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COMPLEX ANALYSIS

Hardy–Stein Identities and Littlewood–Paley Inequalities in Polydisc

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Hardy–Stein Identities and Littlewood–Paley Inequalities in Polydisc*

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Abstract—The paper generalizes the well-known Littlewood-Paley inequality and Hardy-Stein identity. As an application, some area inequalities and quasinorm representations in the space $A^p_\omega$ over the polydisc are obtained.

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Key words: Hardy–Stein identity; Littlewood–Paley inequalities.

1. The classical Littlewood-Paley inequality (see [1]) holds for any function $f(z)$, holomorphic in the unit disc $D \subset \mathbb{C}$ that belongs to the Hardy class $H^p(D), 2 \leq p < \infty$:

$$\int_D |f'(z)|^p (1-|z|)^{p-1} dm_2(z) \leq C_p \|f\|_{H^p}^p,$$  \hspace{1cm} (0.1)

where $m_2$ is the Lebesgue measure in $D$. There are numerous generalizations of the inequality (0.1), see [2]-[13], among them the following one due to Luecking [4].

**Theorem A** If $0 < p, s < \infty$, then the inequality

$$\int_D |f(z)|^p |f'(z)|^s (1-|z|)^{s-1} dm_2(z) \leq C(p, s) \|f\|_{H^p}^p$$  \hspace{1cm} (0.2)

holds for all $f \in H^p(D)$ if and only if $2 \leq s < p + 2$.

Our purpose is to obtain some versions of (0.2) for $0 < s < 2$ in the unit polydisc of the $n$-dimensional complex space $\mathbb{C}^n$. The methods used in Luecking’s proof of (0.2) such as distribution of zeros, Blaschke product etc. can hardly be applied in the multidimensional case.

2. By $D^n = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j| < 1, 1 \leq j \leq n\}$ we denote the unit polydisc with the distinguished boundary $T^n = \{\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n : |\xi_j| = 1, 1 \leq j \leq n\}$ ($T^n$ is an $n$-dimensional torus), by $H(D^n)$ the set of all holomorphic functions in $D^n$. For a measurable in $D^n$ function $f(z) = f(r\xi)$ we will consider the integral means

$$M_p(f, r) = \left[\frac{1}{(2\pi)^n} \int_{T^n} |f(r\xi)|^p dm_n(\xi)\right]^{1/p}, \quad r = (r_1, \ldots, r_n) \in I^n,$$

where $0 < p < \infty$, and $m_n$ is the Lebesgue measure on $T^n$. We note that the ordinary Hardy space over $D^n$ is the set of those functions holomorphic in $D^n$, for which

$$\|f\|_{H^p} = \sup_{r \in I^n} M_p(f, r) < +\infty.$$
For a radial weight function \( \omega(r) = \prod_{j=1}^{n} \omega_j(r_j) \), by \( L^p_\omega \) \((0 < p < \infty)\) we denote the space of those functions \( f(z) \) measurable in \( \mathbb{D}^n \), for which the following (quasi-)norm is finite:

\[
\|f\|_{L^p_\omega} = \left( C_\omega \int_{\mathbb{D}^n} |f(z)|^p \prod_{j=1}^{n} \omega_j(|z_j|) \, dm_{2n}(z) \right)^{1/p},
\]

where \( dm_{2n}(z) = r \, dr \, dm_n(\xi) \) is the Lebesgue measure in \( \mathbb{D}^n \) and the normalizing constant \( C_\omega \) is chosen to provide \( \|1\|_{L^p_\omega} = 1 \). By \( A^\alpha_\omega = H(\mathbb{D}^n) \cap L^p_\omega \) we denote the holomorphic subspace of \( L^p_\omega \), and if \( \omega_j(r_j) = (1 - r_j)^{\alpha_j} \) \((\alpha_j > -1, 1 \leq j \leq n)\), then we write \( L^p_\omega \) and \( A^\alpha_\omega \) for the spaces \( L^p_\omega \) and \( A^\alpha_\omega \) respectively.

