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Continuous inclusions and Bergman type operators in n-harmonic mixed norm spaces on the polydisc

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Abstract

We study anisotropic mixed norm spaces of n-harmonic functions in the unit polydisc of $\mathbb{C}^n$. Bergman type reproducing integral formulas are established by means of fractional derivatives and some continuous inclusions. It gives us a tool to construct corresponding projections and related operators and prove their boundedness on the mixed norm and Besov spaces.

0. Introduction

Let $U^n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n\colon\ |z_j| < 1,\ 1 \leqslant j \leqslant n\}$ be the unit polydisc in $\mathbb{C}^n$, and let $T^n = \{w = (w_1, \dots, w_n) \in \mathbb{C}^n\colon\ |w_j| = 1,\ 1 \leqslant j \leqslant n\}$ be the n-dimensional torus, the distinguished boundary of U^n. We shall deal with n-harmonic functions on the polydisc U^n, i.e. functions harmonic in each variable z_j separately. Denote by $h(U^n)$ $(H(U^n))$ the set of all n-harmonic (respectively holomorphic) functions in U^n. If $f(z) = f(rw)$ is a measurable function in U^n, then we write

$$M_p(f;r) = \|f(r\cdot)\|_{L^p(T^n;dm_n)}, \quad r = (r_1, \dots, r_n) \in I^n,\ 0 < p \leqslant \infty,$$

where $I^n = [0, 1)^n$, dm_n is the n-dimensional Lebesgue measure on T^n normalized so that $m_n(T^n) = 1$. The collection of n-harmonic (holomorphic) functions $f(z)$, for which $\|f\|_{h^p} = \sup_{r \in I^n} M_p(f;r) < +\infty$, is the usual Hardy space h^p (respectively H^p).

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The quasi-normed space $L(p,q,\alpha)$ $(0 < p, q \leqslant \infty, \alpha = (\alpha_1, \ldots, \alpha_n))$ is the set of those functions $f(z)$ measurable in the polydisk U^n, for which the quasi-norm

$$\|f\|_{p,q,\alpha} = \begin{cases} \left(\displaystyle\int_{I^n} \prod_{j=1}^{n} (1-r_j)^{\alpha_j q - 1} M_p^q(f;r) \prod_{j=1}^{n} dr_j \right)^{1/q}, & 0 < q < \infty, \\ \operatorname*{ess\,sup}_{r \in I^n} \displaystyle\prod_{j=1}^{n} (1-r_j)^{\alpha_j} M_p(f;r), & q = \infty, \end{cases}$$

is finite. For the subspaces of $L(p,q,\alpha)$ consisting of n-harmonic or holomorphic functions let $h(p,q,\alpha) = h(U^n) \cap L(p,q,\alpha)$, $H(p,q,\alpha) = H(U^n) \cap L(p,q,\alpha)$. For $p = q < \infty$, the spaces $h(p,q,\alpha)$ and $H(p,q,\alpha)$ coincide with the well-known weighted Bergman spaces. The first results on mixed norm spaces are contained in classical works of Hardy and Littlewood [10,11], who considered functions holomorphic in the unit disk $\mathbb{D} = U^1$. Later, Flett [8] essentially improved and developed methods of [10,11]. Holomorphic and pluriharmonic mixed norm spaces on the unit ball and bounded symmetric domains of $\mathbb{C}^n$ have been studied, for example, in [14,17,19]. Motivated by papers of Choe [3], Shamoyan [18], and Zhu [21], we are interested in projections in mixed norm and Besov spaces on the polydisc U^n. The paper is organized as follows. First, we prove several continuous inclusions of Hardy, Littlewood, and Flett in Theorem 1 for n-harmonic spaces $h(p,q,\alpha)$ and Hardy spaces on the polydisc. These inclusions are used in further theorems. A Poisson–Bergman type reproducing integral formula is stated in Theorem 2 for n-harmonic functions in $h(p,q,\alpha)$. Corresponding integral operators $T_{\beta,\lambda}$, $\widetilde{T}_{\beta,\lambda}$, $S_{\beta,\lambda}$, $\widetilde{S}_{\beta,\lambda}$ of Bergman type are constructed on the basis of fractional integro-differentiation and Poisson type reproducing kernels. In Theorem 3 of Forelli–Rudin type, given $1 \leqslant p, q < \infty$, we find a necessary and sufficient condition for $T_{\beta,0}$ to be a bounded projection of $L(p,q,\alpha)$ onto $h(p,q,\alpha)$, and also for $T_{\beta,\lambda}$ to be a bounded operator in $L(p,q,\alpha)$. The traditional way of stating the projection results is to use Schur test (see, e.g., [12]). Instead, we use a higher-dimensional version of Hardy's inequality and give a quick elementary proof of projection theorems. Further, Bergman type operators can be considered on other function spaces. In Theorem 4, the action of the operators $T_{\beta,0}$ and $\widetilde{T}_{\beta,0}$ is studied on mixed norm spaces $L(p,q,\alpha)$ for multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ with non-positive entries α_j. It turns out that the image of $L(p,q,\alpha)$ with $\alpha_j \leqslant 0$ under $T_{\beta,0}$ and $\widetilde{T}_{\beta,0}$ is the Besov space $h\Lambda_\alpha^{p,q}$ of n-harmonic functions. On the other hand, it is known that Bergman projection preserves Lipschitz spaces in the setting of the unit ball of $\mathbb{C}^n$ or $\mathbb{R}^n$ and in strictly pseudoconvex domains of $\mathbb{C}^n$ (see [4,16,20]). One may ask whether this is still true for Besov spaces. In Theorem 5 we generalize the preservation property to Besov spaces under a Bergman type operator which projects the Besov space $\Lambda_\alpha^{p,q}$ onto its n-harmonic subspace $h\Lambda_\alpha^{p,q}$. Theorem 5 seems to be new even for one-variable case. Finally, as an application we give in Theorem 6 a duality result for spaces $h(p,q,\alpha)$.

Note that many particular results of the theorems are well known especially for holomorphic Bergman spaces on the unit disk, the unit ball or the polydisc in $\mathbb{C}^n$, see [3,5,8, 10–12,14,17–19,21]. Observe that in Theorems 1–6 for $p \neq q$, an iteration of one-variable case does not work. There is an additional difficulty in the proof of Theorem 1 connected with non-n-subharmonicity of $|u|^p$ and non-monotonicity of integral means $M_p(u;r)$ with respect to r for $0 < p < 1$. On the other hand, a passage from n-harmonic functions to holomorphic ones is impossible because n-harmonic functions need not be real parts of holomorphic functions.

1. Main theorems

We shall use the conventional multi-index notations: $\bar{\zeta} = (\bar{\zeta}_1, \ldots, \bar{\zeta}_n)$, $r\zeta = (r_1\zeta_1, \ldots, r_n\zeta_n)$, $dr = dr_1 \cdots dr_n$, $(1-|\zeta|^2)^\alpha = \prod_{j=1}^n (1-|\zeta_j|^2)^{\alpha_j}$, $\Gamma(\alpha+|k|) = \prod_{j=1}^n \Gamma(\alpha_j+|k_j|)$ for $\zeta \in \mathbb{C}^n$, $r \in I^n$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $k = (k_1, \ldots, k_n)$.

