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Weighted Integrals and Bloch Spaces of *n*-Harmonic Functions on the Polydisc

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Abstract We study anisotropic mixed norm spaces $h(p, q, \alpha)$ consisting of n-harmonic functions on the unit polydisc of $\mathbb{C}^n$ by means of fractional integro-differentiation including small $0 < p < 1$ and multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ with non-positive $\alpha_j \le 0$. As an application, two different Bloch spaces of n-harmonic functions are characterized.

Keywords n-Harmonic and pluriharmonic functions · Mixed-norm · Bloch · Bergman spaces · Polydisc

Mathematics Subject Classifications (2000) Primary 32U05 · Secondary 32A18 · 32A36 · 32A37

1 Introduction and Main Theorems

Let $U^n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_j| < 1, 1 \le j \le n\}$ be the unit polydisc in $\mathbb{C}^n$, and let $\mathbb{T}^n = \{w = (w_1, \dots, w_n) \in \mathbb{C}^n : |w_j| = 1, 1 \le j \le n\}$ be the n-dimensional torus, the distinguished boundary of U^n. We will deal with n-harmonic functions on the polydisc U^n, i.e. functions harmonic in each variable z_j separately. Denote by $H(U^n)$, $h(U^n)$ the sets of holomorphic and n-harmonic functions in U^n, respectively.

If $f(z) = f(r\zeta)$ is a measurable function in U^n, then we write

$$M_p(f; r) = \| f(r\cdot)\|_{L^p(\mathbb{T}^n; dm_n)}, \quad r = \big(r_1, \dots, r_n\big) \in I^n, \quad 0 < p \le \infty$$

where $I^n = [0, 1)^n$, dm_n is the n-dimensional Lebesgue measure on $\mathbb{T}^n$ normalized so that $m_n(\mathbb{T}^n) = 1$. The collection of n-harmonic (holomorphic) functions $f(z)$, for which $\|f\|_{h^p} = \sup_{r\in I^n} M_p(f; r) < +\infty$, is the usual Hardy space h^p (resp. H^p).

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Definition 1 The quasi-normed space $L(p,q,\alpha)$ $\big(0<p,q\le\infty, \alpha=(\alpha_1,\dots,\alpha_n), \alpha_j\in\mathbb{R}\big)$ is the set of those functions $f(z)$ measurable in the polydisc U^n, for which the quasi-norm

$$\|f\|_{p,q,\alpha}=\begin{cases}\left(\displaystyle\int_{I^n}\prod_{j=1}^{n}(1-r_j)^{\alpha_j q-1}M_p^q(f;r)\prod_{j=1}^{n}dr_j\right)^{1/q}, & 0<q<\infty,\\ \operatorname*{ess\,sup}_{r\in I^n}\displaystyle\prod_{j=1}^{n}(1-r_j)^{\alpha_j}M_p(f;r), & q=\infty,\end{cases}$$

is finite. For the subspaces of $L(p,q,\alpha)$ consisting of holomorphic or n-harmonic functions let $H(p,q,\alpha)=H(U^n)\cap L(p,q,\alpha)$ and $h(p,q,\alpha)=h(U^n)\cap L(p,q,\alpha)$.

For $p=q<\infty$ the spaces $H(p,q,\alpha), h(p,q,\alpha)$ coincide with the well-known weighted Bergman spaces, while for $q=\infty$ they are known as growth spaces. The first results on mixed norm spaces are contained in classical works of Hardy and Littlewood [9, 10] who considered functions holomorphic in the unit disc $\mathbb{D}=U^1$. They established in particular that

$$\mathcal{D}^{\beta}\big(H(p,q,\alpha)\big)=H(p,q,\alpha+\beta),\quad 0<p,q\le\infty,\ \alpha>0,\ \alpha+\beta>0,\qquad (1)$$

where $\mathcal{D}^{\beta}$ is an operator of fractional integro-differentiation. Later, Flett [8] essentially improved and developed methods of [9, 10]. The relation Eq. 1 was discovered many times and generalized to various domains in $\mathbb{C}^n$ and $\mathbb{R}^n$ and also for general weight functions. Holomorphic and pluriharmonic mixed norm spaces on the unit ball and more general domains of $\mathbb{C}^n$ have been studied, for example, in [12, 13, 16, 18, 19, 23]. For the polydisc case, see [2, 3, 11, 17, 20–22, 25].

The purpose of the present paper is to extend and generalize the relation Eq. 1 to functions n-harmonic in the polydisc. It should be noted several important differences with earlier known cases.

If a function $f(z)$ is holomorphic, then $|f|^p$ is n-subharmonic for any $p>0$ and its integral means $M_p(f;r)$ are increasing in r. This fact makes the proof of Eq. 1 much easier for holomorphic functions.

If a function $u(z)$ is pluriharmonic, i.e. the real part of a holomorphic function, then as is well known, the operator of pluriharmonic conjugation is bounded in $h(p,q,\alpha)$ for all $0<p,q\le\infty$, see [12, 13, 17–19, 23]. Therefore for pluriharmonic functions, the proofs in fact reduce to those of holomorphic ones.

