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Mathematics

ON ONE URYSOHN TYPE NONLINEAR INTEGRAL EQUATION WITH NONCOMPACT OPERATOR

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In the present paper the Urysohn type nonlinear integral equation with noncompact operator on the half-line is considered. It is assumed that the Wiener–Hopf–Hankel type operator is a local minorant for the initial Urysohn operator. The existence of a positive and bounded solution is proved. The limit of constructed solution at infinity is calculated. At the end of the work a list of examples is given.

***Keywords*:** nonlinearity, iterative methods, Urysohn operator, Caratheodory condition.

§ 1. Introduction and Formulation of Theorem. In the present work the following nonlinear integral equation

$$f(x)=\int_0^\infty K(x,t,f(t))dt, \quad x\in R^+=(0,+\infty), \qquad (1)$$

is considered. Here $f(x)$ is an unknown real measurable function, satisfying equation (1) almost everywhere, $K(x,t,\tau)$ is defined on $R^+\times R^+\times R$ and satisfies the following conditions: there exists a number $\eta>0$, such that

a) $K(x,t,\tau)\ge 0, \quad (x,t,\tau)\in R^+\times R^+\times[0,\eta]=\Omega_\eta$.

b) $K\uparrow$ in τ on interval $[0,\eta]$ for each fixed $(x,t)\in R^+\times R^+$.

c) $K(x,t,\tau)\in Carat(\Omega_\eta)$, i.e. function $K(x,t,\tau)$ satisfies the Caratheodory condition in τ on Ω_η . The latter means that for each fixed $\tau\in[0,\eta]$ the function $K(x,t,\tau)$ is measurable in $(x,t)\in R^+\times R^+$ and for almost all $(x,t)\in R^+\times R^+$ the function $K(x,t,\tau)$ is continuous in τ on the interval $[0,\eta]$ (concerning this condition we refer to [1]).

d) $\int_0^\infty K(x,t,\eta)dt\le\eta, \quad x\in R^+$. (2)

Let $K_0(x)$ and $K^*(x)$ are given measurable functions on sets R and R^+ respectively, satisfying

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- $0 \le K_0 \in L_1(R),\ K_0(-x) > K_0(x),\ \ x \in R^+,$ (3)
- $\int\limits_{-\infty}^{+\infty} K_0(\tau)d\tau = 1,\quad \nu(K_0) \equiv \text{v.p.} \int\limits_{-\infty}^{+\infty} \tau K_0(\tau)d\tau > -\infty,$ (4)
- $0 \le K^*(x) < K_0(x),\ x \in R^+\ ,\ K^*(x) \downarrow \text{in } x \text{ on } R^+,$ (5)
- $m_2 = \int\limits_0^{+\infty} x^2 K^*(x)dx < +\infty.$ (6)

We assume that

e) $K(x,t,\tau) \ge \lambda(x)(K_0(x-t) - K^*(x+t))\tau,\qquad (x,t,\tau) \in \Omega_\eta,$ (7)

where $\lambda(x)$ is a measurable function on R^+, and besides

$$0 \le \lambda(x) \le 1,\ \lambda \uparrow \text{ in } x,\ \ (1-\lambda(x))x^j \in L_1(R^+),\ j = 0,1. \tag{8}$$

f) We also assume that for each measurable function $\varphi(x)$, $0 \le \varphi(x) \le \eta$, $x > 0$, the functions $K(x,t,\varphi(t))$ and $\int\limits_0^\infty K(x,t,\varphi(t))dt$ are measurable with respect to $t > 0$ and $x > 0$.

Remark 1. It is easy to check that, if $K(x,t,\tau)$ is continuous in the totality of all arguments on set Ω_η, then the conditions c) and f) are fulfilled automatically.

It should be noted, that the equation (1) has been recently investigated by the author in [2] for the particular case $\lambda(x) \equiv 1,\ K^*(x) \equiv 0$.

In the present paper the following result is proved.

Theorem. Let the conditions a)–f) are fulfilled. Then the equation (1) has a nonnegative and bounded solution $f(x) \le \eta,\ x \in R^+,$ and besides

$$\lim_{x\to+\infty} f(x) = \eta\,. \tag{9}$$

Moreover, if $\inf\limits_{x\in R^+} \lambda(x) = \varepsilon_0 > 0$, then $f(x) > 0.$

§ 2. The Proof of Theorem.