The following notation is standard for \( \mathbb{C}^n \):

\[
r_\zeta = (r_1 \zeta_1, \ldots, r_n \zeta_n), \quad dr = dr_1 \cdots dr_n,
\]

\[
(1 - |\zeta|^2)^\alpha = \prod_{j=1}^{n} (1 - |\zeta_j|^2)^{\alpha_j}, \quad \zeta^\alpha = \prod_{j=1}^{n} \zeta_j^{\alpha_j}, \quad \alpha q + 1 = (\alpha_1 q + 1, \ldots, \alpha_n q + 1)
\]

for \( \zeta \in \mathbb{C}^n \), \( r \in \mathbb{I}^n \), \( q \in \mathbb{R} \) and \( \alpha = (\alpha_1, \ldots, \alpha_n) \) for multiindices.

By \( \mathbb{Z}^n_+ \) we denote the set of all multiindices \( k = (k_1, \ldots, k_n) \) with nonnegative integers \( k_j \in \mathbb{Z}_+ \). We write \( A \lesssim B \) if there is a constant \( c > 0 \) such that \( A \leq cB \). The notation \( A \asymp B \) will mean that \( A \lesssim B \) and \( B \lesssim A \). For any \( p, 1 \leq p \leq \infty \) we put \( p'] = p/(p - 1) \) (the conjugate index).

Further, for any function \( f \in H(\mathbb{D}^n) \) with expansion \( f(z) = \sum_{k \in \mathbb{Z}^n_+} a_k r^k \xi^k \), where \( r = r_\xi \), \( r \in \mathbb{I}^n \) and \( \xi \in \mathbb{T}^n \), we introduce the radial fractional integro-differential operator of order \( \alpha = (\alpha_1, \ldots, \alpha_n) \), \( \alpha_j \in \mathbb{R} \), as follows:

\[
\mathcal{D}^\alpha f(z) = \mathcal{D}^\alpha_{r_1} \mathcal{D}^\alpha_{r_2} \cdots \mathcal{D}^\alpha_{r_n} f, \quad \text{where } \mathcal{D}^\alpha_{r_j} \text{ means the same operator acting only in the variable } r_j.
\]

3. Theorem 1 which follows yields a family of inequalities which generalize the Littlewood-Paley type inequalities the unit disc proved by Yamashita [2] and Luecking [4].

**Theorem 1.** If \( 0 < \alpha < s < 2 \) and \( s < p \), then for any \( \lambda > (p - s)/\alpha \)

\[
\int_{\mathbb{D}^n} |f(z)|^{p-s} |f(z)|^s \left( \prod_{j=1}^{n} (1 - |z_j|^2)^{\alpha_j} \right) dm_{2n}(z) \lesssim \|f\|_{L^\lambda_\omega}^{p-s} \|f\|_{L^s_\omega}^s.
\]

(0.3)

**Remark 1.** For \( p = 2 \), the formal limits as \( s \to 2^+ \) and \( \alpha \to +0 \), reduce (0.3) to the classical Littlewood-Paley inequality (0.1) in the polydisc for \( p = 2 \).

Without loss of generality, we can simplify the further proofs by considering only the case \( n = 2 \).

**Proof of Theorem 1:** We start with some additional notation and statements. For a fixed \( \delta > 1 \), let \( \Gamma_\delta(\xi) = \{ z \in \mathbb{D} : |1 - \xi z| \leq \delta (1 - |z|) \} \) denote the admissible approach domain with the vertex at \( \xi \in \mathbb{T} \).

Further, for any arc \( I \subset \mathbb{T} \) of the length \( |I| \) we define the Carleson square \( [I] = \{ z \in \mathbb{D} : |1 - z| \leq 1/|I| \} \) over \( I \). Following [14], we consider the functions

\[
A_p(f)(\xi) = \left( \int_{\Gamma_\delta(\xi)} |f(z)|^p \, dm_2(z) \right)^{1/p}, \quad p < \infty,
\]

\[
C_p(f)(\xi) = \sup_{I \ni \xi} \left( \frac{1}{|I|} \int_{[I]} |f(z)|^p \, dm_2(z) \right)^{1/p}, \quad p < \infty, \quad \xi \in \mathbb{T}.
\]

