

Mathematics

ON THE ANISOTROPIC BOUNDARY VALUE PROBLEM, CONNECTED WITH HELMHOLTZ-SCHRÖDINGER EQUATION, UNDER THE BOUNDARY CONDITIONS OF THE FIRST AND SECOND TYPE

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In this paper we consider the solvability of the boundary value problem, connected with the anisotropic Helmholtz-Schrödinger equation, under the boundary conditions of the first and second type on the line $y = 0$.

Keywords: Helmholtz-Schrödinger equation, factorization of matrix-function.

Introduction. The issue of solvability was considered in [1] for a class of boundary value problems, coordinated with the anisotropic Helmholtz-Schrödinger equation in the Sobolev space on the upper and lower half-planes, where the boundary conditions of the first and second type were fulfilled on $y = 0$ line. In [1] it was shown that the solvability of this problem is equivalent to that of some Riemann-Hilbert problem.

Let $\Omega^\pm = \{(x, y) \in R^2 : y > 0 (y < 0)\}$ and $H^{1/2}(\Omega^\pm)$, $H^{-1/2}(\Omega^\pm)$ are the corresponding Sobolev spaces (see [2]). Now consider the following anisotropic Helmholtz-Schrödinger equation

$$\begin{cases} \Delta u + (k^2 + 2\beta_+^2 \sec^2 h^2(\beta_+ y))u = 0 & \text{in } \Omega^+, \\ \Delta u + (k^2 + 2\beta_-^2 \sec^2 h^2(\beta_- y))u = 0 & \text{in } \Omega^-. \end{cases} \quad (1)$$

This equation a particular case of equation (1) from [1], where $\operatorname{Re} k > 0, \operatorname{Im} k > 0$. Note that similar problems in the isotropic case were investigated in [3–6]. We suppose that the following boundary conditions are fulfilled:

$$\begin{cases} \begin{cases} a_0 u(x, +0) + b_0 u(x, -0) = h_0(x), \\ a_1 \frac{\partial u(x, +0)}{\partial y} + b_1 \frac{\partial u(x, -0)}{\partial y} = h_1(x) \end{cases} & \text{in } R^+, \\ \begin{cases} c_0 u(x, +0) + d_0 u(x, -0) = p_0(x), \\ c_1 \frac{\partial u(x, +0)}{\partial y} + d_1 \frac{\partial u(x, -0)}{\partial y} = p_1(x) \end{cases} & \text{in } R^-, \end{cases} \quad (2)$$

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where $h_0 \in H^{1/2}(R^+)$, $h_1 \in H^{-1/2}(R^+)$, $p_0 \in H^{1/2}(R^-)$, $p_1 \in H^{-1/2}(R^-)$, the coefficients $a_0, a_1, b_0, b_1, c_0, c_1, d_0, d_1$ are the complex constants and $a_1 d_1 = b_1 c_1$.

Introduce the functions

$$\begin{cases} u_-(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 [a_0 u(x; +0) + b_0 u(x; -0) - h_0(x)] e^{i\lambda x} dx, \\ w_-(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \left[a_1 \frac{\partial u(x; +0)}{\partial y} + b_1 \frac{\partial u(x; -0)}{\partial y} - h_1(x) \right] e^{i\lambda x} dx, \\ u_+(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} [c_0 u(x; +0) + d_0 u(x; -0) - p_0(x)] e^{i\lambda x} dx, \\ w_+(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \left[c_1 \frac{\partial u(x; +0)}{\partial y} + d_1 \frac{\partial u(x; -0)}{\partial y} - p_1(x) \right] e^{i\lambda x} dx, \end{cases} \quad (3)$$

