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Littlewood–Paley inequalities in uniformly convex and uniformly smooth Banach spaces

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Abstract

It is proved that the inequality $\delta_X(\varepsilon) \geqslant c\varepsilon^p$, $p \geqslant 2$, where δ_X is the modulus of convexity of X, is sufficient and necessary for the inequality

$$\int_{\mathbb{D}} \|\nabla f(z)\|^p (1-|z|)^{p-1}\, dA(z) \leqslant C\big(\|f\|_{p,X}^p - \|f(0)\|^p\big),$$

where f is an X-valued harmonic function belonging to the Hardy space $h^p(X)$. The reverse inequality $(1 < p \leqslant 2)$ holds if and only if $\rho_X(\tau) \leqslant C\tau^p$, where ρ_X is the modulus of smoothness of X.

Keywords: Uniform p-convexity; Uniform p-smoothness; Littlewood–Paley inequality; Hardy space; Banach space; Harmonic function

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1. Introduction

Let $h(\mathbb{D})$ denote the class of all real-valued functions harmonic in the open unit disk $\mathbb{D}$ of the complex plane. The harmonic Hardy class h^p, $1 \leqslant p \leqslant \infty$, consists of those $f \in h(\mathbb{D})$ for which $\|f\|_p := \sup_{0<r<1} M_p(r, f) < \infty$, where

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f\left(re^{i\theta}\right)\right|^p \, d\theta \right)^{1/p}.$$

In this note we are concerned with an extension to vector-valued functions of the following theorem of Littlewood and Paley.

Theorem 1.1. *If $p \geqslant 2$, $f \in h^p$, then there is a constant C_p such that*

$$\int_{\mathbb{D}} \left|\nabla f(z)\right|^p \left(1 - |z|\right)^{p-1} \, dA(z) \leqslant C_p\left(\|f\|_p^p - \left|f(0)\right|^p\right). \tag{1.1}$$

If $1 \leqslant p \leqslant 2$, $f \in h(\mathbb{D})$, and

$$\int_{\mathbb{D}} \left|\nabla f(z)\right|^p \left(1 - |z|\right)^{p-1} \, dA(z) < \infty,$$

then $f \in h^p$, and there is a constant $c_p > 0$ such that

$$c_p\left(\|f\|_p^p - \left|f(0)\right|^p\right) \leqslant \int_{\mathbb{D}} \left|\nabla f(z)\right|^p \left(1 - |z|\right)^{p-1} \, dA(z). \tag{1.2}$$

Here dA denotes the Lebesgue measure in the plane normalized so that the measure of $\mathbb{D}$ is equal to 1.

Usually this result is stated in the less precise form in which (1.1), respectively (1.2), are replaced by

$$\int_{\mathbb{D}} \left|\nabla f(z)\right|^p \left(1 - |z|\right)^{p-1} \, dA(z) + \left|f(0)\right|^p \leqslant C_p \|f\|_p^p, \tag{1.3}$$

respectively

$$c_p \|f\|_p^p \leqslant \left|f(0)\right|^p + \int_{\mathbb{D}} \left|\nabla f(z)\right|^p \left(1 - |z|\right)^{p-1} \, dA(z). \tag{1.4}$$

However (1.1) shows also how much the subharmonicity of $|f|$ is improved when passing to $|f|^p$, $p \geqslant 2$. To see this it suffices to rewrite (1.1) as

$$\left|f(0)\right|^p \leqslant \|f\|_p^p - (1/C) \int_{\mathbb{D}} \left|\nabla f(z)\right|^p \left(1 - |z|\right)^{p-1} \, dA(z). \tag{1.5}$$

Inequalities (1.1) and (1.2) are special cases of a Littlewood–Paley theorem for subharmonic functions [12]. It should be noted however that (1.1) also follows from Luecking's proof of (1.3) (see [10]). For a very short and elementary proof of (1.1) we refer the reader to [11].

That Theorem 1.1 holds for harmonic functions with values in a Hilbert space was observed in [12]. Assuming that (1.1) (with the obvious change of notation) holds in a Banach space X we take $f(\xi + i\eta) = x + \eta y$, where $x, y \in X$ are fixed, to get

$$\|y\|^p \int_{\mathbb{D}} \left(1 - |z|\right)^{p-1} dA(z) \leqslant C_p \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left\| x + (\cos\theta) y \right\|^p d\theta - \|x\|^p \right).$$

This implies

$$\max_{\theta \in [-\pi,\pi]} \left\| x + (\cos\theta) y \right\|^p \geqslant \|x\|^p + b_p \|y\|^p \quad (b_p \text{ a positive constant})$$

whence, by the convexity of the norm,

$$\max\left\{ \|x + y\|^p, \|x - y\|^p \right\} \geqslant \|x\|^p + b_p \|y\|^p. \tag{1.6}$$

From this we can conclude that the modulus of convexity of X (see below) satisfies $\delta_X(\varepsilon) \geqslant c\varepsilon^p$ ($c = \text{const} > 0$). In other words, the validity of (1.1) implies that X is "p-uniformly convex;" we will prove that the converse holds as well (see Theorem 6.1).