Step I. First let us consider the following homogeneous linear equation with sum – difference kernel

$$S(x) = \int\limits_0^{+\infty} [K_0(x-t) - K^*(x+t)]S(t)dt,\quad x > 0\,, \tag{10}$$

with respect to an unknown function $S(x)$. From results of [3] it follows that the equation (10) has nontrivial (possessing both positive and negative values) and bounded solution $\tilde{S}(x)$. Below it will be proved that besides the solution $\tilde{S}(x)$, equation (10) has a positive non-decreasing and bounded solution $S^*(x)$ with

$$\inf_{x\in R^+} S^*(x) > 0. \tag{11}$$

First we show that

$$K_0(x-t) > K^*(x+t),\quad (x,t) \in R^+ \times R^+. \tag{12}$$

Indeed, let $x \ge t$, then from (5) it follows that $K_0(x-t) > K^*(x-t) \ge K^*(x+t).$
If we assume that $x < t$, then taking into account (3)–(5) we'll have

$$K_0(x-t) > K_0(t-x) > K^*(t-x) \geq K^*(x+t).$$

Thus, the inequality (12) is established. Now consider the following iteration:

$$S_{n+1}(x) = \int_0^{+\infty} [K_0(x-t) - K^*(x+t)] S_n(t)dt, \quad x > 0, \qquad (13)$$

$$S_0(x) \equiv c = \sup_{x \in R^+} \left|\tilde{S}(x)\right|, \quad n = 0,1,2,...$$

It is easy to check by induction that

$$i_1)\ S_n(x) \downarrow \text{in } n;\ \ i_2)\ S_n(x) \uparrow \text{in } x;\ \ i_3)\ S_n(x) \geq \left|\tilde{S}(x)\right|, n = 0,1,2... \qquad (14)$$

For example, let's prove $i_3)$: for $n = 0$ it follows from (13). Assuming that $S_n(x) \geq \left|\tilde{S}(x)\right|$ for any $n \in N$**,** and taking into account (12) we have

$$S_{n+1}(x) \geq \int_0^{+\infty} [K_0(x-t) - K^*(x+t)] \left|\tilde{S}(t)\right| dt \geq \left| \int_0^{+\infty} [K_0(x-t) - K^*(x+t)] \tilde{S}(t)dt \right| = \left|\tilde{S}(x)\right|.$$

The statement $i_1)$ is proved in the same way. Now let us consider the statement.

The monotonicity of sequences $\{S_n(x)\}_{n=0}^{\infty}$ in x is easy to check, if the iteration (13) is rewritten in the following form

$$S_{n+1}(x) = \int_{-\infty}^{x} K_0(t) S_n(x-t)dt - \int_x^{+\infty} K^*(t) S_n(t-x)dt, \quad x > 0, \qquad (15)$$

$$S_0(x) \equiv c, \quad n = 0,1,2...$$

Thus, the sequence of functions $\{S_n(x)\}_{n=0}^{\infty}$ has a pointwise limit:

$$\lim_{n \to +\infty} S_n(x) = S^*(x) \leq c. \qquad (16)$$

Besides that, in accordance with the B. Levis theorem [4], the limit function $S^*(x)$ satisfies equation (10). From (14) it follows that

$$\left|\tilde{S}(x)\right| \leq S^*(x),\ S^* \uparrow \text{by } x, \text{ on } R^+. \qquad (17)$$

Now prove formulae (11). As $S^*(x) \geq 0,\ S^*(x) \neq 0,$ then there exists $x_0 \geq 0$ such that

$$\alpha_0 \equiv S^*(x_0) > 0. \qquad (18)$$

Then taking into account (12), (17), (18), we obtain from (10)

$$S^*(x) \geq \int_{x_0}^{+\infty} [K_0(x-t) - K^*(x+t)] S^*(t)dt \geq \alpha_0 \Big(\int_{-\infty}^{x-x_0} K_0(\tau)d\tau - \int_{x+x_0}^{\infty} K^*(\tau)d\tau \Big) \geq$$

$$\geq \alpha_0 \int_{x_0}^{\infty} (K_0(-t) - K^*(t)dt) > 0.$$

Therefore, the formulae (11) is true.

Step II. Now we consider the following more general linear homogenous equation:

$$\varphi(x) = \lambda(x) \int_0^{+\infty} [K_0(x-t) - K^*(x+t)] \varphi(t)dt, \quad x > 0, \qquad (19)$$

with respect to an unknown real function $\varphi(x)$.