(0.4)
Besides, we note that for any measurable in \( \mathbb{D} \) functions \( f(z) \) and \( g(z) \)
\[
\int_{\mathbb{D}} \frac{|f(z)|^q |g(z)|}{1 - |z|} \, dm(z) \lesssim \int_{\mathbb{T}} A_p(f)(\xi) C_{\nu}(g)(\xi) \, dm(\xi), \quad 1 < p < \infty,
\]  
(0.5)
where \( dm(\xi) = dm_1(\xi) \) is the Lebesgue measure on \( \mathbb{T} \) (see [14], [7]). In addition, for any numbers \( 0 < q < \infty, \alpha > 0, \beta > 0 \) and any function \( f(z) \) measurable in \( \mathbb{D} \) (see [7])
\[
\left\| C_q(|f(z)|(1 - |z|)^\alpha) \right\|_{L^\infty}^q \lesssim \sup_{w \in \mathbb{D}} (1 - |w|)^\beta \int_{\mathbb{D}} \frac{|f(z)|^q (1 - |z|)^{\alpha q - 1}}{|1 - wz|^{1 + \beta}} \, dm(z).
\]  
(0.6)
We consider a version of Luzin area integral
\[
S(f)(\xi) = \left( \int_{\Gamma_0(\xi)} |D^1 f(z)|^2 \, dm(z) \right)^{1/2}, \quad \xi \in \mathbb{T}, \quad \delta > 1,
\]
and recall Luzin’s well-known inequality (see [1])
\[
\left\| S(f) \right\|_{L_p(\mathbb{T})} \lesssim \| f \|_{H^p}, \quad 0 < p < \infty.
\]  
(0.7)
By \( L \) we denote the left–hand side integral in (0.3) and write it in the form
\[
L = \int_{\mathbb{D}} (1 - |z_2|)^{s-1} \left[ \int_{\mathbb{D}} |f(z)|^{p-s} |D^1 f(z)|^s |1 - |z_1||^{s-1} \, dm(z_1) \right] \, dm(z_2).
\]  
(0.8)
Further, by \( J \) we denote the inner integral in (0.8) and fixing any \( \alpha, 0 < \alpha < s \), we use (0.5) to estimate \( J \):
\[
J = \int_{\mathbb{D}} |D^1 f(z)|^s |1 - |z_1||^{s-\alpha} \cdot |f(z)|^{p-s} |1 - |z_1||^{\alpha} \, dm(z_1) \]
\[
\lesssim \int_{\mathbb{T}} A_2/s \left( |D^1 f(z)|^s |1 - |z_1||^{s-\alpha} \right)(\xi_1)
\times C_{(2/s)^r} \left( |f(z)|^{p-s} |1 - |z_1||^{\alpha} \right)(\xi_1) \, dm(\xi_1)
\leq \left\| C_{(2/s)^r} \left( |f(z)|^{p-s} |1 - |z_1||^{\alpha} \right) \right\|_{L^\infty}
\times \int_{\mathbb{T}} A_2/s \left( |D^1 f(z)|^s |1 - |z_1||^{s-\alpha} \right)(\xi_1) \, dm(\xi_1).
\]  
(0.9)
Separately estimating the last integral
\[
J_1 = \int_{\mathbb{T}} \left[ \int_{\Gamma_0(\xi_1)} |D^1 f(z)|^2 (1 - |z_1|)^{-2\alpha/s} \, dm(z_1) \right]^{s/2} \, dm(\xi_1),
\]
according to (0.7) and the fractional differentiation rule [6] (pp. 179 and 186) we get
\[
J_1 \lesssim \int_{\mathbb{T}} \left[ \int_{\Gamma_0(\xi_1)} |D^{\alpha/s}_1 D^1 f(z)|^2 \, dm(z_1) \right]^{s/2} \, dm(\xi_1) \lesssim \| D^{\alpha/s}_2 D^{\alpha/s}_1 f \|_{H^s_{12}},
\]  
(0.10)
where \( H^s_{12} \) means the Hardy class by the variable \( z_1 \). Then, uniting the inequalities (0.8) - (0.10) and applying Fatou’s lemma and (0.5) we get
\[
L \lesssim \liminf_{r \to 1^{-1}} \int_{\mathbb{D}} (1 - |z_2|)^{s-1} \left\| C_{(2/s)^r} \left( |f(z)|^{p-s} |1 - |z_1||^{\alpha} \right) \right\|_{L^\infty}
\times \left\| D^1 f D^{\alpha/s}_1 f \right\| \, dm(z_1) \, dm(z_2)
\lesssim \left\| C_{(2/s)^r} \left( |f(z)|^{p-s} |1 - |z_1||^{\alpha} \right) \right\|_{L^\infty} \left( 1 - |z_2| \right)^{\alpha} \, dm(z_2).
\]
Theorem 2. Our next theorem establishes some weighted analogs of the Hardy-Stein identity (see \cite{15}) and the equality £