In view of the duality between uniform convexity and uniform smoothness it is natural to expect that the validity of (1.2) (for a fixed $p \in (1, 2]$) is equivalent to p-uniform smoothness of X. This does hold (see Theorem 7.2 and Remark 8.1) but for the proof that (1.2) implies uniform smoothness we need a nontrivial result of Arregui and Blasco on duality for vector-valued Bergman spaces (see [1, Theorem 3.9] and [2, Theorem 3.6]).

It is worthwhile to note that the "vector-valued" approach provides a unified treatment of the Littlewood–Paley inequalities and some inequalities of Hardy and Littlewood, see (8.4) and (8.5).

2. Moduli of convexity and smoothness

Let X be a real Banach space. The modulus of convexity and the modulus of smoothness of X are defined by

$$\delta_X(\varepsilon) = \inf\left\{ 1 - \left\| \frac{x + y}{2} \right\| : x, y \in X,\ \|x\| = 1,\ \|y\| = 1,\ \|x - y\| \geqslant \varepsilon \right\}, \quad 0 < \varepsilon < 2,$$

$$\rho_X(\tau) = \sup\left\{ \frac{1}{2}\left(\|x + \tau y\| + \|x - \tau y\| \right) : x, y \in X,\ \|x\| = 1,\ \|y\| = 1 \right\}, \quad \tau > 0.$$

The space X is said to be uniformly convex (respectively uniformly smooth) if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0, 2)$ (respectively $\rho_X(\tau)/\tau \to 0$ as $\tau \to 0$).

If there is a constant $p \geqslant 2$ such that $\delta_X(\varepsilon) \geqslant c\varepsilon^p$, where c is a positive constant, then X is said to be p-uniformly convex. It is well known and easy to see that X is p-uniformly convex if

$$\frac{\|x + y\|^p + \|x - y\|^p}{2} \geqslant \|x\|^p + \lambda \|y\|^p, \quad x, y \in X, \tag{2.1}$$

for some positive constant λ. It follows from a result of Figiel and Pisier [6] that the converse is true as well.

For a fixed $p \geqslant 2$ we denote by $I_p(X)$ the largest $\lambda \geqslant 0$ satisfying (2.1). Therefore X is p-uniformly convex if and only if $I_p(X) > 0$.

A Banach space X is said to be q-uniformly smooth ($1 < q \leqslant 2$) if $\rho_X(\tau) \leqslant C\tau^q$ ($0 < \tau < 1$), where C is a positive constant.

It is known that

X is uniformly convex (respectively smooth) if and only if the dual space X^* is uniformly smooth (respectively convex).

The quantitative form of these facts is expressed by Lindenstrauss' formulae:

$$\rho_{X^*}(\tau) = \sup\left\{\frac{\tau\varepsilon}{2} - \delta_X(\varepsilon)\colon\ 0 < \varepsilon < 2\right\}, \tag{2.2}$$

$$\rho_X(\tau) = \sup\left\{\frac{\tau\varepsilon}{2} - \delta_{X^*}(\varepsilon)\colon\ 0 < \varepsilon < 2\right\} \tag{2.3}$$

(see [8,9]).

It is an old result of Milman (see [9, Proposition 1.e.3]) that

every uniformly convex (and hence every uniformly smooth) space is reflexive.

The most important examples are L^p-spaces. Hanner [7] computed the precise values of the moduli of convexity of L^p. It follows from his result that

the space L^s is p-uniformly convex, $p = \max\{s, 2\}$, and is q-uniformly smooth, $q = \min\{s, 2\}$.

3. Harmonic functions

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$. A function $f : \mathbb{D} \to X$ is said to be harmonic if

$$\frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 f}{\partial \eta^2} = 0, \quad \text{for all } \xi + i\eta \in \mathbb{D}. \tag{3.1}$$

This is equivalent to the requirement that for every $\Phi \in X^*$ the real-valued function $\Phi f = \Phi \circ f$ is harmonic in $\mathbb{D}$.