$$\begin{cases} m_1(\lambda) = \frac{\hat{h}_0(\lambda)}{\Delta(\lambda)} \left\{ a_1 d_0 \frac{\gamma^2(\lambda) - \beta_+^2}{\gamma(\lambda)} + b_1 c_0 \frac{\gamma^2(\lambda) - \beta_-^2}{\gamma(\lambda)} \right\} + \frac{a_0 d_0 - b_0 c_0}{\Delta(\lambda)} \hat{h}_1(\lambda) - \hat{p}_0(\lambda), \\ m_2(\lambda) = \frac{\hat{h}_1(\lambda)}{\Delta(\lambda)} \left\{ b_0 c_1 \frac{\gamma^2(\lambda) - \beta_+^2}{\gamma(\lambda)} + a_0 d_1 \frac{\gamma^2(\lambda) - \beta_-^2}{\gamma(\lambda)} \right\} - \hat{p}_1(\lambda) \end{cases} \quad (4)$$

and

$$\begin{cases} a(\lambda) = \frac{1}{\Delta(\lambda)} \left\{ b_1 \frac{\gamma^2(\lambda) - \beta_-^2}{\gamma(\lambda)} (u_-(\lambda) + \hat{h}_0(\lambda)) - b_0 (w_-(\lambda) + \hat{h}_1(\lambda)) \right\}, \\ b(\lambda) = \frac{1}{\Delta(\lambda)} \left\{ a_1 \frac{\gamma^2(\lambda) - \beta_+^2}{\gamma(\lambda)} (u_-(\lambda) + \hat{h}_0(\lambda)) + a_0 (w_-(\lambda) + \hat{h}_1(\lambda)) \right\}, \end{cases} \quad (5)$$

where

$$\begin{aligned} \hat{h}_0(\lambda) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 h_0(x) e^{i\lambda x} dx, & \hat{h}_1(\lambda) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 h_1(x) e^{i\lambda x} dx, \\ \hat{p}_0(\lambda) &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} p_0(x) e^{i\lambda x} dx, & \hat{p}_1(\lambda) &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} p_1(x) e^{i\lambda x} dx \end{aligned}$$

and

$$\Delta(\lambda) = a_0 b_1 \frac{\gamma^2(\lambda) - \beta_-^2}{\gamma(\lambda)} + a_1 b_0 \frac{\gamma^2(\lambda) - \beta_+^2}{\gamma(\lambda)}.$$

As was noted above, the solvability of the boundary value problem (1)–(2) is equivalent to the solvability of the Riemann-Hilbert problem (see [1])

$$\bar{u}_+(\lambda) = L(\lambda) \bar{u}_-(\lambda) + \bar{m}(\lambda), \quad (6)$$

where the vector-functions

$$\bar{u}_+(\lambda) = \begin{pmatrix} u_+(\lambda) \\ w_+(\lambda) \end{pmatrix}, \quad \bar{u}_-(\lambda) = \begin{pmatrix} u_-(\lambda) \\ w_-(\lambda) \end{pmatrix}, \quad \bar{m}(\lambda) = \begin{pmatrix} m_1(\lambda) \\ m_2(\lambda) \end{pmatrix} \quad (7)$$

and the matrix-function

$$L(\lambda) = \begin{pmatrix} \frac{a_1 d_0 (\gamma^2(\lambda) - \beta_+^2) + b_1 c_0 (\gamma^2(\lambda) - \beta_-^2)}{a_1 b_0 (\gamma^2(\lambda) - \beta_+^2) + a_0 b_1 (\gamma^2(\lambda) - \beta_-^2)} & \frac{(a_0 d_0 - b_0 c_0) \gamma(\lambda)}{a_1 b_0 (\gamma^2(\lambda) - \beta_+^2) + a_0 b_1 (\gamma^2(\lambda) - \beta_-^2)} \\ 0 & \frac{b_0 c_1 (\gamma^2(\lambda) - \beta_+^2) + a_0 d_1 (\gamma^2(\lambda) - \beta_-^2)}{a_1 b_0 (\gamma^2(\lambda) - \beta_+^2) + a_0 b_1 (\gamma^2(\lambda) - \beta_-^2)} \end{pmatrix}. \quad (8)$$