Throughout the paper, the letters $C(\alpha, \beta, \ldots)$, C_α, etc., will denote positive constants possibly different at different places and depending only on the parameters indicated. For $A, B > 0$, the notation $A \approx B$ denotes the two-sided estimate $c_1 A \leqslant B \leqslant c_2 A$ with some inessential positive constants c_1 and c_2 independent of the variable involved. For any p, $1 \leqslant p \leqslant \infty$, we define the conjugate index p' as $p' = p/(p-1)$ (we interpret $1/\infty = 0$ and $1/0 = +\infty$). The symbol dm_{2n} means the Lebesgue measure on the polydisc U^n normalized so that $m_{2n}(U^n) = 1$. We shall write $T: X \to Y$, if T is a bounded operator mapping X to Y, i.e. $\|Tf\|_Y \leqslant C\|f\|_X$ $\forall f \in X$.

We now formulate main theorems of the paper. Starting from the Hardy–Littlewood–Flett inclusions in $h(p, q, \alpha)$, we present them by the following table.

Theorem 1. *Let* $0 < p, q \leqslant \infty$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_0 = (\alpha_{01}, \ldots, \alpha_{0n})$, $\beta = (\beta_1, \ldots, \beta_n)$, $\alpha_j, \alpha_{0j}, \beta_j \in \mathbb{R}$, $1 \leqslant j \leqslant n$. *Then the following inclusions are continuous*:

(i) $h(p, q, \alpha) \subset h(p, q, \beta)$, $\quad \beta_j \geqslant \alpha_j$,

(ii) $h(p, q, \alpha) \subset h(p_0, q, \alpha)$, $\quad 0 < p_0 < p \leqslant \infty$,

(iii) $h(p, q, \alpha) \subset h(p, q_0, \alpha)$, $\quad 0 < q < q_0 \leqslant \infty$,

(iv) $h(p, q, \alpha) \subset h(p_0, q, \alpha_0)$, $\quad \alpha_{0j} \geqslant \alpha_j + 1/p - 1/p_0,\ 0 < p \leqslant p_0 \leqslant \infty$,

(v) $h(p, q, \alpha) \subset h(p_0, q_0, \beta)$, $\quad \beta_j > \alpha_j + 1/p,\ 0 < p_0, q_0 \leqslant \infty$,

(vi) $h(p, q, \alpha) \subset h(p, q_0, \beta)$, $\quad \beta_j > \alpha_j,\ 0 < q_0 \leqslant \infty$,

(vii) $H^p \subset H\left(p_0, q, \dfrac{1}{p} - \dfrac{1}{p_0}\right)$, $\quad 0 < p < p_0 \leqslant \infty,\ 0 < p \leqslant q \leqslant \infty$,

(viii) $h^p \subset h\left(p_0, q, \dfrac{1}{p} - \dfrac{1}{p_0}\right)$, $\quad 1 \leqslant p < p_0 \leqslant \infty,\ 1 \leqslant p \leqslant q \leqslant \infty$,

(ix) $h^p \subset h(p_0, q, \beta)$, $\quad \beta_j > \dfrac{1}{p} - \dfrac{1}{p_0},\ 0 < p < p_0 \leqslant \infty$.

(x) *Besides, if* $u \in h(p, q, \alpha)$, $0 < q < \infty$, *then* $(1-r)^\alpha M_p(u; r) = o(1)$ *as* $r_j \to 1-$ *for each* $j \in [1, n]$.

The next theorem establishes a reproducing integral formula of Poisson–Bergman type for functions in $h(p, q, \alpha)$.

Theorem 2. *Let* $\alpha_j > 0$ *and* $u \in h(p,q,\alpha)$. *If either* $0 < p, q \leqslant \infty$, $\beta_j > \max\{\alpha_j + 1/p - 1, \alpha_j\}$, *or* $1 \leqslant p \leqslant \infty$, $0 < q \leqslant 1$, $\beta_j \geqslant \alpha_j$ $(1 \leqslant j \leqslant n)$, *then for* $z \in U^n$

$$u(z) = \frac{1}{\prod_{j=1}^n \Gamma(\beta_j)} \int\limits_{U^n} \prod_{j=1}^n \bigl(1-|\zeta_j|^2\bigr)^{\beta_j - 1} P_\beta(z,\zeta) u(\zeta)\, dm_{2n}(\zeta), \tag{1.1}$$

where the kernel P_β *of Poisson type is defined in Section* 3.

The representation (1.1) induces linear integral operators of Bergman type:

$$T_{\beta,\lambda}(f)(z) = \frac{(1-|z|^2)^\lambda}{\Gamma(\beta+\lambda)} \int\limits_{U^n} \bigl(1-|\zeta|^2\bigr)^{\beta-1} P_{\beta+\lambda}(z,\zeta)\, f(\zeta)\, dm_{2n}(\zeta),$$

$$S_{\beta,\lambda}(f)(z) = \frac{(1-|z|^2)^\lambda}{\Gamma(\beta+\lambda)} \int\limits_{U^n} \bigl(1-|\zeta|^2\bigr)^{\beta-1} \bigl|P_{\beta+\lambda}(z,\zeta)\bigr|\, f(\zeta)\, dm_{2n}(\zeta),$$

where $\beta = (\beta_1, \ldots, \beta_n)$, $\lambda = (\lambda_1, \ldots, \lambda_n)$. Also, we introduce similar integral operators with "conjugate" kernel Q_β of Poisson type (see Section 3):

$$\widetilde{T}_{\beta,\lambda}(f)(z) = \frac{(1-|z|^2)^\lambda}{\Gamma(\beta+\lambda)} \int\limits_{U^n} \bigl(1-|\zeta|^2\bigr)^{\beta-1} Q_{\beta+\lambda}(z,\zeta)\, f(\zeta)\, dm_{2n}(\zeta),$$

$$\widetilde{S}_{\beta,\lambda}(f)(z) = \frac{(1-|z|^2)^\lambda}{\Gamma(\beta+\lambda)} \int\limits_{U^n} \bigl(1-|\zeta|^2\bigr)^{\beta-1} \bigl|Q_{\beta+\lambda}(z,\zeta)\bigr|\, f(\zeta)\, dm_{2n}(\zeta).$$

It is natural here to ask whether these operators are bounded in mixed norm spaces. The next theorem of Forelli–Rudin type answers to this question.

Theorem 3. (i) *Let* $1 \leqslant p, q \leqslant \infty$, $\beta_j > \alpha_j > -\lambda_j$ $(1 \leqslant j \leqslant n)$. *Then each of the operators* $T_{\beta,\lambda}$, $\widetilde{T}_{\beta,\lambda}$, $S_{\beta,\lambda}$, $\widetilde{S}_{\beta,\lambda}$ *continuously maps the space* $L(p,q,\alpha)$ *into itself. Moreover, the operator* $T_{\beta,0}$ $(\lambda_j = 0)$ *projects* $L(p,q,\alpha)$ *onto* $h(p,q,\alpha)$.

