) = K_{5,3}(B_2) = \cdots = \{1\}.
\]

We again have \( d = 3 \); \( e(1) = 5 \), since \(|K_{1,3}(B_2)/K_{2,3}(B_2)| = 3^5 \), \( e(2) = 0 \), since \(|K_{2,3}(B_2)/K_{3,3}(B_2)| = 1 = 3^0 \), and \( e(3) = 1 \), since \(|K_{3,3}(B_2)/K_{4,3}(B_2)| = 3^1 \).

So we have \( a = 1 + 2(1 \cdot 5 + 2 \cdot 0 + 3 \cdot 1) = 17 \), \( b = (3 - 1)3 = 6 \). Again \( s(1) = 1 \).

So the nilpotency class of \( A \text{Wr} B_2 \) is \( 17 \cdot 1 + (1 - 1)6 = 17 \).

We see that \( A \text{Wr} B_1 \) and \( A \text{Wr} B_2 \) have the same nilpotency class 17. Moreover, these wreath products both have the same length of solubility 2, and the same
exponent $27 = 3 \cdot 3^2$. So, based only on nilpotency class, length of solubility, and exponent, we cannot yet deduce if the varieties $\text{var}(A \ Wr B_1)$ and $\text{var}(A \ Wr B_2)$ are distinct subvarieties in $\mathfrak{V}_3 \mathfrak{V}_9$.

However, by Theorem 2.3 (in fact, by Lemma 3.2 already), we have sharper estimates to deduce that $\text{var}(A \ Wr B_1) \neq \text{var}(A \ Wr B_2)$; the first factors in which $B_1$ and $B_2$ differ are the initial factors $C_{2^3}^{m_{14}} = C_3^2$ for $B_1$, and $C_{2^3}^{m_{14}} = C_3^2$ for $B_2$. Repeating some steps of the above proofs for this example, we would find that $A \ Wr B_2$ belongs to the variety $\mathfrak{V}_3 \mathfrak{V}_9$ which does not contain the group $A \ Wr B_1$.

**Example 4.2.** Let $A_1$ be the Dihedral group

$$D_4 = \langle a, b : a^4 = b^2 = 1, a^b = a^{-1} \rangle,$$

and let $A_2$ be the Quaternion group

$$Q_8 = \langle i, j, k : i^2 = j^2 = k^2 = ijk \rangle.$$

These groups are not isomorphic, but they are both of order 8 and of class 2, and, moreover, they both generate the variety $\mathfrak{V}_2^2 \cap \mathfrak{V}_2$ (see [9, 20]). As active groups take $B_1 = C_{2^2}^3 \times C_2$ and $B_2 = C_{2^2} \times C_{2^2}^7$.

Clearly, $\gamma_1(D_4) = D_4, \gamma_2(D_4) = D_4 = \langle a^2 \rangle \cong C_2$, and $\gamma_3(D_4) = \{1\}$. For the group $B_1$ we have

$$K_{1,2}(B_1) = B_1 = C_{2^2}^3 \times C_2,$$

$$K_{2,2}(B_1) = C_2^3,$$

$$K_{3,2}(B_1) = \{1\}.$$

Then we have $d = 2; e(1) = 4$, since $|K_{1,2}(B_1)/K_{2,2}(B_1)| = 2^4$, and $e(2) = 3$, since $|K_{2,2}(B_1)/K_{3,2}(B_1)| = 2^3$. So we have $a = 1 + 1(1 \cdot 4 + 2 \cdot 3) = 11$ and $b = (2 - 1)2 = 2$. Also, $h = 2$ (the class of $A$) and the exponent of $\gamma_1(A) = A$ is $3^1$, and we have $s(1) = 1$. Thus the nilpotency class of $A_1 \ Wr B_1$ is

$$\max\{11 \cdot 1 + (2 - 1)2, 11 \cdot 2 + (1 - 1)2\} = 22.$$

Furthermore, $\gamma_1(Q_8) = Q_8, \gamma_2(Q_8) = Q_8' = \{\pm 1\} \cong C_2$ and $\gamma_3(Q_8) = \{1\}$. Then for $B_2$ we have

$$K_{1,2}(B_2) = B_2 = C_{2^2} \times C_{2^2}^7, \quad K_{2,2}(B_2) = C_2, \quad K_{3,1}(B_2) = \{1\}.$$