\[ J \equiv J_2 J_3. \]

To estimate \( J_2 \) and \( J_3 \), we again use fractional differentiation rule and Fatou’s lemma along with (0.7) and the equality \( D_r^\alpha D_r^\beta = D_r^\gamma D_r^\tau \):

\[
J_3 = \lim_{r_1 \to 1} \int_\mathbb{T} \left[ \int_{\mathbb{T}} D_r^\alpha D_r^\beta f^2 (1 - |z|)^{-2\alpha/\beta} dm_2(z_2) \right]^{s/2} dm(\xi_2) dm(\xi_1)
\]

\[
\leq \lim_{r_1 \to 1} \int_\mathbb{T} \left[ \int_{\mathbb{T}} D_r^\alpha D_r^\beta f^2 dm_2(z_2) \right]^{s/2} dm(\xi_2) dm(\xi_1)
\]

\[
\leq \lim_{r_1 \to 1} \int_\mathbb{T} \left\| D_r^\alpha f \right\|_{H^s_{r_2}}^s dm(\xi_1) = \left\| D_r^\alpha f \right\|_{H^s}.\]

To estimate \( J_2 \) with \( \beta > 0 \) great enough, we observe that by (0.6) the inner norm in \( J_2 \) can be estimated as follows:

\[
\left\| C_{2/(2-s)} \left( |f(z)|^{p-s}(1 - |z_1|)^\alpha \right) \right\|_{L^\infty}^{2/(2-s)} \leq \sup_{w \in \mathbb{D}} (1 - |w|)^\beta \int_\mathbb{D} |f(z_1, z_2)|^{2(p-s)/(2-s)} \frac{(1 - |z_1|)^{2\alpha/(2-s)-1}}{|1 - \overline{w}z_1|^{\beta+1}} dm_2(z_1)
\]

\[
\leq \left\| f \right\|_{H^s_{r_1}}^{2(p-s)/(2-s)} \sup_{w \in \mathbb{D}} (1 - |w|)^\beta \int_\mathbb{D} \frac{(1 - |z_1|)^{2\alpha/(2-s)-1}}{|1 - \overline{w}z_1|^{\beta+1}} dm_2(z_1) \leq \left\| f \right\|_{H^s_{r_1}}^{2(p-s)/(2-s)}.\]

Consequently,

\[
J_2 \leq \left\| C_{2/(2-s)} \left( \left\| f \right\|_{H^s_{r_1}}^{p-s}(1 - |z_2|)^\alpha \right) \right\|_{L^\infty} \leq \left[ \sup_{w \in \mathbb{D}} (1 - |w|)^\beta \int_\mathbb{D} \left\| f \right\|_{H^s_{r_1}}^{2(p-s)/(2-s)} \right. \]

\[
\times \left. \frac{(1 - |z_2|)^{2\alpha/(2-s)-1}}{|1 - \overline{w}z_2|^{\beta+1}} dm_2(z_2) \right]^{(2-s)/2} \leq \left\| f \right\|_{H^s_{r_1}}^{p-s} \sup_{w \in \mathbb{D}} (1 - |w|)^\beta \int_\mathbb{D} \frac{(1 - |z_2|)^{2\alpha/(2-s)-1}}{|1 - \overline{w}z_2|^{\beta+1}} dm_2(z_2) \right]^{(2-s)/2} \leq \left\| f \right\|_{H^s_{r_1}}^{p-s}.\]

Thus, \( L \leq \left\| f \right\|_{H^s_{r_1}}^{p-s} \left\| D_r^\alpha f \right\|_{H^s}^s \), for any \( \lambda > (p - s)/\alpha \), and the proof is complete.