(ii) *Let* $1 \leqslant p, q < \infty$, $\alpha_j, \beta_j, \lambda_j \in \mathbb{R}$. *Then each of the operators* $T_{\beta,\lambda}$, $S_{\beta,\lambda}$ *is bounded in* $L(p,q,\alpha)$ *if and only if* $\beta_j > \alpha_j > -\lambda_j$ $(1 \leqslant j \leqslant n)$.

Remark. For functions holomorphic in the unit disk or the ball of $\mathbb{C}^n$ as well as for holomorphic Bergman spaces $(p = q)$ the results of Theorems 2 and 3 are known even for general weights; see, e.g., [5,12,14,17,18] and references therein.

Further, a question arises: What is the image of $L(p,q,\alpha)$ with negative α_j under the mappings $T_{\beta,\lambda}$ and $\widetilde{T}_{\beta,\lambda}$? To answer we introduce Besov spaces of smooth enough and n-harmonic functions.

Definition. The function $f(z)$ given in U^n, is said to belong to Besov space $\Lambda_\alpha^{p,q}$ $(0 < p, q \leqslant \infty$, $\alpha_j \geqslant 0)$ if $\mathcal{D}^{\widetilde{\alpha}} f(z) \in L(p,q,\widetilde{\alpha} - \alpha)$, where $\widetilde{\alpha} = (\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_n)$, $\widetilde{\alpha}_j$ is the least integer greater than α_j, and $\mathcal{D}^\alpha$ is a Riemann–Liouville integro-differential operator defined in Section 3. The Besov space $\Lambda_\alpha^{p,q}$ is equipped with a norm (quasinorm) $\|f\|_{\Lambda_\alpha^{p,q}} = \|\mathcal{D}^{\widetilde{\alpha}} f\|_{p,q,\widetilde{\alpha}-\alpha}$.

Let $h\Lambda_\alpha^{p,q}$ be the subspace of $\Lambda_\alpha^{p,q}$ consisting of n-harmonic functions. For a function $f \in h\Lambda_\alpha^{p,q}$, the multi-index $\widetilde{\alpha}$ may be replaced by any multi-index $\gamma = (\gamma_1, \ldots, \gamma_n)$, $\gamma_j > \alpha_j$, and the corresponding norms are equivalent: $\|f\|_{h\Lambda_\alpha^{p,q}} \approx \|\mathcal{D}^\gamma f\|_{p,q,\gamma-\alpha}$.

Theorem 4. *For $1 \leqslant p, q \leqslant \infty$, $\alpha_j \geqslant 0$, $\beta_j > 0$ $(1 \leqslant j \leqslant n)$, the operators*

$$T_{\beta,0} : L(p, q, -\alpha) \longrightarrow h\Lambda_\alpha^{p,q}, \tag{1.2}$$

$$\widetilde{T}_{\beta,0} : L(p, q, -\alpha) \longrightarrow h\Lambda_\alpha^{p,q}, \tag{1.3}$$

are bounded. Moreover, the map (1.2) *is surjective.*

Remark. For $p = q = \infty$, $\alpha_j = 0$, Theorem 4 asserts the boundedness of $T_{\beta,0}$ from $L^\infty(U^n)$ onto the Bloch space $\mathcal{B}h = h\Lambda_0^{\infty,\infty}$ of n-harmonic functions. This is familiar for the (weighted) Bergman projection and holomorphic functions in various domains, see, e.g., [3,5,12,21], while for $p = q$, $\alpha_j = 1/p$ and holomorphic functions, the relation (1.2) is due to Zhu [21].

In some papers, [4,16,20], preservation of Lipschitz spaces under the Bergman projection is studied. Now, for similar operator

$$\Phi_{\widetilde{\alpha}}(f)(z) = \frac{1}{\Gamma(\widetilde{\alpha})} \int_{U^n} \bigl(1 - |\zeta|^2\bigr)^{\widetilde{\alpha}-1} P(z, \zeta)\, \mathcal{D}^{\widetilde{\alpha}} f(\zeta)\, dm_{2n}(\zeta),$$

we study the same problem.

Theorem 5. *For $1 \leqslant p, q \leqslant \infty$, $\alpha_j > 0$ $(1 \leqslant j \leqslant n)$, the operator $\Phi_{\widetilde{\alpha}}$ continuously projects $\Lambda_\alpha^{p,q}$ onto $h\Lambda_\alpha^{p,q}$.*

Finally, as an application of projection theorems we find the dual space of $h(p, q, \alpha)$ for $1 \leqslant p \leqslant \infty$, $1 \leqslant q < \infty$.

Theorem 6. *For $1 \leqslant p \leqslant \infty$, $1 \leqslant q < \infty$, $\alpha_j > 0$ $(1 \leqslant j \leqslant n)$, we have $(h(p, q, \alpha))^* \cong h(p', q', \alpha q/q')$ under the integral pairing*

$$\langle f, g\rangle = \int_{U^n} f(z)\overline{g(z)}\bigl(1 - |z|^2\bigr)^{\alpha q-1}\, dm_{2n}(z),$$

where $f \in h(p, q, \alpha)$, $g \in h(p', q', \alpha q/q')$.

Remark. For holomorphic Bergman spaces in the polydisc, a duality theorem for more general weights is established by Shamoyan [18].

2. Proof of Theorem 1

First notice that as it follows from Aleksandrov's paper [1, Theorem 2.11], the space $h(p, q, \alpha)$ is trivial if at least one of the entries α_j is less than -1 (or clearly $\alpha_j < 0$ for $1 \leqslant$

$p \leqslant \infty$). The most of the inclusions in Theorem 1 are known for functions holomorphic in the unit disk (see [8]). For n-harmonic functions u, some difficulties appear because of non-n-subharmonicity of $|u|^p$ $(0 < p < 1)$. Without loss of generality and to simplify notation, we may assume that $n = 2$.