In the present paper we deal with n-harmonic functions u for which the function $|u|^p$ $(0<p<1)$ need not be n-subharmonic, and $M_p(u;r)$ in general not necessarily monotonic in r. A passage from n-harmonic functions to holomorphic ones is impossible because n-harmonic functions need not be real parts of holomorphic functions. So, we should construct an independent theory for n-harmonic mixed norm spaces $h(p,q,\alpha)$. Another feature is the fact that the operator $\mathcal{D}^{\beta}$ of fractional integro-differentiation (defined in Section 2) may act as a differential operator in some variables z_j, and at the same time as an integral one in other variables z_k.

It is known a phenomenon that in contrast to $H(p,q,\alpha)$, the spaces $h(p,q,\alpha)$ are not trivial for $0<p<1$ and certain multi-indices $\alpha=(\alpha_1,\dots,\alpha_n)$ with non-positive components $\alpha_j\le 0$ (cf. [1, 14, 15]). This case is also involved.

Theorem 1 *If* $\alpha_j > 0, -\infty < \beta_j < \alpha_j \ (1 \le j \le n), 0 < p \le \infty, \ \ 0 < q \le \infty$, *then for all* n*-harmonic functions in* U^n

$$\left\| \mathcal{D}^{-\beta} u \right\|_{p,q,\alpha-\beta} \approx \|u\|_{p,q,\alpha}. \tag{2}$$

Note that Theorem 1 includes both fractional integration and differentiation. One may ask whether the relation Eq. 2 is still true for multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ with non-positive $\alpha_j \le 0$. The next theorem gives a partial answer to this question.

Theorem 2 *If* $\alpha_j \le 0 \le \beta_j \ (1 \le j \le n), 0 < p < 1, \ 0 < q \le \infty$, *then for all* n*-harmonic functions in* U^n

$$\left\| \mathcal{D}^{\beta} u \right\|_{p,q,\alpha+\beta} \le C(p, q, \alpha, \beta, n) \|u\|_{p,q,\alpha}. \tag{3}$$

Remark 1 Theorem 2 seems to be new even for one variable case, while for growth spaces, $q = \infty$, $\alpha + \beta > 0$ and functions harmonic in the unit ball of $\mathbb{R}^n$, Theorem 2 is due to Pavlović [15, Theorem 1].

As an application, we get the boundedness of the operator of pluriharmonic conjugation in $h(p, q, \alpha)$ for all $0 < p, q \le \infty$, as well as in the little subspaces of $h(p, \infty, \alpha)$ where "big oh" is replaced by "little oh". Finally, we give in Section 4 some applications to Bloch spaces of n-harmonic functions in U^n.

2 Notation and Preliminaries

We will use the conventional multi-index notations: $\bar{\zeta} = (\bar{\zeta}_1, \ldots, \bar{\zeta}_n)$, $r\zeta = (r_1\zeta_1, \ldots, r_n\zeta_n)$, $r^\alpha = r_1^{\alpha_1} \cdots r_n^{\alpha_n}$, $dr = dr_1 \cdots dr_n$, $(1 - |\zeta|^2)^\alpha = \prod_{j=1}^n (1 - |\zeta_j|^2)^{\alpha_j}$, $\Gamma(\alpha + |k|) = \prod_{j=1}^n \Gamma(\alpha_j + |k_j|)$ for $\zeta \in \mathbb{C}^n, r \in I^n, \alpha = (\alpha_1, \ldots, \alpha_n), k = (k_1, \ldots, k_n)$. Let $\mathbb{Z}_+^n$ denote the set of all n-tuples of nonnegative integers.

Throughout the paper, the letters $C(\alpha, \beta, \ldots)$, C_α etc. stand for positive different constants depending only on the parameters indicated. For $A, B > 0$ the notation $A \approx B$ denotes the two-sided estimate $c_1 A \le B \le c_2 A$ with some inessential positive constants c_1 and c_2 independent of the variable involved. The symbol dm_{2n} means the Lebesgue measure on the polydisc U^n normalized so that $m_{2n}(U^n) = 1$.

Given a function $f(z) = f(r\zeta), r \in I^n, \zeta \in \mathbb{T}^n$, we will use Riemann–Liouville integro-differential operator $D^\alpha \equiv D_r^\alpha$ with respect to variable r:

$$D^{-\alpha} f(z) = \frac{r^\alpha}{\Gamma(\alpha)} \int_{I^n} (1 - \eta)^{\alpha-1} f(\eta z) d\eta, \quad D^\alpha f(z) = \left(\frac{\partial}{\partial r}\right)^m D^{-(m-\alpha)} f(z),$$

where $\left(\frac{\partial}{\partial r}\right)^m = \left(\frac{\partial}{\partial r_1}\right)^{m_1} \cdots \left(\frac{\partial}{\partial r_n}\right)^{m_n}$, $m = (m_1, \ldots, m_n) \in \mathbb{Z}_+^n$, $\alpha = (\alpha_1, \ldots, \alpha_n), \alpha_j > 0$, $m_j - 1 < \alpha_j \le m_j \ (1 \le j \le n)$. Denote

$$\mathcal{D}^{-\alpha} f(r\zeta) = r^{-\alpha} D^{-\alpha} f(r\zeta), \qquad \mathcal{D}^\alpha f(r\zeta) = D^\alpha \left\{ r^\alpha f(r\zeta) \right\}.$$