Consider the matrices $J_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$, $i = 0, 1$, generated from the coefficients

of the boundary conditions (2). It is easy to verify that without the loss of the generality the non-degenerate cases (i.e. $\Delta(\lambda) \neq 0, \det(L(\lambda)) \neq 0$) are possible only in the following cases:

$$\begin{aligned} 1) J_0 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, J_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, & 2) J_0 &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, J_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \\ 3) J_0 &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, J_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, & 4) J_0 &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, J_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \\ 5) J_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, J_1 = \begin{pmatrix} \nu & 1 \\ \nu & 1 \end{pmatrix}, & 6) J_0 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, J_1 = \begin{pmatrix} \nu & 1 \\ \nu & 1 \end{pmatrix}, \\ 7) J_0 &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, J_1 = \begin{pmatrix} \nu & 1 \\ \nu & 1 \end{pmatrix}, & 8) J_0 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, J_1 = \begin{pmatrix} \nu & 1 \\ \nu & 1 \end{pmatrix}, \\ 9) J_0 &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, J_1 = \begin{pmatrix} \nu & 1 \\ \nu & 1 \end{pmatrix}, & 10) J_0 &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, J_1 = \begin{pmatrix} \nu & 1 \\ \nu & 1 \end{pmatrix}, \end{aligned}$$

where $\nu \neq 0, 1$ is a constant.

In all cases 1)–10) the matrix-function $L(\lambda)$ can be presented in the form

$$L(\lambda) = \begin{pmatrix} \frac{\alpha \gamma^2(\lambda) - \chi}{\delta \gamma^2(\lambda) - \rho} & \frac{\varepsilon \gamma(\lambda)}{\delta \gamma^2(\lambda) - \rho} \\ 0 & 1 \end{pmatrix}, \quad (9)$$

where in the case

$$\begin{aligned} 1) \alpha &= \delta = 1, \chi = \rho = \beta_+^2, \varepsilon = 1, & 6) \alpha &= \nu, \delta = \nu + 1, \chi = \beta_+^2, \rho = \beta_-^2 + \nu \beta_+^2, \varepsilon = 1, \\ 2) \alpha &= \delta = 1, \chi = \rho = \beta_+^2, \varepsilon = -1, & 7) \alpha &= 1, \delta = \nu + 1, \chi = \beta_-^2, \rho = \beta_-^2 + \nu \beta_+^2, \varepsilon = -1, \\ 3) \alpha &= \delta = 1, \chi = \rho = \beta_-^2, \varepsilon = -1, & 8) \alpha &= 1, \delta = \nu, \chi = \beta_+^2, \rho = \nu \beta_+^2, \varepsilon = -1, \\ 4) \alpha &= \delta = 1, \chi = \rho = \beta_-^2, \varepsilon = 1, & 9) \alpha &= \nu + 1, \delta = \nu, \chi = \beta_-^2 + \nu \beta_+^2, \rho = \nu \beta_+^2, \varepsilon = -1, \\ 5) \alpha &= \nu, \delta = 1, \chi = \beta_+^2, \rho = \beta_-^2, \varepsilon = 1, & 10) \alpha &= \nu + 1, \delta = \nu, \chi = \beta_-^2 + \nu \beta_+^2, \rho = \beta_-^2, \varepsilon = -1. \end{aligned}$$

Recall that the generalized factorization of matrix $L(\lambda)$ in space $L^2(R)$ is the following representation:

$$L(\lambda) = L_+(\lambda) \begin{pmatrix} \left(\frac{\lambda - i}{\lambda + i} \right)^{x_1} & 0 \\ 0 & \left(\frac{\lambda - i}{\lambda + i} \right)^{x_2} \end{pmatrix} L_-(\lambda), \quad (10)$$

where

a) $\chi_1, \chi_2 \in Z$, $L_{\pm}^{\pm 1} \in L^2(R, \rho)$ (i.e. each component of the matrix belongs to $L^2(R, \rho)$), where $\rho(\lambda) = \frac{1}{\sqrt{\lambda^2 + 1}}$. The matrix-functions $L_{+}^{\pm 1}(\lambda)$ are analytically