Proof of (iii). We begin by proving the case $q_0 = \infty$ and show that

$$h(p,q,\alpha) \subset h(p,\infty,\alpha). \tag{2.1}$$

Note that for $p \geqslant 1$ or holomorphic functions, the inclusion (2.1) is elementary in view of monotonicity of integral means $M_p(u;r)$ in each radial variable r_j. For $0 < p < 1$, take any function $u \in h(p,q,\alpha)$ and fix a point $z = (z_1, z_2) = (r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) \in U^2$. For the point z and bidisk $B_z = B_{z_1} \times B_{z_2}$, where $B_{z_j} = \{\zeta \in \mathbb{C}\colon |\zeta_j - z_j| < (1-r_j)/2\}$, $j = 1, 2$, write Hardy–Littlewood inequality on subharmonic behavior of $|u|^p$:

$$\bigl|u(z_1,z_2)\bigr|^p \leqslant \frac{C_p}{(1-r_1)^2(1-r_2)^2} \iint\limits_{B_{z_1}\times B_{z_2}} \bigl|u(\zeta_1,\zeta_2)\bigr|^p \, dm_2(\zeta_1)\, dm_2(\zeta_2). \tag{2.2}$$

If $\zeta = (\zeta_1, \zeta_2) \in B_z$, then $\rho_j' < |\zeta_j| = \rho_j < \rho_j''$, $j = 1, 2$, where

$$\rho_j' = \max\left\{0, \frac{3r_j - 1}{2}\right\}, \qquad \rho_j'' = \frac{1 + r_j}{2}.$$

Hence

$$\frac{1}{2}(1-r_j) < 1 - |\zeta_j| < \frac{3}{2}(1-r_j), \quad j = 1, 2. \tag{2.3}$$

From (2.2), (2.3), and a simple inequality $|1 - \zeta_j \bar{z}_j| < 3(1 - |\zeta_j|)$, $|z_j| < 1$, $\zeta_j \in B_{z_j}$, we obtain

$$\bigl|u\bigl(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}\bigr)\bigr|^p \leqslant C_p \int\limits_{B_{z_1}} \int\limits_{B_{z_2}} \bigl|u(\zeta_1,\zeta_2)\bigr|^p \frac{dm_2(\zeta_1)}{|1-\zeta_1\bar{z}_1|^2} \frac{dm_2(\zeta_2)}{|1-\zeta_2\bar{z}_2|^2}. \tag{2.4}$$

Next, we extend the domain of integration in (2.4) to the rings $\rho_j' < |\zeta_j| < \rho_j''$ $(j = 1, 2)$ and integrate over the torus T^2:

$$M_p^p(u; r_1, r_2) \leqslant \frac{C_p}{(1-r_1)(1-r_2)} \int_{\rho_1'}^{\rho_1''} \int_{\rho_2'}^{\rho_2''} M_p^p(u; \rho_1, \rho_2)\, d\rho_1\, d\rho_2.$$

If $0 < p < q < \infty$, then by Hölder inequality with indices q/p and $q/(q-p)$,

$$\prod_{j=1}^{2} (1-r_j)^{\alpha_j q} M_p^q(u;r) \leqslant C \int_{\rho_1'}^{\rho_1''} \int_{\rho_2'}^{\rho_2''} \prod_{j=1}^{2} (1-\rho_j)^{\alpha_j q - 1} M_p^q(u;\rho)\, d\rho_1\, d\rho_2, \tag{2.5}$$

and therefore $(1-r)^\alpha M_p(u;r) \leqslant C(p,q,\alpha)\|u\|_{p,q,\alpha}$, $r \in I^2$.

If $0 < q \leqslant p \leqslant \infty$, then write (2.4) with q instead of p, and apply Minkowski's inequality with exponent $p/q \geqslant 1$:

$$M_p^q(u; r_1, r_2) \leqslant \frac{C_q}{(1-r_1)(1-r_2)} \int_{\rho_1'}^{\rho_1''} \int_{\rho_2'}^{\rho_2''} M_p^q(u; \rho_1, \rho_2)\, d\rho_1\, d\rho_2.$$

Then (2.5) follows. Thus, in both cases the inclusion (2.1) is continuous. The general case in (iii) reduces to (2.1). Indeed, let $0 < q < q_0 < \infty$. Then by (2.1)

$$\|u\|_{p,q_0,\alpha}^{q_0} \leqslant \|u\|_{p,\infty,\alpha}^{q_0-q} \|u\|_{p,q,\alpha}^{q} \leqslant C\|u\|_{p,q,\alpha}^{q_0-q} \|u\|_{p,q,\alpha}^{q} = C\|u\|_{p,q,\alpha}^{q_0}.$$

Thus, the inclusion (iii) is proved. □

The inequality (2.5) implies also the assertion (x) of Theorem 1.

Proof of (iv). Actually the condition $\alpha_{0j} + 1/p_0 \geqslant \alpha_j + 1/p$ is not only sufficient for the inclusion $h(p,q,\alpha) \subset h(p_0,q,\alpha_0)$, but is necessary as well. That follows from the next lemma.

Lemma 1. *Let* $0 < p \leqslant p_0 \leqslant \infty$, $\alpha_j > 0$. *Then* $h(p,q,\alpha) \subset h(p_0,q,\alpha_0)$ *if and only if* $\alpha_{0j} + 1/p_0 \geqslant \alpha_j + 1/p$ $(1 \leqslant j \leqslant n)$.

Proof. Let $\alpha_{0j} + 1/p_0 = \alpha_j + 1/p$ $(1 \leqslant j \leqslant 2)$, and first show the case $p_0 = \infty$

$$h(p,q,\alpha) \subset h(\infty, q, \alpha + 1/p). \tag{2.6}$$

If $0 < p < q < \infty$, then it follows from (2.2)–(2.3) that for any $r = (r_1, r_2) \in I^2$,

$$M_\infty^q(u; r) \leqslant \frac{C(p,q)}{\prod_{j=1}^2 (1-r_j)^{q/p}} \left(\int_{\rho_1'}^{\rho_1''} \int_{\rho_2'}^{\rho_2''} M_p^p(u; \rho) \frac{d\rho_1\, d\rho_2}{\prod_{j=1}^2 (1-\rho_j)} \right)^{q/p}.$$

Applying Hölder inequality with indices q/p, $1/(1-p/q)$ and integrating over I^2, and then interchanging the order of integrating, we get $\|u\|_{\infty,q,\alpha+1/p}^q \leqslant C(p,q,\alpha)\|u\|_{p,q,\alpha}^q$.

If $0 < q \leqslant p \leqslant \infty$, then we use the inequality (2.2) with q instead of p. The same method as above leads to (2.6). Thus, the inclusion (iv) is proved for both $p_0 = \infty$ and $p_0 = p$. For all values $p_0 \in [p, \infty]$ the inclusion (iv) follows from a version of Riesz–Thorin interpolation theorem for quasi-normed spaces [2,13].

Conversely, suppose there exists an index $j \in [1, n]$, say $j = 1$, such that $\alpha_{01} + 1/p_0 < \alpha_1 + 1/p$. For an arbitrary point $a = (a_1, \ldots, a_n) \in U^n$ and a multi-index $\gamma = (\gamma_1, \ldots, \gamma_n)$, $\gamma_j > \max\{\alpha_{0j} + 1/p_0, \alpha_j + 1/p\}$, $1 \leqslant j \leqslant n$, define the function $f_{\gamma,a}(z) = 1/(1-\bar{a}z)^\gamma$. A simple estimation shows that

$$\frac{\|f_{\gamma,a}\|_{p_0,q,\alpha_0}}{\|f_{\gamma,a}\|_{p,q,\alpha}} \approx (1-|a|)^{(\alpha_0+1/p_0)-(\alpha+1/p)}.$$

Letting $|a_1| \to 1$, we get a contradiction with $h(p,q,\alpha) \subset h(p_0,q,\alpha_0)$. The proof of Lemma 1 and the inclusion (iv) is complete. □

Proof of (v), (vi) can be obtain by (iii) and the inclusion $h(p,q,\alpha) \subset h(\infty,\infty,\alpha+1/p)$ which is contained in (iv).