Now along with equation (19) we consider the following non-homogenous integral equation

$$\psi(x)=(1-\lambda(x))S^*(x)+\lambda(x)\int_0^{+\infty}[K_0(x-t)-K^*(x+t)]\psi(t)dt,\ x\in R^+. \quad (20)$$

As it follows from [5], equation (20) has a nonnegative nontrivial solution $\psi_0(x)\in L_1(R^+)\cap M(R^+)$ and besides $\psi_0(x)\le S^*(x)$. It can be easily shown that the function $\psi_1(x)\equiv S^*(x)$ also satisfies the equation (20). Note that $\psi_0(x)\ne\psi_1(x)$ as $\psi_0(x)\in L_1(R^+)\cap M(R^+)$ and $\inf\limits_{x\in R^+}\psi_1>0$.

It is obvious that the function $\varphi(x)=\psi_1(x)-\psi_0(x)\ge 0\ (\ne 0)$ will satisfy the equation (19).

It is noteworthy that such an approach to the solution of equation (19) in case when $K^*(x)\equiv 0$ was suggested in [5].

Now we consider the following iteration:

$$\varphi_{n+1}(x)=\lambda(x)\int_0^{+\infty}[K_0(x-t)-K^*(x+t)]\varphi_n(t)dt, \quad (21)$$

$$\varphi_0(x)\equiv\sup_{x\in R^+}\varphi(x),\quad n=0,1,2...,\quad x>0.$$

By the analogy of Step I the following facts can be established by induction:

$$j_1)\ \varphi_n(x)\downarrow \text{in } n;\ \ j_2)\ \varphi_n(x)\uparrow \text{in } x;\ \ j_3)\ \varphi_n(x)\ge\varphi(x),\ \ n=0,1,2... \quad (22)$$

Therefore, there exists the limit

$$\lim_{n\to+\infty}\varphi_n(x)=\varphi^*(x)\le\sup_{x\in R^+}\varphi(x), \quad (23)$$

and in addition $\varphi^*(x)$ satisfies the equation (19) and $\varphi^*(x)\uparrow$ in x. By analogy with formulae (11), it can be shown, that if $\inf\limits_{x\in R^+}\lambda(x)>0$, then

$$\beta_0=\inf_{x\in R^+}\varphi^*(x)>0. \quad (24)$$

Step III. At this last stage a nontrivial solution of basic equation (1) will be constructed using formulae (24) and monotonicity of function $\varphi^*(x)$.

Let us consider the following iteration

$$f_{n+1}(x)=\int_0^{\infty}K(x,t,f_n(t))dt,\quad f_0(x)\equiv\eta,\ n=0,1,2...,\ \ x\in R^+. \quad (25)$$

From condition f) it follows that each function $f_n(x)$ is measurable.

Below we prove that

$$p_1)\ f_n(x)\downarrow\ \text{in } n;\ \ p_2)\ f_n(x)\ge\frac{\eta}{\sup\limits_{x\in R^+}\varphi^*(x)}\varphi^*(x),\ \ n=0,1,2.....,x\in R^+.$$

First let us prove $p_1)$: in the light of the properties a) and b) we have

$$f_1(x)=\int_0^{\infty}K(x,t,\eta)dt\le\eta=f_0(x).$$

Assuming that $f_n(x) \le f_{n-1}(x)$ and taking into account (25) we obtain

$$f_{n+1}(x) \le f_n(x).$$

Now we prove the inequality $p_2)$: for $n=0$ it is obvious. Assume that $p_2)$ is true for $n=m\in N$ and prove it in case of $n=m+1$. In consequence of inequality (7) we obtain

$$f_{m+1}(x) \ge \int_0^\infty K\Big(x,t,\frac{\eta}{\sup\limits_{x\in R^+}\varphi^*}\varphi^*(t)\Big)dt \ge \frac{\eta\lambda(x)}{\sup\limits_{x\in R^+}\varphi^*}\int_0^\infty (K_0(x-t)-K^*(x+t))\varphi^*(t)dt = \frac{\eta\varphi^*(x)}{\sup\limits_{x\in R^+}\varphi^*}.$$

Using on $p_1)$, $p_2)$ we conclude that the sequence of functions $\{f_n(x)\}_{n=0}^\infty$ has a limit

$$\lim_{n\to+\infty} f_n(x) = f(x),\ x\in R^+, \qquad (26)$$

satisfying

$$\frac{\eta\varphi^*(x)}{\sup\limits_{x\in R^+}\varphi^*} \le f(x) \le \eta,\ x\in R^+. \qquad (27)$$

As $f_n \downarrow$ by n, $K\in Carat(\Omega_\eta)$ and $f(x) \le \int_0^\infty K(x,t,f_n(t))dt$, then from B. Levis theorem we get that limit function $f(x)$ satisfies the equation (1).