continued in the upper half-plane $\text{Im } \lambda > 0$, and $L_{-}^{\pm 1}(\lambda)$ are analytically continued in the lower half-plane $\text{Im } \lambda < 0$;

b) the components of the matrix-function $\rho(\lambda)L_{+}^{\pm 1}(\lambda)$ ($\rho(\lambda)L_{-}^{\pm 1}(\lambda)$) belong to the $L^2(R)$ and have analytic continuations in the upper half-plane $\text{Im } \lambda > 0$ (respectively the lower half-plane $\text{Im } \lambda < 0$).

The factorization is called canonical when $\chi_1 = \chi_2 = 0$. Let

$$R_{+}(\lambda) = \frac{\alpha \left(\lambda + \sqrt{k^2 + \frac{\chi}{\alpha}} \right)}{\delta \left(\lambda + \sqrt{k^2 + \frac{\rho}{\delta}} \right)}, \quad R_{-}(\lambda) = \frac{\lambda - \sqrt{k^2 + \frac{\chi}{\alpha}}}{\lambda - \sqrt{k^2 + \frac{\rho}{\delta}}}. \quad (11)$$

It is easy to see that the components $\rho(\lambda)L_{\pm}^{\pm 1}(\lambda)$ belong to $L^2(R)$ and $\rho(\lambda)L_{+}^{\pm 1}(\lambda)$ ($\rho(\lambda)L_{-}^{\pm 1}(\lambda)$) have the analytic continuations in the upper half-plane $\text{Im } \lambda > 0$ (the lower half-plane $\text{Im } \lambda < 0$).

Denote by

$$r_{+}(\lambda) = \frac{1}{2\pi i} \int_{-\infty+ic}^{\infty+ic} \frac{r(\xi)d\xi}{\xi - \lambda}, \quad r_{-}(\lambda) = \frac{1}{2\pi i} \int_{-\infty+id}^{\infty+id} \frac{r(\xi)d\xi}{\xi - \lambda},$$

where $r(\lambda) = \frac{\varepsilon \sqrt{\lambda^2 - k^2}}{\alpha \left(\lambda + \sqrt{k^2 + \frac{\chi}{\alpha}} \right) \left(\lambda - \sqrt{k^2 + \frac{\rho}{\delta}} \right)}$, $k_* < c < d < k_{**}$.

Here $k_* = \max \left\{ -\text{Im} \sqrt{k^2 + \frac{\chi}{\alpha}}; -\text{Im} \sqrt{k^2 + \frac{\rho}{\delta}} \right\}$, $k_{**} = \min \left\{ \text{Im} \sqrt{k^2 + \frac{\chi}{\alpha}}; \text{Im} \sqrt{k^2 + \frac{\rho}{\delta}} \right\}$.

It is evident that $r_{\pm}(\lambda)$ belong to the $L^2(R)$ and $r_{+}(\lambda)$ ($r_{-}(\lambda)$) have analytic continuation in the upper (lower) half-plane (see [7]). Since

$$R_{+}(\lambda)R_{-}(\lambda) = R(\lambda) = \frac{\alpha\gamma^2(\lambda) - \chi}{\delta\gamma^2(\lambda) - \rho}, \quad r_{+}(\lambda) + r_{-}(\lambda) = r(\lambda) \quad (12)$$

and

$$\begin{pmatrix} 1 & r_{\pm}(\lambda) \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -r_{\pm}(\lambda) \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} R_{\pm}(\lambda) & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{R_{\pm}(\lambda)} & 0 \\ 0 & 1 \end{pmatrix}, \quad (13)$$

therefore, we have the equality

$$L(\lambda) = \begin{pmatrix} R_{+}(\lambda) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & r_{+}(\lambda) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & r_{-}(\lambda) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R_{-}(\lambda) & 0 \\ 0 & 1 \end{pmatrix}. \quad (14)$$

The representation (14) is the canonical factorization of the matrix-function $L(\lambda)$.