4. Our next theorem establishes some weighted analogs of the Hardy–Stein identity (see \cite{15}) and the corresponding Littlewood–Paley inequalities.

**Theorem 2.** If \( f(z) \in H(\mathbb{D}^n) \), \( 0 < p < \infty \), and \( \omega_j(r_j) \in C^1[0, 1], j = 1, \ldots, n \), is such that

\[
\omega_j(r_j) \frac{\partial}{\partial r_j} M_p^2(f, r) = o(1) \quad as \quad r_j \to 1^-,
\]

then:

\[ J \equiv J_2 J_3. \]
a) The following identity is true:

\[
\int_{\mathbb{D}^n} \prod_{j=1}^{n} \omega_j(r_j) \cdot f^\#(z) \, dm_{2n}(z) = (-1)^n \int_{\mathbb{D}^n} \prod_{j=1}^{n} \omega_j(r_j) \frac{\partial^n}{\partial r_1 \cdots \partial r_n} |f(z)|^p \, dm_{2n}(z),
\]

(0.12)

where \( f^\#(z) = \Delta z_1 \Delta z_2 \cdots \Delta z_n |f(z)|^p \) and \( \Delta z_j \) is the ordinary Laplacian in the variable \( z_j \).

The conditions (11) can be omitted in case of standard weight functions \( \omega_j(r_j) = (1 - r_j)^{\alpha_j} \) \((\alpha_j > 0)\).

b) For \( n = 1 \), (0.12) is refined by the identity

\[
\int_{\mathbb{D}} (1 - |z|)^\alpha f^\#(z) \, dm_2(z) = \alpha \int_{\mathbb{D}} (1 - |z|)^{\alpha-1} \frac{\partial}{\partial r} |f(z)|^p \, dm_2(z),
\]

(0.13)

which is true for any \( p > 0 \) and \( \alpha > 0 \), if at least one side integral exists. Here

\[
f^\#(z) = \Delta |f(z)|^p = p^2 |f(z)|^{p-2} |f'(z)|^2.
\]

(0.14)

c) The integrals

\[
A(f; p, \alpha) = \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 (1 - |z|)^\alpha \, dm_2(z),
\]

\[
B(f; p, \alpha) = \int_{\mathbb{D}} |f(z)|^{p-1} |f'(z)| (1 - |z|)^{\alpha-1} \, dm_2(z)
\]

satisfy the inequalities: if \( p > 0 \) and \( \alpha > 0 \), then

\[
A(f; p, \alpha) \leq \frac{\alpha}{p} B(f; p, \alpha),
\]

(0.15)

where \( \alpha/p \) is the best constant; if \( p > 0 \) and \( \alpha > 1 \), then there exists some constant \( C_{\alpha,p} > 0 \) such that

\[
B(f; p, \alpha) \leq C_{\alpha,p} A(f; p, \alpha).
\]

(0.16)

Remark 2. For \( p = 2 \), the inequalities (0.15) and (0.16) are proved in [10]. Their analogs for integer values of \( p \) \((p \geq 2)\), the unit disc \( \mathbb{D} \) and the ball in \( \mathbb{C}^n \) are proved in [11], [12] by some other method.

Below we will need the following generalization of the Hardy–Stein identity (see [15]) in the polydisc.

Lemma 1. If \( f(z) \in H(\mathbb{D}^n) \), \( 0 < p < \infty \), then for any \( r = (r_1, \ldots, r_n) \in \mathbb{R}^n \)

\[
\prod_{j=1}^{n} r_j \frac{\partial^n}{\partial r_1 \cdots \partial r_n} M_p^\#(f, r) = \frac{1}{(2\pi)^n} \int_{|z_1| < r_1} \cdots \int_{|z_n| < r_n} f^\#(z) \, dm_{2n}(z).
\]

(0.17)

Proof: by successive application of Green’s formula.