It is easily seen that if f is n-harmonic, then so are $\mathcal{D}^{\alpha} f$ and $\mathcal{D}^{-\alpha} f$, and for them the following inversion formulas hold:

$$\mathcal{D}^{\alpha}\mathcal{D}^{-\alpha} f(z) = \mathcal{D}^{-\alpha}\mathcal{D}^{\alpha} f(z) = f(z). \tag{4}$$

Note that unlike D^{α}, for $\mathcal{D}^{\alpha}$ the natural semigroup formula $\mathcal{D}^{-\alpha-\beta} = \mathcal{D}^{-\alpha}\mathcal{D}^{-\beta}$ does not hold, see Lemma 4 below for details. Besides, the expansion formula $\mathcal{D}^{\alpha} f = \mathcal{D}^{\alpha_1}_{r_1}\mathcal{D}^{\alpha_2}_{r_2}\dots\mathcal{D}^{\alpha_n}_{r_n} f$ holds for any $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j \in \mathbb{R}$, where $\mathcal{D}^{\alpha_j}_{r_j}$ means the same operator acting in direction r_j only. Note also that for n-harmonic functions the operators $\mathcal{D}^{\alpha}$ have an equivalent definition through a series expansion (see [2, p.734]).

Lemma 1 *If $\alpha_j > 0$ $(1 \le j \le n)$, then for all n-harmonic functions in U^n*

$$\left\|\mathcal{D}^{-\alpha}u\right\|_{h^p} \le C(p, \alpha, n)\|u\|_{p,p,\alpha}, \qquad 0 < p \le 2, \tag{5}$$

$$\left\|\mathcal{D}^{-\alpha}u\right\|_{h^p} \le C(p, \alpha, n)\|u\|_{p,2,\alpha}, \qquad 2 \le p < \infty. \tag{6}$$

The one variable version of Lemma 1 is known and can be deduced from [7, Theorem 2], [8, Theorem 2(iii)] and the fact that harmonic conjugation is bounded in harmonic Bergman spaces on the unit disc. A polydisc version of Lemma 1 can be found in [4, Lemmas 1 and 2].

As is noted in the Introduction, in contrast to $H(p, q, \alpha)$, the space $h(p, q, \alpha)$ has sense for $0 < p < 1$ and certain $\alpha_j \le 0$. The Poisson kernel for the polydisc $P(z) = \prod_{j=1}^{n} P(z_j) = \prod_{j=1}^{n} \frac{1-|z_j|^2}{|1-z_j|^2}$ provides a good example as a function in $h(p, q, \alpha)$, $\alpha_j \le 0$.

Lemma 2 *Let $0 < p \le \infty$, $0 < q < \infty$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j \in \mathbb{R}$. Then*

(a) $P(z) \in h(p, q, \alpha)$ *if and only if* $\alpha_j > \max\{-1, 1 - 1/p\}$, $1 \le j \le n$.
(b) $P(z) \in h(p, \infty, \alpha)$ *for* $p \ne \frac{1}{2}$ *if and only if* $\alpha_j \ge \max\{-1, 1 - 1/p\}$.
(c) $P(z) \in h\left(\frac{1}{2}, \infty, \alpha\right)$ *if and only if* $\alpha_j > -1$, $1 \le j \le n$.

Proof The result immediately follows from the sharp estimates

$$M_p(P; r) \approx \prod_{j=1}^{n} \left(1 - r_j\right)^{\min\{1, 1/p-1\}}, \qquad p \ne \frac{1}{2}, \qquad r \in I^n,$$

$$M_{1/2}(P; r) \approx \prod_{j=1}^{n} \left(1 - r_j\right)\left(\log \frac{e}{1 - r_j}\right)^2, \qquad r \in I^n. \qquad \square$$

However, the space $h(p, q, \alpha)$ is trivial, that is, it consists of zero function only, if at least one component α_j is less than -1. Moreover, a stronger result holds.

Lemma 3

(a) $H(p, q, \alpha) = \{0\}$ *if one of the following statements holds:*

- $0 < p \le \infty, 0 < q < \infty$, *and there exists $j \in [1, n]$ such that* $\alpha_j \le 0$;
- $0 < p \le \infty, 0 < q \le \infty$, *and there exists $j \in [1, n]$ such that* $\alpha_j < 0$.