Let $h(\mathbb{D}, X)$ denote the class of all X-valued functions harmonic in $\mathbb{D}$. If $f \in h(\mathbb{D}, X)$, then there exists a unique sequence $\{a_n\}_{n=-\infty}^{\infty} \subset X$ such that

$$f\left(re^{i\theta}\right) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\theta}, \quad re^{i\theta} \in \mathbb{D},$$

the series being uniformly convergent in every compact subset of $\mathbb{D}$.

The gradient of f is defined as

$$(\nabla f)(a) = \big((D_1 f)(a), (D_2 f)(a)\big),$$

where

$$(D_1 f)(\xi + i\eta) = \frac{\partial f}{\partial \xi}, \qquad (D_2 f)(\xi + i\eta) = \frac{\partial f}{\partial \eta}.$$

The gradient at a fixed point $a \in \mathbb{D}$ can be treated as a linear operator from $\mathbb{R}^2$ into X, namely

$$\mathbb{R}^2 \ni (t_1, t_2) \mapsto (D_1 f)(a)t_1 + (D_2 f)(a)t_2 \ \in X.$$

It is easy to see that the norm of this operator is equivalent to

$$\left(\|D_1 f(a)\|_X^2 + \|D_2 f(a)\|_X^2\right)^{1/2} =: \|\nabla f(a)\|_X.$$

Lemma 3.1. *Let $u:\mathbb{D}\mapsto\mathbb{R}$ be a harmonic function such that $|u|\leqslant 1$ in $\mathbb{D}$. Then*

$$\bigl|\nabla u(0)\bigr|\leqslant 2\bigl(1-\bigl|u(0)\bigr|\bigr).$$

Proof. If $u(0)\geqslant 0$, then the function $v=1-u$ is positive and hence $|\nabla v(0)|\leqslant 2v(0)$, i.e.

$$\bigl|\nabla u(0)\bigr|\leqslant 2\bigl(1-u(0)\bigr)=2\bigl(1-\bigl|u(0)\bigr|\bigr).$$

If $u(0)<0$, we consider the function $v=1+u$. The result follows. □

Lemma 3.2. *Let $f:\mathbb{D}\mapsto X$ be a bounded harmonic function, where X is p-uniformly convex ($p\geqslant 2$). Then*

$$\|f\|_{\infty,X}\geqslant\bigl(\bigl\|f(0)\bigr\|^p+c\bigl\|\nabla f(0)\bigr\|^p\bigr)^{1/p},$$

where $c>0$ is a constant independent of f.

Proof. Let $\|f\|_{\infty,X}=1$ and $\phi\in X^*$, $\|\phi\|=1$. By Lemma 3.1,

$$\bigl|\phi D_1 f(0)\bigr|=\bigl|D_1\phi f(0)\bigr|\leqslant 2\bigl(1-\bigl|\phi\bigl(f(0)\bigr)\bigr|\bigr),$$

whence

$$\bigl|\phi\bigl(f(0)\bigr)\bigr|+(1/2)\bigl|\phi D_1 f(0)\bigr|\leqslant 1$$

and hence

$$\bigl|\phi\bigl(f(0)\pm D_1 f(0)/2\bigr)\bigr|\leqslant 1.$$

Hence

$$\bigl\|f(0)\pm D_1 f(0)/2\bigr\|\leqslant 1.$$

It follows that

$$\begin{aligned}1&\geqslant\frac{\|f(0)+D_1 f(0)/2\|^p+\|f(0)-D_1 f(0)/2\|^p}{2}\\&\geqslant\bigl\|f(0)\bigr\|^p+I_p(X)\bigl\|D_1 f(0)/2\bigr\|^p.\end{aligned}$$

Using the analogous inequality for D_2 we get the desired result. □

4. Hardy spaces

If X is an arbitrary Banach space, then we define $h^p(X)$, $0<p<\infty$, to be the space of those X-valued functions f harmonic in the unit disk such that

$$\|f\|_{p,X}:=\sup_{0<r<1}M_p(r,f)<\infty,$$

where

$$M_p(r,f)=M_{p,X}(r,f)=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\bigl\|f\bigl(re^{i\theta}\bigr)\bigr\|^p\,d\theta\right)^{1/p}.$$

Lemma 4.1. *Let X be p-uniformly convex, $p \geqslant 2$. If $f \in h^p(X)$, then*

$$\|f\|_{p,X} \geqslant \left(\|f(0)\|^p + c\|\nabla f(0)\|^p\right)^{1/p}, \tag{4.1}$$

where $c > 0$ is a constant independent of f.