The inclusion (vii) is due to Frazier [9], and **the inclusion (viii)** follows from [9] in view of n-subharmonicity of $|u|^p$, $p \geqslant 1$.

Finally, **the inclusion (ix)** is a combination of (vi), (iv). Indeed, for any $\alpha = (\alpha_1,\ldots,\alpha_n)$, $\alpha_j > 0$, we have $h^p \subset h(p,q,\alpha) \subset h(p_0,q,\alpha+1/p-1/p_0)$.

Remark. For holomorphic Bergman spaces on the unit ball of $\mathbb{C}^n$, Lemma 1 can be found in [15]. The inclusion (ix) for $n=1$, $0<p<1$, $p_0=q=1$ is proved by Duren and Shields [7]. They showed also that the limiting inclusion $h^p \subset h(1,1,1/p-1)$ is false.

3. Proof of Theorems 2 and 3

For a function $f(z) = f(rw)$, $r \in I^n$, $w \in T^n$, given on U^n, we shall use Riemann–Liouville integro-differential operator $D^\alpha \equiv D_r^\alpha$ with respect to variable r:

$$D^{-\alpha} f(z) = \frac{r^\alpha}{\Gamma(\alpha)} \int\limits_{I^n} (1-\eta)^{\alpha-1} f(\eta z)\,d\eta, \qquad D^\alpha f(z) = \left(\frac{\partial}{\partial r}\right)^m D^{-(m-\alpha)} f(z),$$

where

$$\left(\frac{\partial}{\partial r}\right)^m = \left(\frac{\partial}{\partial r_1}\right)^{m_1} \cdots \left(\frac{\partial}{\partial r_n}\right)^{m_n},$$

$m=(m_1,\ldots,m_n) \in \mathbb{Z}_+^n$, $\alpha=(\alpha_1,\ldots,\alpha_n)$, $\alpha_j>0$, $m_j-1<\alpha_j\leqslant m_j$ $(1\leqslant j\leqslant n)$. It is clear that for any $\alpha=(\alpha_1,\ldots,\alpha_n)$, $\alpha_j \geqslant 0$, $D^{\pm\alpha} f = D_{r_1}^{\pm\alpha_1} D_{r_2}^{\pm\alpha_2} \cdots D_{r_n}^{\pm\alpha_n} f$, where $D_{r_j}^{\alpha_j}$ means the same operator acting in direction r_j only. Denote

$$\mathcal{D}^{-\alpha} f(rw) = r^{-\alpha} D^{-\alpha} f(rw), \qquad \mathcal{D}^\alpha f(rw) = D^\alpha\bigl\{r^\alpha f(rw)\bigr\}.$$

It is easily seen that if f is n-harmonic, then so are $\mathcal{D}^\alpha f$ and $\mathcal{D}^{-\alpha} f$, and for them the following inversion formulas hold:

$$\mathcal{D}^\alpha \mathcal{D}^{-\alpha} f(z) = \mathcal{D}^{-\alpha}\mathcal{D}^\alpha f(z) = f(z). \tag{3.1}$$

For n-harmonic functions the operators $\mathcal{D}^{-\alpha}$ and $\mathcal{D}^\alpha$ have an equivalent definition. Every function $f \in h(U^n)$ has a series expansion $f(z) = \sum_{k\in\mathbb{Z}^n} a_k r^{|k|} e^{ik\cdot\theta}$, where $r^{|k|} = r_1^{|k_1|}\cdots r_n^{|k_n|}$, $k\cdot\theta = k_1\theta_1+\cdots+k_n\theta_n$, and we can present

$$\mathcal{D}^{-\alpha} f(z) = \sum_{k\in\mathbb{Z}^n} \prod_{j=1}^n \frac{\Gamma(|k_j|+1)}{\Gamma(|k_j|+1+\alpha_j)} a_k r^{|k|} e^{ik\cdot\theta},$$

$$\mathcal{D}^{\alpha} f(z) = \sum_{k\in\mathbb{Z}^n} \prod_{j=1}^n \frac{\Gamma(|k_j|+1+\alpha_j)}{\Gamma(|k_j|+1)} a_k r^{|k|} e^{ik\cdot\theta}.$$

We shall consider kernels P_α and conjugate kernels Q_α of Poisson type for the unit disk $\mathbb{D}$ (see [6, Chapter IX]):

$$P_\alpha(z) = \Gamma(\alpha+1)\left[\operatorname{Re}\frac{2}{(1-z)^{\alpha+1}} - 1\right], \quad z \in \mathbb{D},\ \alpha \geqslant 0,$$

$$Q_\alpha(z) = \Gamma(\alpha+1)\operatorname{Im}\frac{2}{(1-z)^{\alpha+1}}, \quad z \in \mathbb{D},\ \alpha \geqslant 0.$$

It is easily seen that $P_0(z) = P(z)$ and $Q_0(z) = Q(z)$ are the usual Poisson and conjugate Poisson kernels. Denote also $P_\alpha(z,\zeta) = P_\alpha(z\bar{\zeta})$, $Q_\alpha(z,\zeta) = Q_\alpha(z\bar{\zeta})$, $z, \zeta \in \mathbb{D}$. For the polydisc U^n the kernels P_α and Q_α are defined as $P_\alpha(z,\zeta) = \prod_{j=1}^n P_{\alpha_j}(z_j,\zeta_j)$, $Q_\alpha(z,\zeta) = \prod_{j=1}^n Q_{\alpha_j}(z_j,\zeta_j)$, where $\alpha = (\alpha_1,\dots,\alpha_n)$, $\alpha_j \geqslant 0$, $z, \zeta \in U^n$. Kernels P_α and Q_α are n-harmonic both in z and in ζ. Clearly, $P_\alpha(z,\zeta) = P_\alpha(\zeta, z) = P_\alpha(\bar{z},\bar{\zeta})$. Before passing to the proofs of Theorems 2 and 3, we give two auxiliary lemmas which are proved by direct computation and estimation.

Lemma 2. *For any $z, \zeta \in U^n$, $\alpha_j \geqslant 0$ $(1 \leqslant j \leqslant n)$,*

$$P_0(z,\zeta) = \mathcal{D}^{-\alpha} P_\alpha(z,\zeta), \qquad Q_0(z,\zeta) = \mathcal{D}^{-\alpha} Q_\alpha(z,\zeta),$$
$$P_\alpha(z,\zeta) = \mathcal{D}^{\alpha} P_0(z,\zeta), \qquad Q_\alpha(z,\zeta) = \mathcal{D}^{\alpha} Q_0(z,\zeta).$$

This lemma enables us to extend the definition of the kernels P_α and Q_α to negative $\alpha_j < 0$. We assume that $P_\alpha = \mathcal{D}^\alpha P_0$ and $Q_\alpha = \mathcal{D}^\alpha Q_0$ for any $\alpha_j \in \mathbb{R}$.