Again $d = 2; e(1) = 8$, since $|K_{1,2}(B_2)/K_{2,2}(B_2)| = 2^8$, and $e(2) = 1$, since $|K_{2,2}(B_2)/K_{3,2}(B_2)| = 2^1$. Thus, we have $a = 1 + 1(1 \cdot 8 + 2 \cdot 1) = 11$ and $b = (2 - 1)2 = 1$. Using the values $s(1) = 2$, and $s(2) = 1$ again, we find
that the nilpotency class of $A_2 \text{ Wr } B_2$ is

$$\max\{11 \cdot 1 + (2 - 1)2, 11 \cdot 2 + (1 - 1)2\} = 22.$$  

We have that $A_1 \text{ Wr } B_1$ and $A_2 \text{ Wr } B_2$ have the same nilpotency class 22. These groups both have the same length of solubility 3, and the same exponent $16 = 4 \cdot 2^2$. Thus, based only on nilpotency class, length of solubility, and exponent, we cannot determine whether or not $\text{var}(A_1 \text{ Wr } B_1)$ and $\text{var}(A_2 \text{ Wr } B_2)$ are distinct subvarieties in $\mathcal{R}_3 \mathcal{R}_4$.

However, by Theorem 2.3 or by Lemma 3.2, $\text{var}(A_1 \text{ Wr } B_1) \neq \text{var}(A_2 \text{ Wr } B_2)$. Moreover, $A_2 \text{ Wr } B_2$ does belong to the variety $\mathcal{R}_4 \mathcal{R}_2$ which does not contain the group $A_1 \text{ Wr } B_1$.

These examples involved relatively uncomplicated groups, all of which were finite. Nevertheless, since the criterion of Theorem 2.3 is simple for applications, we can easily construct examples with infinite groups also. For example, if in the last example we replace $B_1 = C_3^3 \times C_2$ by $B_1 = C_2^3 \times C_2^3$ and $B_2 = C_2 \times C_2^7$ by $B_2 = C_2 \times C_2^8$, we obtain wreath products $A_1 \text{ Wr } B_1$ and $A_2 \text{ Wr } B_2$ which are non-nilpotent by a theorem of Baumslag [2], and which generate distinct varieties by Theorem 2.3.

**Example 4.3.** We consider groups which, unlike the ones in previous examples, are not $p$-groups. Let $A_1 = A_2 = A = D_4 \times Q_8 \times C_3 \times C_5 \times C_7^2$. Take

$$B_1 = C_2^3 \times C_2^8 \times C_3^2 \times C_5^2 \times C_7^8$$

and

$$B_2 = C_2^3 \times C_2^8 \times C_3^2 \times C_5^2 \times C_7^8.$$  

The prime 5 divides the exponent $m = 4 \cdot 3 \cdot 5 \cdot 7 = 420$ of $A$ but not the exponent $n = 32 \cdot 3 \cdot 7 = 672$ of $B_1$ and of $B_2$. So we can ignore the prime 5 and in Theorem 2.3 apply the primes $p_1 = 2$, $p_2 = 3$, and $p_3 = 7$ only.

We have

$$B_1(p_1) = B_1(2) = C_2^3 \times C_2^8$$

and

$$B_2(p_1) = B_2(2) = C_2^3 \times C_2^8.$$  

Although these 2-primary components are evidently non-isomorphic, they are equivalent, i.e., $B_1(2) \equiv B_2(2)$. So, for $p_1 = 2$, the condition of Theorem 2.3 is satisfied, and we can proceed to the next prime.

Note that $B_1(p_2) = B_1(3) = C_3^3$ and $B_2(p_2) = B_2(3) = C_3^3$. These 3-primary components again are non-isomorphic, but they are equivalent, i.e., $B_1(3) \equiv B_2(3)$. So, for $p_2 = 3$, the condition of Theorem 2.3 again is satisfied.
Finally, we consider $B_1(p_3) = B_1(7) = C_7^8$ and $B_2(p_3) = B_2(7) = C_7^9$. These 7-primary components are not only non-isomorphic but also are non-equivalent, i.e., $B_1(7) \not\cong B_2(7)$. So, for $p_3 = 7$, the condition of Theorem 2.3 is not satisfied, and we have $\text{var}(A_1 \text{ Wr } B_1) \neq \text{var}(A_2 \text{ Wr } B_2)$. Notice how the difference of factors $C_7^8$ and $C_7^9$ matters for equality (***), whereas the difference of other factors, such as $C_2^8$ and $C_2^9$, does not matter.

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Bibliography


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