Proof. Assume that f is a harmonic polynomial and define the harmonic function $g : \mathbb{D} \mapsto L^p(\partial\mathbb{D}, X) =: Y$ by

$$g(z)\left(e^{i\theta}\right) = f\left(ze^{i\theta}\right) \quad (z \in \mathbb{D}). \tag{4.2}$$

Then g is bounded and

$$\|g(z)\|_Y = M_{p,X}(r, f), \qquad \|g(0)\|_Y = \|f(0)\|_X, \qquad \|g\|_{\infty,Y} = \|f\|_{p,X}.$$

On the other hand, we have

$$D_1 g(0)\left(e^{i\theta}\right) = D_1 f(0)\cos\theta + D_2 f(0)\sin\theta, \tag{4.3}$$

$$D_2 g(0)\left(e^{i\theta}\right) = D_2 f(0)\cos\theta - D_1 f(0)\sin\theta, \tag{4.4}$$

from which it follows that

$$\|D_1 g(0)\|_Y + \|D_2 g(0)\|_Y \leqslant 2\left(\|D_1 f(0)\|_X + \|D_2 f(0)\|_X\right).$$

The converse inequality can be stated similarly. Therefore $\|\nabla f(0)\|_X \asymp \|\nabla g(0)\|_Y$, i.e. $\|\nabla f(0)\|_X / K \leqslant \|\nabla g(0)\|_Y \leqslant K\|\nabla f(0)\|_X$, where K is an absolute constant. Now inequality (4.1) follows from Lemma 3.2 and the easily verified fact that Y is p-uniformly convex with $I_p(Y) = I_p(X)$. $\square$

5. Riesz measure

We continue to denote by X a p-uniformly convex space.

Lemma 5.1. *Let $f : \mathbb{D} \mapsto X$ be a harmonic function. Then*

$$\|\nabla f(z)\|^p \left(1 - |z|\right)^p \leqslant C \int\limits_{D_\varepsilon(z)} d\mu(w),$$

where $d\mu(w)$ is the Riesz measure of the subharmonic function $w \mapsto \|f(w)\|^p$ and

$$D_\varepsilon(z) = \left\{w\colon |w - z| < \varepsilon\left(1 - |z|\right)\right\}.$$

Here $\varepsilon > 0$ is sufficiently small and C depends on ε, not on f.

Proof. Let u be an arbitrary subharmonic function on $\mathbb{D}$. Then there holds the (Poisson–Jensen) formula

$$\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} u\left(re^{i\theta}\right) d\theta - u(0) = \frac{1}{2\pi}\int\limits_{|w|<r} \log\frac{r}{|w|}\, d\mu(w) \quad (0 < r < 1).$$

Putting $u(z) = \|f(z)\|^p$ and using the p-uniform convexity of X and Lemma 4.1 we get

$$\|\nabla f(0)\|^p \leqslant C \int\limits_{|w|<r} \log\frac{r}{|w|}\, d\mu(w).$$

Applying this to the function $w \mapsto u(z+w)$ we get

$$\|\nabla f(z)\|^p \leqslant C \int\limits_{|w-z|<r} \log\frac{r}{|w-z|}\,d\mu(w),$$

which holds for small r. Hence

$$\|\nabla f(z)\|^p \leqslant C \int\limits_{|w-z|<r} \frac{1}{|w-z|}\,d\mu(w).$$

We now integrate this inequality over the disc $|z|<r$ and use Fubini's theorem and the subharmonicity of $\|\nabla f(z)\|^p$ to get

$$\begin{aligned}
\|\nabla f(0)\|^p &\leqslant C \int\limits_{|z|<r} \|\nabla f(z)\|^p\,dA(z) \\
&\leqslant C \int\limits_{|z|<r} \left(\int\limits_{|w|<2r} \frac{1}{|w-z|}\,d\mu(w)\right) dA(z) \\
&= C \int\limits_{|w|<2r} \left(\int\limits_{|z|<r} \frac{1}{|w-z|}\,dA(z)\right) d\mu(w) \leqslant C_r \int\limits_{|w|<2r} d\mu(w).
\end{aligned}$$

Taking small $r = \varepsilon/2$ we obtain the desired result for $z = 0$. In the general case we can apply this special case to the function $w \mapsto f(z + \varepsilon(1-|z|)w)$. $\square$