(b) $h(p, q, \alpha) = \{0\}$ *if one of the following statements holds:*

- $1 \le p \le \infty, 0 < q < \infty$, *and there exists* $j \in [1, n]$ *such that* $\alpha_j \le 0$;
- $1 \le p \le \infty, 0 < q \le \infty$, *and there exists* $j \in [1, n]$ *such that* $\alpha_j < 0$;
- $0 < p \le 1, 0 < q < \infty$, *and* $\exists\ j \in [1, n]$ *such that* $\alpha_j \le \max\left\{-1, 1 - \frac{1}{p}\right\}$;
- $p = 1/2,\ 0 < q \le \infty$, *and* $\exists\ j \in [1, n]$ *such that* $\alpha_j \le -1$;
- $0 < p \le 1, 0 < q \le \infty$, *and* $\exists\ j \in [1, n]$ *such that* $\alpha_j < \max\left\{-1, 1 - \frac{1}{p}\right\}$.

Proof We only illustrate here the proof of the last typical case. Suppose that $u \in h(p, q, \alpha)$ for $0 < p < 1/2,\ 0 < q \le \infty$ and a component α_j less than -1, say $\alpha_1 < -1$. According to the inclusions (vi), (x) of Theorem 1 in [2], $h(p, q, \alpha) \subset h(p, 1, -1)$ and $(1 - r_1)^{-1} M_p(u; r_1) = o(1)$ as $r_1 \to 1-$. It follows from Aleksandrov's result [1, Theorem 2.11] that $u \equiv 0$. □

3 Proof of Theorems 1 and 2

We begin with some semigroup type formulas for the operator $\mathcal{D}^{\alpha}$ and another similar fractional integral on the unit disc $\mathbb{D}$:

$$\widetilde{\mathcal{D}}^{-\gamma} f(z) := \frac{1}{\Gamma(\gamma)} \int_0^1 (1 - \eta)^{\gamma - 1} \eta^{\beta} f(\eta z) d\eta, \qquad \gamma, \beta > 0, \quad z = r\zeta \in \mathbb{D}. \tag{7}$$

We can extend it to U^n by the expansion formula $\widetilde{\mathcal{D}}^{\alpha} f = \widetilde{\mathcal{D}}^{\alpha_1}_{r_1} \widetilde{\mathcal{D}}^{\alpha_2}_{r_2} \ldots \widetilde{\mathcal{D}}^{\alpha_n}_{r_n} f$.

Lemma 4 *For a function $f(z)$ continuous in the unit disc $\mathbb{D}$ the following semigroup type formulas hold:*

$$\text{(i)}\quad \mathcal{D}^{-\alpha-\beta} f = r^{-\beta} \mathcal{D}^{-\alpha}\left\{r^{\beta} \mathcal{D}^{-\beta} f\right\} = \widetilde{\mathcal{D}}^{-\alpha} \mathcal{D}^{-\beta} f, \qquad \alpha, \beta > 0, \tag{8}$$

$$\text{(ii)}\quad \mathcal{D}^{-\alpha} \mathcal{D}^{\beta} f = r^{-\beta} \mathcal{D}^{-(\alpha-\beta)}\left\{r^{\beta} f\right\} = \widetilde{\mathcal{D}}^{-(\alpha-\beta)} f, \qquad \alpha > \beta > 0, \tag{9}$$

$$\text{(iii)}\quad \mathcal{D}^{-\alpha} \mathcal{D}^{\beta} f = r^{-\alpha} \mathcal{D}^{\beta-\alpha}\left\{r^{\alpha} f\right\}, \qquad \beta > \alpha > 0, \tag{10}$$

$$\text{(iv)}\quad \mathcal{D}^{-\alpha} \mathcal{D}^{m} f = \mathcal{D}^{m} \mathcal{D}^{-\alpha} f, \qquad m \in \mathbb{Z}_+,\ \alpha > 0. \tag{11}$$

Proof

(i) By the definitions of the operators $D^{\alpha}, \mathcal{D}^{\alpha}, \widetilde{\mathcal{D}}^{-\alpha}$, we have

$$\mathcal{D}^{-\alpha-\beta} f = r^{-\alpha} r^{-\beta} D^{-\alpha} D^{-\beta} f = r^{-\beta} \mathcal{D}^{-\alpha}\left\{r^{\beta} r^{-\beta} D^{-\beta} f\right\} = r^{-\beta} \mathcal{D}^{-\alpha}\left\{r^{\beta} \mathcal{D}^{-\beta} f\right\}$$
$$= r^{-\beta} \frac{1}{\Gamma(\alpha)} \int_0^1 (1 - \eta)^{\alpha - 1} \eta^{\beta} r^{\beta} \mathcal{D}^{-\beta} f(\eta z)\, d\eta = \widetilde{\mathcal{D}}^{-\alpha} \mathcal{D}^{-\beta} f.$$

(ii) Using inversion formulas Eq. 4 and Eqs. 7–8 we obtain for $\alpha > \beta > 0$

$$\mathcal{D}^{-\alpha} \mathcal{D}^{\beta} f = \mathcal{D}^{-(\alpha-\beta)-\beta} \mathcal{D}^{\beta} f = r^{-\beta} \mathcal{D}^{-(\alpha-\beta)}\left\{r^{\beta} \mathcal{D}^{-\beta} \mathcal{D}^{\beta} f\right\} = \widetilde{\mathcal{D}}^{-(\alpha-\beta)} f.$$