Remark Due to (0.14), for \( n = 1 \) the formula (0.17) coincides with the well-known Hardy–Stein identity [15].

Proof of Theorem 2: An application of Lemma 1 leads to the identity

\[
r_1 r_2 \int_{\mathbb{T}^2} \Delta z_1 \Delta z_2 |f(z_1, z_2)|^p \, dm(\xi_1) \, dm(\xi_2) =
\]

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constant $r$

This equality shows that if $H_\kappa$.

Hence, it remains only to verify that

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By the Cauchy criterion,

Hardy-Stein identity

If the right-hand side integral in

To prove the inequality (0.16), we observe that by the Cauchy-Schwarz inequality

Turning to the proof of the inequality (0.15), observe that the example $f(z) = z$ shows that the constant $\alpha/p$ is the best possible. Further, the identity (0.13) can be written in the form

$$A(f; p, \alpha) = \frac{\alpha}{p} \int_0^1 \int_{-\pi}^\pi |f(re^{i\theta})|^p \left(\frac{\partial}{\partial r} |f(re^{i\theta})| \right) (1 - r)^{\alpha - 1} rdrd\theta.$$ 

It remains to check that $|\frac{\partial}{\partial r} |f(re^{i\theta})|| \leq |f'(re^{i\theta})|$ since $|f(re^{i\theta})| - |f(\rho e^{i\theta})| \leq |f(re^{i\theta}) - f(\rho e^{i\theta})|$. 

To prove the inequality (0.16), we observe that by the Cauchy-Schwarz inequality

$$B(f; p, \alpha) \leq \sqrt{A(f; p, \alpha)} \left( \int_\mathbb{D} |f(z)|^p (1 - |z|)^{\alpha - 2} dm_2(z) \right)^{1/2}.$$ 

Hence, it remains only to verify that

$$\int_\mathbb{D} |f(z)|^p (1 - |z|)^{\alpha - 2} dm_2(z) \leq A(f; p, \alpha), \quad p > 0, \quad \alpha > 1.$$ 

Integration by parts yields

$$\frac{p^2}{2\pi \alpha} A(f; p, \alpha) = \left. \int_0^1 (1 - r)^{\alpha - 1} r \frac{\partial}{\partial r} M_p^\alpha(f, r) dr \right|_{r=0}^{r=1} + \frac{1}{2\pi} \int_0^1 \int_{-\pi}^\pi |f(re^{i\theta})|^p (1 - r)^{\alpha - 2} dm_2(z) drd\theta.$$ 

This equality shows that if $A(f; p, \alpha)$ exists, then $f(z) \in A^p_{\alpha - 2}(\mathbb{D})$. Consequently, $\lim_{r \to 1} (1 - r)^{\alpha - 1} M_p^\alpha(f, r) = 0$, and hence

$$A(f; p, \alpha) = \frac{\alpha}{p^2} \int_0^1 \int_{-\pi}^\pi |f(re^{i\theta})|^p (1 - r)^{\alpha - 2} dm_2(z) drd\theta \geq \frac{\alpha(\alpha - 1)}{2p^2} \int_0^1 \int_{-\pi}^\pi |f(re^{i\theta})|^p (1 - r)^{\alpha - 2} drd\theta \geq$$

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Proof of Theorem 3:

\[ \geq C(\alpha, p) \int_{\mathbb{D}} |f(z)|^p (1 - |z|)^{\alpha - 2} dm_2(z), \]

which leads to the desired result. Thus, the statement c) is proved. The statement a) is proved similarly, by using (0.18).

5. Our last theorem gives a characterization of the weighted spaces \( \mathcal{A}^p_\alpha \) with general weights in the bidisc and some representations of (quasi-)norms in \( \mathcal{A}^p_\alpha \) by means of Luecking type integrals (0.2).