6. The case of uniformly convex spaces

Theorem 6.1. *Let X be a p-uniformly convex Banach space ($p \geqslant 2$). If $f \in h^p(X)$, then there is a constant C such that*

$$\int\limits_{\mathbb{D}} \|\nabla f(z)\|^p \big(1-|z|\big)^{p-1}\,dA(z) \leqslant C\big(\|f\|_{p,X}^p - \|f(0)\|^p\big). \tag{6.1}$$

Proof. Assume first that f is harmonic in a neighborhood of the closed disk. By Lemma 5.1

$$\int\limits_{\mathbb{D}} \|\nabla f(z)\|^p \big(1-|z|\big)^{p-1}\,dA(z) \leqslant C \int\limits_{\mathbb{D}} d\mu(w) \int\limits_{E_\varepsilon(w)} \big(1-|z|\big)^{-1}\,dA(z),$$

where $E_\varepsilon(w) = \{z\colon\ |z-w| < \varepsilon(1-|z|)\}$. We now use the inequality

$$\frac{1}{1+\varepsilon}\big(1-|w|\big) < 1-|z| < \frac{1}{1-\varepsilon}\big(1-|w|\big), \quad w \in E_\varepsilon(z),$$

to get

$$\int\limits_{E_\varepsilon(w)} \big(1-|z|\big)^{-1}\,dA(z) \leqslant C_\varepsilon\big(1-|w|\big),$$

and hence

$$\int_{\mathbb{D}} \|\nabla f(z)\|^p (1-|z|)^{p-1}\, dA(z) \leqslant C \int_{\mathbb{D}} (1-|w|)\, d\mu(w)$$
$$\leqslant C \int_{\mathbb{D}} \log \frac{1}{|w|}\, d\mu(w)$$
$$= 2\pi C \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \|f(e^{i\theta})\|^p\, d\theta - \|f(0)\|^p \right),$$

where we have used the Poisson–Jensen formula. This proves the desired result in the special case. If $f \in h^p(X)$ is arbitrary we apply the result to the functions $f_r(z) = f(rz)$; we get

$$\int_{\mathbb{D}} r^p \|\nabla f(rz)\|^p (1-|z|)^{p-1}\, dA(z) \leqslant C_p \left(\|f\|_{p,X}^p - \|f(0)\|^p \right), \quad 0 < r < 1.$$

Now the proof can be completed by letting r tend to 1 and using Fatou's lemma. □

7. The case of uniformly smooth spaces

A consequence of the formula (2.3) is the following well known fact.

Lemma 7.1. *Let $1/p + 1/q = 1$. The space X is q-uniformly smooth if and only if its dual is p-uniformly convex.*

We need the following theorem of Phillips [13].

Phillips' theorem. *If X is reflexive, then the dual of $L^q(X, S, d\sigma)$ is isometrically isomorphic to $L^p(X^*, S, d\sigma)$, with the duality pairing given by*

$$(f, g) = \int_S \langle f(s), g(s) \rangle\, d\sigma(s), \quad f \in L^q(X, S, d\sigma),\ g \in L^p(X^*, S, d\sigma),$$

where $\langle\, ,\rangle$ stands for the duality pairing between X and X^*.

The following theorem can be deduced by duality.

Theorem 7.2. *Let X be a q-uniformly smooth Banach space $(1 < q \leqslant 2)$. If*

$$\int_{\mathbb{D}} \|\nabla f(z)\|^q (1-|z|)^{q-1}\, dA(z) < \infty,$$

then $f \in h^q(X)$, and there is a constant C such that

$$\|f\|_{q,X}^q - \|f(0)\|^q \leqslant C \int_{\mathbb{D}} \|\nabla f(z)\|^q (1-|z|)^{q-1}\, dA(z). \tag{7.1}$$

To deduce this theorem from Theorem 6.1 we need a lemma.

Let $\mathcal{T}(X)$ denote the class of all trigonometric polynomials with X-valued coefficients.

Lemma 7.3. *If X is separable and reflexive, then $\mathcal{T}(X)$ is dense in $L^p(\partial\mathbb{D}, X)$, for $1 < p < \infty$.*

Proof. It is well known that a necessary and sufficient condition that a subset $T \subset L$ of a normed linear space L be fundamental, i.e. $\overline{T} = L$, is that every linear bounded functional which vanishes on T vanishes identically.

So we have to prove that if a functional $\Lambda \in (L^p(\partial\mathbb{D}, X))^*$ satisfies the condition $\Lambda f = 0$ for all $f \in \mathcal{T}(X)$, then $\Lambda \equiv 0$.