(iii) Again by Eq. 4 and for $\beta > \alpha > 0$

$$\mathcal{D}^{-\alpha}\mathcal{D}^{\beta} f = r^{-\alpha} D^{\beta-\alpha} D^{-(\beta-\alpha)} D^{-\alpha}\left\{\mathcal{D}^{\beta} f\right\} = r^{-\alpha} D^{\beta-\alpha} D^{-\beta}\left\{\mathcal{D}^{\beta} f\right\}$$
$$= r^{-\alpha} D^{\beta-\alpha}\left\{r^{\beta}\mathcal{D}^{-\beta}\mathcal{D}^{\beta} f\right\} = r^{-\alpha} D^{\beta-\alpha}\left\{r^{\beta} f\right\} = r^{-\alpha}\mathcal{D}^{\beta-\alpha}\left\{r^{\alpha} f\right\}.$$

(iv) The result follows by expanding both sides of Eq. 11 and using the commutation relation $\mathcal{D}^{-\alpha}\left\{r^m D^m f\right\} = r^m D^m \mathcal{D}^{-\alpha} f$. We omit the routine details. □

Remark 2 We can easily deduce higher-dimensional analogs for Eqs. 8–11 by an iteration of those in one variable.

Now without loss of generality we may assume that n=2 everywhere in proofs below. First, note that for the Bergman spaces $h(p, p, \alpha)$ the results of Theorem 1 immediately follow by an iteration of those in one variable. But for $p \neq q$ the iteration doesn't work. Second, we focus our attention mainly in small values of p, q ($0 < p < 1$ or $0 < q < 1$), because for $p, q \geq 1$ there are several known proofs suitable for our context. However, we give a simple and short proof also for $p, q \geq 1$.

We begin with fractional integrals, that is $\beta_j > 0$ ($1 \leq j \leq 2$).

Case $1 \leq p \leq \infty, 1 \leq q < \infty$ We exploit an old result of Hardy and Littlewood [9], [6, p.490] for integration in weighted Lebesgue spaces

$$\int_{I^2} (1-r)^{(\alpha-\beta)q-1}\left(\mathcal{D}^{-\beta} g(r)\right)^q dr \leq C(\alpha, \beta, q)\int_{I^2} (1-r)^{\alpha q-1} g(r)^q dr, \qquad (12)$$

where $g(r) \geq 0\,(r \in I^2),\ 1 \leq q < \infty,\ \alpha_j > \beta_j > 0$.

In view of Minkowski's inequality and Eq. 12, we get

$$\begin{aligned}\left\|\mathcal{D}^{-\beta} u\right\|^q_{p,q,\alpha-\beta} &= \int_{I^2} (1-r)^{(\alpha-\beta)q-1}\left\|\mathcal{D}^{-\beta} u\right\|^q_{L^p(\mathbb{T}^2)} dr \\ &\leq \int_{I^2} (1-r)^{(\alpha-\beta)q-1}\left(\mathcal{D}^{-\beta}\|u\|_{L^p(\mathbb{T}^2)}\right)^q dr \\ &\leq C\int_{I^2} (1-r)^{(\alpha-\beta)q-1}\|u\|^q_{L^p(\mathbb{T}^2)} dr = \|u\|^q_{p,q,\alpha}.\end{aligned}$$

Case $1 \leq p \leq \infty, q = \infty$ Assuming that $u \in h(p, \infty, \alpha)$, we have

$$(1-r)^{\alpha} M_p(u; r) \leq \|u\|_{p,\infty,\alpha}, \qquad r = (r_1, r_2) \in I^2.$$

By Minkowski's inequality,

$$\begin{aligned}M_p(\mathcal{D}^{-\beta} u; r) &\leq \frac{1}{\Gamma(\beta)}\int_{I^2} (1-\eta)^{\beta-1} M_p(u; \eta r) d\eta \\ &\leq \|u\|_{p,\infty,\alpha}\frac{1}{\Gamma(\beta)}\int_{I^2}\frac{(1-\eta)^{\beta-1}}{(1-r\eta)^{\alpha}} d\eta \leq C(\alpha, \beta)\frac{\|u\|_{p,\infty,\alpha}}{(1-r)^{\alpha-\beta}}.\end{aligned}$$

Hence $\left\|\mathcal{D}^{-\beta} u\right\|_{p,\infty,\alpha-\beta} \leq C\|u\|_{p,\infty,\alpha}$, as desired.