**Theorem 3.** Let \( 0 < p < \infty, f(z) \in H(\mathbb{D}^2), \omega_j(r_j) \in L^1(0, 1), \omega_j(r_j) > 0, j = 1, 2. \) Then the following representations are valid:

\[
\|f\|_{\mathcal{A}^p_\alpha(\mathbb{D}^2)}^p \leq \|f(0, 0)\|^p + \int_{\mathbb{D}^2} \Delta_{z_1} \Delta_{z_2} |f(z_1, z_2)|^p + \Delta_{z_2} |f(z_1, 0)|^p + \\
\delta \sum_{j=1}^2 h_{\omega_j}(|z_j|) dm_4(z), \tag{20}
\]

\[
\|f\|^p_{\mathcal{A}^p_\alpha(\mathbb{D}^2)} + |f(0, 0)|^p = \|f(\cdot, 0)\|^p_{\mathcal{A}^p_0(\mathbb{D})} + \|f(0, \cdot)\|^p_{\mathcal{A}^p_0(\mathbb{D})} + \\
C_\alpha \int_{\mathbb{D}^2} f^#(z_1, z_2) \prod_{j=1}^2 h_{\omega_j}(|z_j|) dm_4(z), \tag{21}
\]

where \( \mathcal{A}^p_\alpha \) are one-dimensional spaces by the variables \( z_j \), while \( h_{\omega_j} \) are the weight functions

\[ h_{\omega_j}(|z_j|) = \int_{|z_j|}^1 \left( \int_{\rho_j}^1 \omega_j(x) dx \right) d\rho_j. \]

In particular, \( f \in \mathcal{A}^p_\alpha(\mathbb{D}^2) \) if and only if \( f^# \in L^1_{\alpha+2}(\mathbb{D}^2) \) (\( \alpha > -1 \)).

**Remark** If \( n = 1 \) and \( \omega(r) = (1 - r)^\alpha (\alpha > -1) \), then by (0.14) the limit of the relation (20) as \( \alpha \to -1 \) coincides with Yamashita’s characterization [2] of the Hardy classes \( H^p(\mathbb{D}) \). Some analogs of (20) and (21) for the ball in \( \mathbb{C}^n \) can be found in [3], [5] and [8].

Proof of Theorem 3: The integrated Hardy-Stein identity (see Lemma 1)

\[ M^p_\alpha(f, r_1, r_2) + |f(0, 0)|^p = M^p_\alpha(f, r_1, r_2) + M^p_\alpha(f, r_1, 0) + \\
\delta \sum_{j=1}^2 h_{\omega_j}(|z_j|) dm_4(z) \]

can be integrated with the weight \( (2\pi)^2 C_\omega C_{\omega_1} \omega_1(r_1) \omega_2(r_2) r_1 r_2 dr_1 dr_2 \). This gives

\[ \|f\|^p_{\mathcal{A}^p_\alpha} + |f(0, 0)|^p = J_1 + J_2 + J_3, \]

where

\[ J_1 = \|f(0, 0)\|^p_{\mathcal{A}^p_\alpha(\mathbb{D})} + \\
\delta \sum_{j=1}^2 h_{\omega_j}(|z_j|) dm_4(z) \]

\[ J_2 = \|f(0, z_2)\|^p_{\mathcal{A}^p_\alpha(\mathbb{D})} + \\
\delta \sum_{j=1}^2 h_{\omega_j}(|z_j|) dm_4(z) \]

\[ J_3 = C_\omega C_{\omega_2} \int_{\mathbb{D}} \left( \int_{|z_1| < \rho_1} \int_{|z_2| < \rho_2} f^#(z_1, z_2) dm_4(z) \right) d\rho_1 d\rho_2 \]
\[ \omega(r)rdr \times \omega(r)rdr = (2\pi)^2 C_{\omega_1} C_{\omega_3} \int_0^1 \int_0^1 M_1(f^#(z_1, z_2), r_1, r_2) h_{\omega_1}(r_1) h_{\omega_2}(r_2) r_1 r_2 dr_1 dr_2, \]

and the proof is complete.

REFERENCES