Lemma 3. *Let $\alpha_j \geqslant 0$, $1/(1+\alpha_j) < p \leqslant \infty$ $(1 \leqslant j \leqslant n)$ and let K be either of the kernels P_α and Q_α. Then*

$$\left|K(z,\zeta)\right| \leqslant C(\alpha,n)\prod_{j=1}^n \frac{1}{|1-\bar{\zeta}_j z_j|^{\alpha_j+1}}, \quad z,\zeta \in U^n,$$

$$M_p(K;r) \leqslant C(\alpha,n,p)\prod_{j=1}^n \frac{1}{(1-r_j)^{\alpha_j+1-1/p}}, \quad r \in I^n.$$

Proof of Theorem 2. Let first $p = q = 1$, $\beta_j = \alpha_j$ $(1 \leqslant j \leqslant n)$ and let $u(z) \in h(1,1,\alpha)$. Applying the inversion formula (3.1) and then changing the variables, we get

$$\begin{aligned}
u(z) &= \frac{1}{\Gamma(\alpha)}\int_{I^n}\left(1-\rho^2\right)^{\alpha-1}\mathcal{D}_r^\alpha u\left(\rho^2 z\right)2^n \rho\, d\rho \\
&= \frac{1}{\Gamma(\alpha)}\int_{I^n}\left(1-\rho^2\right)^{\alpha-1}\mathcal{D}_r^\alpha\left\{\int_{T^n} P(z,\rho\eta)u(\rho\eta)\,dm_n(\eta)\right\}2^n\rho\,d\rho \\
&= \frac{1}{\Gamma(\alpha)}\int_{I^n}\int_{T^n}\left(1-\rho^2\right)^{\alpha-1}\mathcal{D}_r^\alpha P(z,\rho\eta)u(\rho\eta)2^n\rho\,d\rho\,dm_n(\eta),
\end{aligned}$$

where the integral converges absolutely by Lemma 3. For other admissible p, q, β the proof follows from the inclusion $h(p,q,\alpha) \subset h(1,1,\beta)$ (see Theorem 1). □

The representation (1.1) suggests corresponding integral operators $T_{\beta,\lambda}$, $\widetilde{T}_{\beta,\lambda}$, $S_{\beta,\lambda}$, $\widetilde{S}_{\beta,\lambda}$ (see Section 1). It is natural to ask whether they are bounded in $L(p,q,\alpha)$. For proving Theorem 3, we need a higher-dimensional version of Hardy's inequality.

Lemma 4. *If $g(t) \geqslant 0$, $t \in I^n$, $1 \leqslant q < \infty$, $\beta_j < -1 < \alpha_j$ $(1 \leqslant j \leqslant n)$, then*

$$\int_{I^n} (1-r)^{\alpha} \left(\int_0^{r_1} \cdots \int_0^{r_n} g(t)\,dt \right)^q dr \leqslant C \int_{I^n} (1-r)^{\alpha+q} g^q(r)\,dr, \tag{3.2}$$

$$\int_{I^n} x^{\beta} \left(\int_0^{x_1} \cdots \int_0^{x_n} g(t)\,dt \right)^q dx \leqslant C \int_{I^n} x^{\beta+q} g^q(x)\,dx, \tag{3.3}$$

where the constants C may depend only on α, β, q, n.

The inequalities (3.2) and (3.3) are proved by iteration of those in one variable.

Proof of Theorem 3. (i) It is enough to prove the boundedness of $S_{\beta,\lambda}$. Instead of applying the standard Schur test (see, e.g., [12]), we use Lemma 4. Let $f(z) \in L(p,q,\alpha)$, $1 \leqslant q < \infty$. By Minkowski's inequality and Lemma 3,

$$\begin{aligned} M_p(S_{\beta,\lambda} f; r) &\leqslant \frac{(1-r^2)^{\lambda}}{\Gamma(\beta+\lambda)} \int_{U^n} \left(1-|\zeta|^2\right)^{\beta-1} \left|P_{\beta+\lambda}(r,\zeta)\right| M_p(f;\rho)\,dm_{2n}(\zeta) \\ &\leqslant C(1-r)^{\lambda} \left(\int_0^{r_1} \cdots \int_0^{r_n} + \int_{r_1}^{1} \cdots \int_{r_n}^{1} \right) M_p(f;\rho) \frac{(1-\rho)^{\beta-1}}{(1-r\rho)^{\beta+\lambda}}\,d\rho. \end{aligned}$$

By the triangle inequality and Lemma 4,

$$\begin{aligned} \|S_{\beta,\lambda} f\|_{p,q,\alpha} &= \left\|(1-r)^{\alpha} M_p(S_{\beta,\lambda} f; r)\right\|_{L^q(dr/(1-r))} \\ &\leqslant C \left\| (1-r)^{\alpha+\lambda} \int_0^{r_1} \cdots \int_0^{r_n} M_p(f;\rho) \frac{d\rho}{(1-\rho)^{1+\lambda}} \right\|_{L^q(dr/(1-r))} \\ &\quad + C \left\| (1-r)^{\alpha-\beta} \int_{r_1}^{1} \cdots \int_{r_n}^{1} (1-\rho)^{\beta-1} M_p(f;\rho)\,d\rho \right\|_{L^q(dr/(1-r))} \\ &\leqslant C \left[\int_{I^n} (1-r)^{(\alpha+\lambda)q-1} \left(\frac{1-r}{(1-r)^{1+\lambda}} M_p(f;r) \right)^q dr \right]^{1/q} \\ &\quad + C \left[\int_{I^n} x^{(\alpha-\beta)q-1} \left(\int_0^{x_1} \cdots \int_0^{x_n} \eta^{\beta-1} M_p(f;1-\eta)\,d\eta \right)^q dx \right]^{1/q} \\ &\leqslant C\|f\|_{p,q,\alpha}. \end{aligned}$$

The case $q=\infty$ can be proved easier. Of course, the boundedness of the operator $T_{\beta,0}$ $(\lambda_j=0)$ means that $T_{\beta,0}$ is a n-harmonic projection of $L(p,q,\alpha)$ onto $h(p,q,\alpha)$. This completes the proof of part (i) of Theorem 3.