Case $0 < q < 1,\ 0 < q \le p \le \infty$ Let $u(z_1, z_2) \in h(p, q, \alpha)$ and let u_ρ be the dilated function defined by $u_\rho(z) = u(\rho z) = u(\rho_1 z_1, \rho_2 z_2),\ \ \rho \in I^2$. Since $q \le \min\{2, p\}$ and the spaces $h(p, q, \alpha)$ increase in q ([2, Theorem 1(iii)]), by Lemma 1, we get

$$\left\|\mathcal{D}^{-\beta} u_\rho\right\|_{h^p} \le C\|u_\rho\|_{p,q,\beta}, \qquad \rho \in I^2,$$

or equivalently

$$M_p\big(\mathcal{D}^{-\beta} u; \rho r\big) \le C\|u_\rho\|_{p,q,\beta}, \qquad r, \rho \in I^2,$$

for any $\beta_j > 0,\ j = 1, 2$. By Fatou's lemma,

$$\begin{aligned} M_p^q\big(\mathcal{D}^{-\beta} u; \rho\big) &\le \liminf_{r_1, r_2 \to 1-} M_p^q\big(\mathcal{D}^{-\beta} u; \rho r\big) \\ &\le C \int_{I^2} (1 - r)^{\beta q - 1} M_p^q(u; \rho r) dr = C\,\mathcal{D}^{-\beta q}\big\{M_p^q(u; \rho)\big\}. \end{aligned}$$

Weighted integration by means of the inequality Eq. 12 leads to

$$\left\|\mathcal{D}^{-\beta} u\right\|^q_{p,q,\alpha-\beta} \le C \int_{I^2} (1 - \rho)^{(\alpha-\beta)q-1}\, \mathcal{D}^{-\beta q}\big\{M_p^q(u; \rho)\big\} d\rho \le C\|u\|^q_{p,q,\alpha}.$$

Case $0 < p < 1, 0 < p \le q < \infty$. The inequality Eq. 5 of Lemma 1 gives

$$M_p\big(\mathcal{D}^{-\beta} u; \rho r\big) \le C\|u_\rho\|_{p,p,\beta}, \qquad r, \rho \in I^2.$$

for any $\beta_j > 0,\ j = 1, 2$. By Fatou's lemma

$$M_p^p\big(\mathcal{D}^{-\beta} u; \rho\big) \le C \int_{I^2} (1 - r)^{\beta p - 1} M_p^p(u; \rho r) dr = C\,\mathcal{D}^{-\beta p}\big\{M_p^p(u; \rho)\big\}.$$

Raising both parts of this inequality to the power $q/p \ge 1$ and then integrating with an application of Eq. 12, we obtain

$$\left\|\mathcal{D}^{-\beta} u\right\|^q_{p,q,\alpha-\beta} \le C \int_{I^2} (1 - \rho)^{p(\alpha-\beta)q/p-1} \left[\mathcal{D}^{-\beta p} M_p^p(u; \rho)\right]^{q/p} d\rho \le C\|u\|^q_{p,q,\alpha}.$$

Case $0 < p < 1, q = \infty$ is simpler. Thus, the proof for fractional integrals is complete.

We now turn to the case of fractional derivatives, that is $\beta_j \ge 0\, (1 \le j \le 2)$. We are going to combine this case with the proof of Theorem 2. For any $\alpha = (\alpha_1, \alpha_2),\ \ \alpha_j \in \mathbb{R}$ and a function $u(z) = u(r\zeta) \in h(p, q, \alpha)$, we have to prove the inequality

$$\left\|\mathcal{D}^{\beta} u\right\|_{p,q,\alpha+\beta} \le C\|u\|_{p,q,\alpha}. \tag{13}$$

First, we prove Eq. 13 for multi-indices $\beta = m = (m_1, m_2)$ with integers $m_j \in \mathbb{Z}_+$.

Case $0 < p \le q < \infty$ Given a point $z=(z_1, z_2)=(r_1 w_1, r_2 w_2)\in U^2$ define the bidisc $B_z = B_{z_1}\times B_{z_2}$, where $B_{z_j}=\{\zeta \in \mathbb{C} : |\zeta_j - z_j| < \left(1-r_j\right)/2\}$, $j=1,2$. Cauchy's estimates for n-harmonic functions and the well-known Hardy–Littlewood–Fefferman–Stein inequality on subharmonic behavior of $|u|^p$ imply a "differentiated" version of that (cf. [1, 7, 20]):

$$|\mathcal{D}^m u(z_1, z_2)|^p \le \frac{C(p,m)}{|B_{z_1}||B_{z_2}|(1-r_1)^{m_1 p}(1-r_2)^{m_2 p}} \iint\limits_{B_{z_1}\times B_{z_2}} |u(\zeta_1, \zeta_2)|^p dm_4(\zeta),$$

where $|B_{z_j}|$ is the area of the disc B_{z_j}. For $\zeta = (\zeta_1, \zeta_2) \in B_z$, we have

$$\rho'_j < |\zeta_j| = \rho_j < \rho''_j, \qquad \text{where} \qquad \rho'_j = \max\left\{0, \frac{3r_j - 1}{2}\right\}, \qquad \rho''_j = \frac{1+r_j}{2},$$

for $j=1,2$. Hence

$$\frac{1}{2}\left(1-r_j\right) < 1 - |\zeta_j| < \frac{3}{2}\left(1-r_j\right), \qquad j=1,2.$$

It follows from this together with the simple inequality

$$|1-\zeta_j \bar{z}_j| < 3(1-|\zeta_j|), \qquad |z_j|<1, \ \zeta_j \in B_{z_j},$$

that

$$\left|\mathcal{D}^m u(z_1, z_2)\right|^p \le C(m,p) \int\limits_{B_{z_1}} \int\limits_{B_{z_2}} \frac{|u(\zeta_1, \zeta_2)|^p \, dm_2(\zeta_1)\, dm_2(\zeta_2)}{|1-\zeta_1\bar{z}_1|^{2+m_1 p}|1-\zeta_2\bar{z}_2|^{2+m_2 p}}. \tag{14}$$