By Phillips' theorem, there is a function $g \in L^q(X^*)$ such that

$$\Lambda f = \frac{1}{2\pi}\int_{-\pi}^{\pi} \langle f(e^{i\theta}), g(e^{i\theta})\rangle\, d\theta.$$

Assuming that $\Lambda = 0$ on $\mathcal{T}(X)$ we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}\langle x, g(e^{i\theta})\rangle\, d\theta = 0$$

for all $x \in X$ and all integers n. Hence for each $x \in X$ we have $\langle x, g(e^{i\theta})\rangle = 0$ for almost all $\theta \in [-\pi, \pi]$.

Since X is separable, we can choose a dense sequence $\{x_n\} \subset X$ such that $\langle x_n, g(e^{i\theta})\rangle = 0$ for $\theta \in E_n \subset [-\pi, \pi]$, where the measure of E_n is equal to 2π. Now let $E = \bigcap_n E_n$. Then, for each $\theta \in E$ we have $\langle x_n, g(e^{i\theta})\rangle = 0$ for all n, and therefore $\langle x, g(e^{i\theta})\rangle = 0$ for all $x \in X$. This implies $g(e^{i\theta}) = 0$ for all $\theta \in E$, and hence $g(e^{i\theta}) = 0$ a.e. because the measure of E is 2π. This completes the proof. □

Proof of Theorem 7.2. Let X be a q-uniformly smooth Banach space, where $1 < q \leqslant 2$, and let $1/p + 1/q = 1$. By Lemma 7.1, the dual space X^* is p-uniformly convex. In order to prove Theorem 7.2 we apply Theorem 6.1 to the space X^*; moreover we write inequality (6.1) in the form

$$\int_{\mathbb{D}} \|\nabla g(z)\|_{X^*}^p \left(\log\frac{1}{|z|}\right)^{p-1} dA(z) \leqslant C_p\left(\|g\|_{p,X^*}^p - \|g(0)\|_{X^*}^p\right), \tag{7.2}$$

whence we get

$$\|g(0)\|_{X^*}^p + c\int_{\mathbb{D}} \|\nabla g(z)\|_{X^*}^p \left(\log\frac{1}{|z|}\right)^{p-1} dA(z) \leqslant \|g\|_{p,X^*}^p, \tag{7.3}$$

where c is a positive constant independent of g. Now we introduce Z to be the space of those harmonic functions $g : \mathbb{D} \mapsto X^*$ such that

$$\|g\|_Z^p := \|g(0)\|_{X^*}^p + c\int_{\mathbb{D}} \|\nabla g(z)\|_{X^*}^p \left(\log\frac{1}{|z|}\right)^{p-1} dA(z) < \infty.$$

Analogously, Y consists of those harmonic functions $f : \mathbb{D} \mapsto X$ for which

$$\|f\|_Y^q := \|f(0)\|_X^q + b\int_{\mathbb{D}} \|\nabla f(z)\|_X^q \left(\log\frac{1}{|z|}\right)^{q-1} dA(z) < \infty,$$

where b is a positive constant independent of f. To continue the proof we have to rewrite the bilinear form $\langle\,,\rangle$ in a suitable form. □

Lemma 7.4. *If f respectively g are X-valued respectively X^*-valued functions harmonic in a neighborhood of the closed unit disk, then there holds the formula*

$$(f,g) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \langle f(e^{i\theta}), g(e^{i\theta})\rangle\, d\theta = \langle f(0), g(0)\rangle + \int_{\mathbb{D}} \langle \nabla f(z), \nabla g(z)\rangle \log\frac{1}{|z|}\, dA(z), \tag{7.4}$$

where

$$\langle \nabla f(z), \nabla g(z)\rangle := \langle D_1 f(z), D_1 g(z)\rangle + \langle D_2 f(z), D_2 g(z)\rangle.$$

Proof. Assuming that X and X^* are complex Banach spaces with the duality pairing

$$\langle \alpha f(z), \beta g(z)\rangle = \alpha\bar{\beta}\langle f(z), g(z)\rangle \quad \text{for any } \alpha,\beta\in\mathbb{C},$$

and using series expansions

$$f(z) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\theta} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n}\bar{z}^n, \quad z = re^{i\theta}\in\mathbb{D},\ a_n\in X,$$

$$g(z) = \sum_{k=-\infty}^{\infty} b_k r^{|k|} e^{ik\theta} = \sum_{k=0}^{\infty} b_k z^k + \sum_{k=1}^{\infty} b_{-k}\bar{z}^k, \quad z = re^{i\theta}\in\mathbb{D},\ b_k\in X^*,$$

we can write the left-hand side of (7.4) in the form

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \langle f(e^{i\theta}), g(e^{i\theta})\rangle\, d\theta = \sum_{n=-\infty}^{\infty} \langle a_n, b_n\rangle.$$