Using equalities (13) and (14) we can write the Riemann-Hilbert problem (6) in the form

$$\begin{pmatrix} \frac{1}{R_+(\lambda)} & -r_+(\lambda) \\ 0 & 1 \end{pmatrix} \bar{u}_+(\lambda) = \begin{pmatrix} R_-(\lambda) & r_-(\lambda) \\ 0 & 1 \end{pmatrix} \bar{u}_-(\lambda) + \begin{pmatrix} \frac{1}{R_+(\lambda)} & -r_+(\lambda) \\ 0 & 1 \end{pmatrix} \bar{m}(\lambda). \quad (15)$$

In an expanded form we get the following system:

$$\begin{cases} \frac{u_+(\lambda)}{R_+(\lambda)} - r_+(\lambda)w_+(\lambda) = R_-(\lambda)u_-(\lambda) + r_-(\lambda)w_-(\lambda) + \frac{m_1(\lambda)}{R_+(\lambda)} - r_+(\lambda)m_2(\lambda), \\ w_+(\lambda) = w_-(\lambda) + m_2(\lambda). \end{cases} \quad (16)$$

Hence, we have

$$\begin{aligned} w_+(\lambda) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty m_2(x)e^{i\lambda x} dx, & w_-(\lambda) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 m_2(x)e^{i\lambda x} dx, \\ u_+(\lambda) &= \frac{r_+(\lambda)R_+(\lambda)}{\sqrt{2\pi}} \int_0^\infty m_2(x)e^{i\lambda x} dx + \frac{1}{\sqrt{2\pi}} \int_0^\infty (m_1(x) + r_+(x)R_+(x)m_2(x))e^{i\lambda x} dx, \\ u_-(\lambda) &= \frac{r_-(\lambda)}{R_-(\lambda)\sqrt{2\pi}} \int_{-\infty}^0 m_2(x)e^{i\lambda x} dx - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \left(\frac{m_1(x)}{R_-(x)} + \frac{r_+(x)m_2(x)}{R_-(x)} \right) e^{i\lambda x} dx. \end{aligned}$$

Using (5) and the following representation

$$\begin{aligned} u(x, y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \left\{ a(\lambda) \frac{\gamma(\lambda) + \beta_+ \tanh(\beta_+ y)}{\gamma(\lambda)} e^{-\gamma(\lambda)y} \chi_+(y) + \right. \\ &\quad \left. + b(\lambda) \frac{\gamma(\lambda) - \beta_- \tanh(\beta_- y)}{\gamma(\lambda)} e^{\gamma(\lambda)y} \chi_-(y) \right\} e^{i\lambda x} d\lambda, \end{aligned} \quad (17)$$

we obtain the solution of the boundary value problem (1)–(2).

Theorem. The boundary value problem (1)–(2) has the unique solution, which is given by the formula (17), where the functions $a(\lambda)$ and $b(\lambda)$ can be reconstructed by formula (5).

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Ս. Ա. Հուսեյն Մաթեկոլաեյի

Հելմհոլց-Շրյոդինգերի հավասարման հետ կապված մի անիզոտրոպ
սահմանային խնդրի մասին առաջին և
երկրորդ սեռի սահմանային պայմանների առկայությամբ

Աշխատանքում քննարկվում է անիզոտրոպ հավասարման հետ կապված
Հելմհոլց-Շրյոդինգերի սահմանային խնդրի լուծելիությունը $y = 0$ ուղղի վրա
առաջին և երկրորդ սեռի սահմանային պայմանների առկայությամբ:

С. А. Усейин Матеклази

Об одной анизотропной граничной задаче, связанной с уравнением Гельмгольца
Шредингера, при граничных условиях первого и второго рода

В данной работе исследуется разрешимость граничной задачи, связанной с анизотропным уравнением Гельмгольца-Шредингера, с граничными условиями первого и второго рода на линии $y = 0$.