If $\varepsilon_0 = \inf\limits_{x\in R^+}\lambda(x) > 0$, then from (27) immediately follows that $\psi(x)>0$. As $\varphi^*(x)\uparrow \sup\limits_{x\in R^+}\varphi^*(x)$, then from (27) we also obtain $\lim\limits_{x\to+\infty} f(x) = \eta$.

Thus, the Theorem is proved.

§ 3. Below we give some particular examples of equation (1):

- $$f(x) = \lambda(x)\int_0^{+\infty}[K_0(x-t)-K^*(x+t)]G(f(t))dt,\quad x>0, \qquad (28)$$

where

$$G\in C[0,\eta],\ G(x)\ge x,\ x\in[0,\eta],\ G\uparrow \text{ by } x \text{ on } [0,\eta],\ G(\eta)=\eta. \qquad (29)$$

- $$f(x) = \int_0^{+\infty} R(x,f(t))[K_0(x-t)-K^*(x+t)]G(f(t))dt,\quad x>0, \qquad (30)$$

where $R(x,t)$ is a measurable function defined on $R^+\times R^+$ with

1) $R(x,t)\in Carat(R^+\times[0,\eta])$,

2) $R(x,t)\uparrow$ in t on $[0,\eta]$ for each fixed $x\in R^+$,

3) $\lambda(x) \le R(x,t) \le \dfrac{1}{\int_{-\infty}^{x} K_0(\tau)d\tau - \int_x^\infty K^*(\tau)d\tau}$, $(x,t)\in R^+\times[0,\eta]$.

As particular examples of such G and R we can take the following functions:

I. $G(x) = x^\alpha$, $\alpha\in(0,1)$, $\eta=1$, $x\in R^+$.

II. $G(x) = x+\sin x$, $\eta=1$, $x\in R^+$.

III. $G(x)=\sqrt{xe^{x-1}}$, $\eta=1,\ x\in R^+$.

$$R(x,t)=\frac{F(x)-\lambda(x)}{2}u(t)+\frac{F(x)+\lambda(x)}{2},$$

where $F(x)=(\int_{-\infty}^{x}K_0(t)dt-\int_{x}^{\infty}K^*(t)dt)^{-1}$, and $u\in C[0,\eta],\ 0\le u(t)\le 1,\ t\in[0,\eta],\ u\uparrow$ in t on $[0,\eta]$.

Remark 2. We note that it can be proved that the solution of equation (28) is increasing.

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REFERENCES

1. **Kufner A. and Fuchik S.** Nonlinear Differential Equations. M.: Nauka, 1988, 304 p.
2. **Khachatryan Kh.A.** Doklady Rossiyskoy Akademii Nauk. Matematica, 2009, v. 425, № 2, p. 462–465 (in Russian).
3. **Yengibaryan B.N.** Izv. NAN Armenii, Matematica, 1997, v. 32, № 1, p. 38–48 (in Russian).
4. **Kolmogorov A.N. and Fomin V.S.** Elements of the Theory of Functions and Functional Analysis. M.: Nauka, 1981, 544 p.
5. **Arabadjyan L.G.** Differential Equations, 1987, v. 23, № 9, p. 1618–1622.

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Խ. Ա. Խաչատրյան

Ոչ կոմպակտ օպերատորով Ուրիսոնի տիպի մի
ոչ գծային ինտեգրալ հավասարման մասին

Աշխատանքում հետազոտվում է ոչ կոմպակտ օպերատորով Ուրիսոնի տիպի ոչ գծային ինտեգրալ հավասարում կիսառանցքի վրա: Ենթադրվում է, որ Վիներ–Հոպֆ–Հանկելի օպերատորը ծառայում է որպես լոկալ մինորանտ Ուրիսոնի սկզբնական օպերատորի համար: Ապացուցվել է դրական և սահմանափակ լուծման գոյությունը: Գտնվել է կառուցված լուծման սահմանն անվերջությունում: Աշխատանքի վերջում բերվել են օրինակներ:

Х. А. Хачатрян.

Об одном нелинейном интегральном уравнении
типа Урысона с некомпактным оператором

В работе исследуется нелинейное интегральное уравнение типа Урысона с некомпактным оператором на полуоси. Предполагается, что оператор Винера–Хопфа–Ганкеля служит локальной минорантой для исходного оператора Урысона. Доказывается существование положительного и ограниченного решения. Вычисляется предел построенного решения в бесконечности. В конце работы приведены примеры.