We now turn to the proof of part (ii) of Theorem 3. It suffices to prove that boundedness of $T_{\beta,\lambda}$ on $L(p,q,\alpha)$ implies $\beta_j>\alpha_j>-\lambda_j$. Let $T_{\beta,\lambda}$ be a bounded operator on $L(p,q,\alpha)$, i.e. $\|T_{\beta,\lambda}\|_{p,q,\alpha}\leqslant C\|f\|_{p,q,\alpha}$ $\forall f\in L(p,q,\alpha)$, where the constant C is independent of f. Taking a multi-index $N=(N_1,\dots,N_n)$ with the components N_j large enough ($N_j+\alpha_j>0$, $N_j+\beta_j>0$) such that $f_N(z)=(1-|z|^2)^N\in L(p,q,\alpha)$, we deduce $T_{\beta,\lambda}(f_N)(z)=C(\beta,\lambda,N)(1-|z|^2)^{\lambda}$. Hence

$$+\infty>\left\|T_{\beta,\lambda}(f_N)\right\|^q_{p,q,\alpha}\geqslant C(\beta,\lambda,q,N,n)\int\limits_{I^n}(1-r)^{(\alpha+\lambda)q-1}\,dr,$$

so the inequality $\alpha_j+\lambda_j>0$ holds for all $j\in[1,n]$. Further, let $T^*_{\beta,\lambda}$ be the adjoint operator of $T_{\beta,\lambda}$. It is given explicitly by

$$T^*_{\beta,\lambda}(f)(z)=\frac{(1-|z|^2)^{\beta-\alpha q}}{\Gamma(\beta+\lambda)}\int\limits_{U^n}\left(1-|\zeta|^2\right)^{\lambda+\alpha q-1}P_{\beta+\lambda}(z,\zeta)f(\zeta)\,dm_{2n}(\zeta).$$

According to [2, p. 304], the dual space $L^*(p,q,\alpha)$ of $L(p,q,\alpha)$ can be identified with $L(p',q',\alpha q/q')$. The boundedness of $T_{\beta,\lambda}$ on $L(p,q,\alpha)$ is equivalent to that of $T^*_{\beta,\lambda}$ on $L^*(p,q,\alpha)\cong L(p',q',\alpha q/q')$, i.e.

$$\left\|T^*_\beta f\right\|_{p',q',\alpha q/q'}\leqslant C\|f\|_{p',q',\alpha q/q'}\quad\forall f\in L(p',q',\alpha q/q'). \tag{3.4}$$

We now distinguish two cases.

Case $1<q<\infty$. The action of $T^*_{\beta,\lambda}$ on a function $f_N(z)=(1-|z|^2)^N\in L(p',q',\alpha q/q')$, with the components N_j large enough, gives $T^*_{\beta,\lambda}(f_N)(z)=C(1-|z|^2)^{\beta-\alpha q}$. Hence

$$+\infty>\left\|T^*_{\beta,\lambda}(f_N)\right\|^{q'}_{p',q',\alpha q/q'}\geqslant C\int\limits_{I^n}(1-r)^{q'(\beta-\alpha q)+\alpha q-1}\,dr,$$

where the constant C depends only on $\alpha,\beta,\lambda,q,N,n$. So it follows that $q'(\beta_j-\alpha_j q)+\alpha_j q>0$, or equivalently, $\beta_j>\alpha_j$ for all j, $1\leqslant j\leqslant n$.

Case $q=1$. Then the inequality (3.4) turns to

$$\left\|T^*_{\beta,\lambda}f\right\|_{p',\infty,0}\leqslant C\|f\|_{p',\infty,0}\quad\forall f\in L(p',\infty,0). \tag{3.5}$$

The action of $T^*_{\beta,\lambda}$ on the function $f_N(z)$ gives

$$+\infty>\left\|T^*_{\beta,\lambda}(f_N)\right\|_{p',\infty,0}=C\sup_{r\in I^n}\left(1-r^2\right)^{\beta-\alpha}.$$

Hence $\beta_j-\alpha_j\geqslant 0$ for all $1\leqslant j\leqslant n$. It remains to show that for $q=1$, $1\leqslant p<\infty$ the equality $\beta_j=\alpha_j$ holds for no index j. Assume $\beta_1=\alpha_1$, say. Then, given parameter $a\in U^n$, we consider functions $g_a(z)=|P_{\beta+\lambda}(a,z)|/P_{\beta+\lambda}(a,z)$, where $\beta_j+\lambda_j\geqslant\alpha_j+$

$\lambda_j > 0$. Clearly, $|g_a(z)| \equiv 1$ and $g_a(z) \in L(p', \infty, 0)$ for each $a \in U^n$. Then by (3.5), $T^*_{\beta,\lambda}(g_a) \in L(p', \infty, 0)$. For $z = a$ we have

$$T^*_{\beta,\lambda}(g_z)(z) = C \prod_{j=2}^{n} \bigl(1 - |z_j|^2\bigr)^{\beta_j - \alpha_j} \prod_{j=1}^{n} \int_{\mathbb{D}} \bigl(1 - |\zeta_j|^2\bigr)^{\lambda_j + \alpha_j - 1} \bigl|P_{\beta_j + \lambda_j}(z_j, \zeta_j)\bigr|\, dm_2(\zeta_j).$$

In view of boundedness of harmonic conjugation in spaces $h(1, 1, \alpha)$ (see, e.g., [5,8]),

$$T^*_{\beta,\lambda}(g_z)(z) \geqslant C(\alpha, \beta, \lambda, n) \log \frac{1}{1 - |z_1|}.$$

Letting here $|z_1| \to 1$, we obtain a contradiction with the boundedness of $T^*_{\beta,\lambda}$ on $L(p', \infty, 0)$. Thus, the equality $\beta_j = \alpha_j$ holds for no index j. This completes the proof of Theorem 3.

4. Proofs of Theorems 4–6

We now briefly sketch proofs of Theorems 4–6.

Proof of Theorem 4. Given a function $\varphi(z) \in L(p, q, -\alpha)$, $1 \leqslant p, q \leqslant \infty$, $\alpha_j \geqslant 0$ $(1 \leqslant j \leqslant n)$ we shall prove that $\|T_{\beta,0}(\varphi)\|_{h\Lambda^{p,q}_\alpha} \leqslant C\|\varphi\|_{p,q,-\alpha}$ for any $\beta = (\beta_1, \ldots, \beta_n)$, $\beta_j > 0$. Let $f(z) = T_{\beta,0}(\varphi)(z)$, then for any $\gamma_j > \alpha_j$ $(1 \leqslant j \leqslant n)$, the desired inequality can be written in the form $\|\mathcal{D}^\gamma f\|_{p,q,\gamma-\alpha} \leqslant C\|\varphi\|_{p,q,-\alpha}$. To prove these inequalities, we differentiate the equality $f(z) = T_{\beta,0}(\varphi)(z)$ by means of the operator $\mathcal{D}^\gamma$ and then, by analogy with the proof of Theorem 3(i), estimate using Minkowski's inequality, Lemmas 3 and 4.

To prove the surjectivity of (1.2), we need several additional lemmas.

Lemma 5. *The inclusions* $h\Lambda^{p,q}_\alpha \subset h(1, 1, \beta)$ *and* $h\Lambda^{p,q}_\alpha \subset h\Lambda^{1,1}_0$ *are continuous for any* $1 \leqslant p \leqslant \infty$, $0 < q \leqslant \infty$, $\alpha_j > 0$, $\beta_j > 0$.