Next, we extend the domain of integration in Eq. 14 up to the rings $\rho'_j < |\zeta_j| < \rho''_j$ $(j=1,2)$ and integrate over the torus $\mathbb{T}^2$:

$$M_p^p(\mathcal{D}^m u; r_1, r_2) \le \frac{C(m,p)}{(1-r_1)^{1+m_1 p}(1-r_2)^{1+m_2 p}} \int\limits_{\rho'_1}^{\rho''_1} \int\limits_{\rho'_2}^{\rho''_2} M_p^p(u; \rho_1, \rho_2) d\rho_1 d\rho_2.$$

By Hölder's inequality with indices $q/p \ge 1$ and $q/(q-p)$

$$M_p^p(\mathcal{D}^m u; r_1, r_2) \le \frac{C(m,p)}{(1-r_1)^{m_1 p}(1-r_2)^{m_2 p}} \left[\int\limits_{\rho'_1}^{\rho''_1} \int\limits_{\rho'_2}^{\rho''_2} \frac{M_p^q(u; \rho_1, \rho_2)\, d\rho_1\, d\rho_2}{(1-\rho_1)(1-\rho_2)}\right]^{p/q},$$

and

$$\prod_{j=1}^{2}\left(1-r_j\right)^{(\alpha_j+m_j)q-1} M_p^q(\mathcal{D}^m u; r) \le C \prod_{j=1}^{2}\left(1-r_j\right)^{\alpha_j q-1} \int\limits_{\rho'_1}^{\rho''_1} \int\limits_{\rho'_2}^{\rho''_2} \frac{M_p^q(u;\rho) d\rho_1\, d\rho_2}{(1-\rho_1)(1-\rho_2)}.$$

Integrate over I^2 and apply Fubini's theorem

$$\|\mathcal{D}^m u\|^q_{p,q,\alpha+m} \le C \int_0^1\int_0^1 \prod_{j=1}^2 \left(1-r_j\right)^{\alpha_j q-1}\left[\int_{\rho_1'}^{\rho_1''}\int_{\rho_2'}^{\rho_2''} \frac{M_p^q(u;\rho)\,d\rho_1\,d\rho_2}{(1-\rho_1)(1-\rho_2)}\right] dr_1 dr_2$$

$$\le C\int_0^1\int_0^1 M_p^q(u;\rho)\prod_{j=1}^2\left[\int_{\max\{0,2\rho_j-1\}}^{(2\rho_j+1)/3} \left(1-r_j\right)^{\alpha_j q-1} dr_j\right]\frac{d\rho_1\,d\rho_2}{(1-\rho_1)(1-\rho_2)}$$

$$\le C\int_0^1\int_0^1 M_p^q(u;\rho)\prod_{j=1}^2 (1-\rho_j)^{\alpha_j q}\frac{d\rho_1\,d\rho_2}{(1-\rho_1)(1-\rho_2)}$$

$$\le C(p,q,\alpha,m)\|u\|^q_{p,q,\alpha}.$$

Case $0<q<p\le\infty$ Write the inequality Eq. 14 with q instead of p

$$|\mathcal{D}^m u(z_1,z_2)|^q \le C(m,q)\int_{B_{z_1}}\int_{B_{z_2}} \frac{|u(\zeta_1,\zeta_2)|^q\,dm_2(\zeta_1)\,dm_2(\zeta_2)}{|1-\zeta_1\bar{z}_1|^{2+m_1 q}\,|1-\zeta_2\bar{z}_2|^{2+m_2 q}}$$

$$\le C(m,q)\int_{\rho_1'}^{\rho_1''}\int_{\rho_2'}^{\rho_2''}\left[\int_{\mathbb{T}^2}\frac{|u(\rho_1 t_1 w_1,\rho_2 t_2 w_2)|^q\ dm_2(t)}{|1-\rho_1 r_1 t_1|^{2+m_1 q}|1-\rho_2 r_2 t_2|^{2+m_2 q}}\right]\rho_1\rho_2 d\rho_1 d\rho_2,$$

where $z=rw,\ \zeta=\rho t,\ r,\rho\in I^2,\ w,t\in\mathbb{T}^2$. By Minkowski's inequality with exponent $p/q\ge 1$

$$M_p^q(\mathcal{D}^m u;r_1,r_2)\le\frac{C(m,q)}{(1-r_1)^{1+m_1 q}(1-r_2)^{1+m_2 q}}\int_{\rho_1'}^{\rho_1''}\int_{\rho_2'}^{\rho_2''} M_p^q(u;\rho_1,\rho_2)d\rho_1 d\rho_2.$$

It remains to integrate and to apply Fubini's theorem.

Case $q=\infty$ is simpler, so we omit it.