In order to transform the right-hand side of (7.4), recall Wirtinger's partial differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}(D_1 - iD_2), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(D_1 + iD_2).$$

Then

$$\begin{aligned}\langle \nabla f(z), \nabla g(z)\rangle &= \langle D_1 f(z), D_1 g(z)\rangle + \langle D_2 f(z), D_2 g(z)\rangle \\ &= \left\langle \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}}, \frac{\partial g}{\partial z} + \frac{\partial g}{\partial \bar{z}}\right\rangle + \left\langle i\left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar{z}}\right), i\left(\frac{\partial g}{\partial z} - \frac{\partial g}{\partial \bar{z}}\right)\right\rangle \\ &= 2\left\langle \frac{\partial f}{\partial z}, \frac{\partial g}{\partial z}\right\rangle + 2\left\langle \frac{\partial f}{\partial \bar{z}}, \frac{\partial g}{\partial \bar{z}}\right\rangle \\ &= 2\sum_{n,k=0}^{\infty} nk\langle a_n, b_k\rangle z^{n-1}(\bar{z})^{k-1} + 2\sum_{n,k=1}^{\infty} nk\langle a_{-n}, b_{-k}\rangle (\bar{z})^{n-1} z^{k-1}.\end{aligned}$$

Integrating we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\big\langle\nabla f(z),\nabla g(z)\big\rangle\,d\theta = 2\sum_{n=1}^{\infty} n^2 r^{2n-2}\big(\langle a_n,b_n\rangle + \langle a_{-n},b_{-n}\rangle\big)$$

and

$$\int_{\mathbb{D}}\big\langle\nabla f(z),\nabla g(z)\big\rangle\log\frac{1}{|z|}\,dA(z) = -\langle a_0,b_0\rangle + \sum_{n=-\infty}^{\infty}\langle a_n,b_n\rangle.$$

This completes the proof of Lemma 7.4. □

Lemma 7.5. *Under the hypotheses of Lemma* 7.4 *we have* $|(f,g)| \leqslant \|f\|_Y\|g\|_Z$.

Proof. This is easily deduced from (7.4) by using the inequality

$$\big|\big\langle\nabla f(z),\nabla g(z)\big\rangle\big| \leqslant \big\|\nabla f(z)\big\|_X\big\|\nabla g(z)\big\|_{X^*}$$

and Hölder's inequality.

We continue the proof of Theorem 7.2. Assume first that f is harmonic in a neighborhood of the closed disk. Then we have to prove that $\|f_1\|_{L^q(\partial\mathbb{D},X)} \leqslant \|f\|_Y$, where f_1 denotes the restriction of f to the unit circle. The image under f of the closed disk is contained in the closed linear span of the coefficients of f, so we can assume that X is separable. Then X^* is separable (because X is reflexive) and therefore $\mathcal{T}(X^*)$ is dense in $L^p(\partial\mathbb{D},X^*)$. From this and Phillips' theorem it follows that

$$\|f_1\|_{L^q(\partial\mathbb{D},X)} = \sup\big\{\big|(f,g)\big|:\ g_1\in\mathcal{T}(X^*),\ \|g_1\|_{L^p(\partial\mathbb{D},X^*)}\leqslant 1\big\}.$$

Since $\|g_1\|_{L^p(\partial\mathbb{D},X^*)} \geqslant \|g\|_Z$, by Theorem 6.1, we get

$$\|f_1\|_{L^q(\partial\mathbb{D},X)} \leqslant \sup\big\{\big|(f,g)\big|:\ g_1\in\mathcal{T}(X^*),\ \|g\|_Z\leqslant 1\big\}.$$

Hence $\|f_1\|_{L^q(\partial\mathbb{D},X)} \leqslant \|f\|_Y$ because $|(f,g)| \leqslant \|f\|_Y\|g\|_Z$ (Lemma 7.5). This proves the theorem in the special case. If $f\in h^p(X)$ is arbitrary, we apply the result to the function $f_r(z) = f(rz)$; it follows that

$$M_q^q(r,f) \leqslant \big\|f(0)\big\|_X^q + b\int_{\mathbb{D}} r^q\big\|\nabla f(rz)\big\|_X^q\left(\log\frac{1}{|z|}\right)^{q-1} dA(z).$$