Proof. Lemma follows from the inclusions (ii), (vi) of Theorem 1 and the definition of Besov spaces. □

Lemma 6. *Suppose that* $u(z)$ *is in* $h\Lambda^{p,q}_\alpha$ *for* $1 \leqslant p \leqslant \infty$, $0 < q \leqslant \infty$, $\alpha_j > 0$, $1 \leqslant j \leqslant n$. *Then for any* $\delta = (\delta_1, \ldots, \delta_n)$, $\delta_j > 0$, $1 \leqslant j \leqslant n$, *the function* u *can be represented in the form* $u(z) = \Phi_\delta(u)(z)$, $z \in U^n$.

Proof. By the second inclusion of the previous lemma, $\mathcal{D}^\delta u(z) \in h(1, 1, \delta)$ for any $\delta_j > 0$. It is enough to represent $\mathcal{D}^\delta u(z) = T_{\delta,0}(\mathcal{D}^\delta u)(z)$ by Theorem 2, and then to integrate by means of $\mathcal{D}^{-\delta}$ using (3.1). □

Lemma 7. *For* $\beta_j > 0$, $\gamma_j \geqslant 0$ $(1 \leqslant j \leqslant n)$, $k \in \mathbb{Z}^n$, $z = rw$, $r \in I^n$, $w \in T^n$, *the following identities hold*:

$$T_{\beta,\gamma}\{r^{|k|}w^k\} = \left(1-|z|^2\right)^\gamma \frac{\Gamma(\beta)\Gamma(|k|+1+\beta+\gamma)}{\Gamma(\beta+\gamma)\Gamma(|k|+1+\beta)} r^{|k|}w^k, \tag{4.1}$$

$$T_{\beta,0}\{\left(1-|z|^2\right)^\gamma r^{|k|}w^k\} = \frac{\Gamma(\beta+\gamma)\Gamma(|k|+1+\beta)}{\Gamma(\beta)\Gamma(|k|+1+\beta+\gamma)} r^{|k|}w^k. \tag{4.2}$$

Proof. Substituting the series expansion of the kernel $P_{\beta+\gamma} = \mathcal{D}^{\beta+\gamma}P$ into the left-hand side of (4.1), we get the identity (4.1). Formula (4.2) can be proved in the same way. □

Lemma 8. *For any $1 \leqslant p \leqslant \infty$, $0 < q \leqslant \infty$, $\alpha_j \geqslant 0$, $\beta_j > 0$, $\gamma_j \geqslant 0$, $1 \leqslant j \leqslant n$, the operator $T_{\beta,0} \circ T_{\beta,\gamma}$ is the identity map on $h\Lambda_\alpha^{p,q}$.*

Proof. If $f(z) = \sum_{k\in\mathbb{Z}^n} a_k r^{|k|}w^k$ is in $h\Lambda_\alpha^{p,q}$, then in view of (4.1), the operator $T_{\beta,\gamma}$ can be written in the form

$$T_{\beta,\gamma}(f)(z) = \frac{(1-|z|^2)^\gamma}{\Gamma(\beta+\gamma)} \sum_{k\in\mathbb{Z}^n} a_k \frac{\Gamma(\beta)\Gamma(|k|+1+\beta+\gamma)}{\Gamma(|k|+1+\beta)} r^{|k|}w^k. \tag{4.3}$$

It follows from (4.2) that $T_{\beta,0}(T_{\beta,\gamma}f(z)) = f(z)$. □

Lemma 9. *For any $1 \leqslant p \leqslant \infty$, $0 < q \leqslant \infty$, $\alpha_j \geqslant 0$, $\beta_j > 0$, $m \in \mathbb{Z}_+^n$, $m_j > \alpha_j$, $1 \leqslant j \leqslant n$, the operator $T_{\beta,m}$ maps $h\Lambda_\alpha^{p,q}$ boundedly into $L(p,q,-\alpha)$.*

Proof. By the representation (4.3), we have

$$\begin{aligned}\frac{T_{\beta,m}f(z)}{(1-|z|^2)^m} &= C\sum_{k\in\mathbb{Z}^n}\left(|k|^m + C|k_1|^{m_1-1}|k_2|^{m_2}\cdots|k_n|^{m_n} + \cdots + C\right)a_k r^{|k|}w^k \\ &= C\left[\mathcal{D}^m f(z) + C_{\beta,m}\mathcal{D}^{(m_1-1,m_2,\ldots,m_n)}f(z) + \cdots + C_{\beta,m}f(z)\right].\end{aligned}$$

Thus, the condition $(1-r)^m\mathcal{D}^m f(z) \in L(p,q,-\alpha)$ implies $T_{\beta,m}f(z) \in L(p,q,-\alpha)$.

Finally, the operator $T_{\beta,0}: L(p,q,-\alpha) \to h\Lambda_\alpha^{p,q}$ is onto by Lemmas 8 and 9. □

Proof of Theorem 5. Given a function (not n-harmonic) $f(z) \in \Lambda_\alpha^{p,q}$, we need to prove $\|\mathcal{D}^\gamma \Phi_{\widetilde{\alpha}}(f)\|_{p,q,\gamma-\alpha} \leqslant C\|\mathcal{D}^{\widetilde{\alpha}}f\|_{p,q,\widetilde{\alpha}-\alpha}$, where $\widetilde{\alpha} \in \mathbb{Z}_+^n$, $\alpha_j < \widetilde{\alpha}_j \leqslant \alpha_j + 1$, $\gamma_j > \alpha_j$ $(1 \leqslant j \leqslant n)$. The rest of the proof runs as before in Theorem 3(i). □

Proof of Theorem 6. A function $g \in h(p',q',\alpha q/q')$ induces a bounded linear functional on $h(p,q,\alpha)$, $F(f) = \langle f,g\rangle$ $\forall f \in h(p,q,\alpha)$. Indeed, applying Hölder's inequality twice, we get $|F(f)| \leqslant C(\alpha,q,n)\|f\|_{p,q,\alpha}\|g\|_{p',q',\alpha q/q'}$. Conversely, let $F \in (h(p,q,\alpha))^*$. Then by the Hahn–Banach extension theorem, F can be extended to a bounded linear functional (still denoted by F) on $L(p,q,\alpha)$ without increasing its norm. By the duality theory of mixed norm spaces, see [2, p. 304], $(L(p,q,\alpha))^* \cong L(p',q',\alpha q/q')$. There exists a function g_0 in $L(p',q',\alpha q/q')$ such that $F(f) = \langle f,g_0\rangle$ and $\|F\| = \|g_0\|_{p',q',\alpha q/q'}$. Writing, by Theorem 2, $f = T_{\alpha q,0}f$, we have $F(f) = \langle T_{\alpha q,0}(f), g_0\rangle = \langle f, T_{\alpha q,0}(g_0)\rangle$. Taking $g = T_{\alpha q,0}(g_0)$ and using Theorem 3, we conclude that g is in $L(p',q',\alpha q/q')$ and $F(f) = \langle f,g\rangle$ $\forall f \in h(p,q,\alpha)$, such that $\|g\|_{p',q',\alpha q/q'} \leqslant C\|g_0\|_{p',q',\alpha q/q'} \leqslant C\|F\|$. This completes the proof of Theorem 6. □

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