The expression on the right-hand side is $\leqslant \|f\|_Z^q$ because $M_q(r,\|\nabla f\|)$ increases with r, which is a consequence of the subharmonicity of $\|\nabla f(z)\|$. Thus the proof is completed. □

8. Remarks

Theorem 8.1. *If there holds the inequality* (7.1)*, then the space* X *is* q*-uniformly smooth.*

Proof. For the proof we need a result of Arregui and Blasco on vector-valued Bergman spaces (see [1, Theorem 3.9] and [2, Theorem 3.6]), in a slightly generalized form. Namely, for a harmonic function $f:\mathbb{D}\mapsto X$ let

$$N_{q,X}(f) = \left(\int_{\mathbb{D}}\big\|\nabla f(z)\big\|_X^q\left(\log\frac{1}{|z|}\right)^{q-1} dA(z)\right)^{1/q}.$$

Then the proof of Theorem 3.6 of [2] can be modified to prove that

$$N_{p,X^*}(g) \asymp \sup\{|(f,g)|:\ N_{q,X}(f) \leqslant 1\}, \tag{8.1}$$

where

$$(f,g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \langle f(e^{i\theta}), g(e^{i\theta})\rangle \, d\theta,$$

and the supremum is taken over the X-valued harmonic "polynomials" f with $f(0) = 0$. Now we write (7.1) as

$$\|f\|_{q,X}^q \leqslant \|f(0)\|_X^q + C N_{q,X}^q(f).$$

Hence, by (8.1) and Hölder's inequality, we find that

$$N_{p,X^*}(g) \leqslant C\|g\|_{(h^q(X))^*}.$$

Since $\|g\|_{(h^q(X))^*} \leqslant \|g\|_{h^p(X^*)}$, we get

$$\|g(0)\|_{X^*} + c_1 N_{p,X^*}(g) \leqslant \|g\|_{p,X^*}.$$

As is shown in Section 1, this implies that X^* is p-uniformly convex, which implies that X is q-uniformly smooth. □

Theorem 6.1 can be used to prove a generalization:

Theorem 8.2. *Let X be a p-uniformly convex space and $1 < q \leqslant p$ ($p \geqslant 2$). If $f \in h^q(X)$, then there holds the inequality*

$$\int_0^1 \{M_{q,X}(r, \nabla f)\}^p (1-r)^{p-1}\, dr \leqslant C\big(\|f\|_{q,X}^p - \|f(0)\|^p\big). \tag{8.2}$$

Proof. The space $Y = L^q(\partial\mathbb{D}, X)$ is p-uniformly convex (see, for example, [5]). Applying Theorem 6.1 to the function g defined by (4.2) we get the result. □

The dual of Theorem 8.2 can be proved in a similar way:

Theorem 8.3. *Let X be a q-uniformly smooth space and $p \geqslant q$ ($1 < q \leqslant 2$). If $f \in h^q(X)$, then there holds the inequality*

$$\|f\|_{p,X}^q - \|f(0)\|^q \leqslant C\int_0^1 \{M_{p,X}(r, \nabla f)\}^q (1-r)^{q-1}\, dr. \tag{8.3}$$

It should be noted that inequalities (8.2) and (8.3) contain, besides the Littlewood–Paley inequalities, the inequalities of Hardy and Littlewood,

$$\int_0^1 \{M_q(r, \nabla f)\}^2 (1-r)\, dr \leqslant C\big(\|f\|_q^2 - |f(0)|^2\big) \quad (1 < q \leqslant 2), \tag{8.4}$$

$$\|f\|_p^2 - \left|f(0)\right|^2 \leqslant C \int_0^1 \left\{M_p(r, \nabla f)\right\}^2 (1-r)\, dr \quad (p \geqslant 2), \tag{8.5}$$

which hold for scalar-valued h^q-functions. This inequalities are obtained from (8.2) and (8.3) by taking $X = \mathbb{R}$.

Theorem 6.1 can be generalized, with the same proof, to obtain the following.

Theorem 8.4. *Let $q \geqslant p$. If $f \in h^q(X)$, where X is p-uniformly convex, then*

$$\int_{\mathbb{D}} \left\|\nabla f(z)\right\|^p \left\|f(z)\right\|^{q-p} \left(1-|z|\right)^{p-1} dA(z) \leqslant C\left(\|f\|_{q,X}^q - \left\|f(0)\right\|^q\right).$$

For holomorphic functions on $\mathbb{D}$ with values in a p-uniformly PL-convex space Theorems 8.2 and 8.3 are proved in [3]. For information on PL-convex spaces we refer to [4].

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