Thus, for $m=(m_1,m_2),\ m_j\in\mathbb{Z}_+$, we have proved that

$$\left\|\mathcal{D}^m u\right\|_{p,q,\alpha+m}\le C\|u\|_{p,q,\alpha}.$$

Take now any $\beta=(\beta_1,\beta_2)$, $\beta_j\ge 0$ and suppose $m_j-1<\beta_j\le m_j$ ($m_j\in\mathbb{Z}_+$).

We will use semigroup type formulas obtained in Lemma 4. As is easily seen, the integral Eq. 7 differs from $\mathcal{D}^{-\gamma}$ only by an inessential factor η^β in the integrand. So, the assertions of the preceding parts of Theorem 1 are valid also for $\widetilde{\mathcal{D}}^{-\beta}$. Therefore

$$\left\|\mathcal{D}^{(\beta_1,\beta_2)}u\right\|_{p,q,\alpha+\beta}=\left\|\widetilde{\mathcal{D}}^{(-(m_1-\beta_1),-(m_2-\beta_2))}\mathcal{D}^{(m_1,m_2)}u\right\|_{p,q,\alpha+\beta}$$
$$\le C\left\|\mathcal{D}^{(m_1,m_2)}u\right\|_{p,q,\alpha+m}\le C\|u\|_{p,q,\alpha}.$$

Finally, consider the mixed case when $\beta_1 \le 0 \le \beta_2$, i.e. the operator $\mathcal{D}^{(\beta_1,\beta_2)}$ acts as a primitive in direction r_1 and as a derivative in direction r_2. Let $0 < p \le \infty$, $0 < q < \infty$, $\beta_j < \alpha_j$ and $v(z_1, z_2) := \mathcal{D}^{-\beta_2} u(z_1, z_2)$. Then by the relations proved above and by Fubini's theorem

$$\begin{aligned}\|\mathcal{D}^{-\beta}u\|^q_{p,q,\alpha-\beta} &= \int_0^1 (1-r_2)^{(\alpha_2-\beta_2)q-1}\left[\int_0^1 (1-r_1)^{(\alpha_1-\beta_1)q-1} M_p^q(\mathcal{D}_{r_1}^{-\beta_1}v; r_1, r_2)dr_1\right]dr_2 \\ &\le C\int_0^1 (1-r_2)^{(\alpha_2-\beta_2)q-1}\left[\int_0^1 (1-r_1)^{\alpha_1 q-1} M_p^q(v; r_1, r_2)dr_1\right]dr_2 \\ &\le C\|u\|^q_{p,q,\alpha}.\end{aligned}$$

Thus, we have proved both Theorems 1 and 2.

The following two theorems can be proved in the same manner. The first of them is a "little oh" version of Theorem 1.

Theorem 3 *Let $u(z)$ be an n-harmonic function in U^n, and $\alpha_j > 0$, $\alpha_j > \beta_j$ $(1 \le j \le n)$, $0 < p \le \infty$.*

(i) *If $0 < q < \infty$ and $u \in h(p, q, \alpha)$, then for each $j \in [1, n]$*

$$(1-r)^{\alpha-\beta} M_p(\mathcal{D}^{-\beta}u; r) = o(1) \qquad \text{as} \qquad r_j \to 1-.$$

(ii) *The following two statements are equivalent for each $j \in [1, n]$*

$$(1-r)^{\alpha} M_p(u; r) = o(1) \qquad \text{as} \qquad r_j \to 1-,$$

$$(1-r)^{\alpha-\beta} M_p\big(\mathcal{D}^{-\beta}u; r\big) = o(1) \qquad \text{as} \qquad r_j \to 1-.$$

Theorem 4 *Theorems 1–3 remain valid for integral operators $D^{-\beta}$ or $\widetilde{\mathcal{D}}^{-\beta}$ in place of $\mathcal{D}^{-\beta}$, and for differential operators D^{β} in place of $\mathcal{D}^{\beta}$, as well as for ordinary partial derivatives. In particular,*

$$\begin{aligned}\|D^{-\beta}u\|_{p,q,\alpha-\beta} &\approx \|u\|_{p,q,\alpha}, && \alpha_j > \beta_j > 0,\ 0 < p, q \le \infty, \\ \|\widetilde{\mathcal{D}}^{-\beta}u\|_{p,q,\alpha-\beta} &\le C\|u\|_{p,q,\alpha}, && \alpha_j > \beta_j > 0,\ 0 < p, q \le \infty, \\ \|\partial^{\lambda}u\|_{p,q,\alpha+\lambda} &\le C\|u\|_{p,q,\alpha}, && \alpha_j \le 0,\ 0 < p < 1,\ 0 < q \le \infty, \\ \|\partial^{\lambda}u\|_{p,q,\alpha+\lambda} &\le C\|u\|_{p,q,\alpha}, && \alpha_j > 0,\ 0 < p, q \le \infty,\end{aligned}$$

where $\partial^{\lambda} = \partial^{\lambda_1} \cdots \partial^{\lambda_n}$, and ∂^{λ_j} means mixed partial derivative of order $\lambda_j \in \mathbb{Z}_+$ in the variables r_j and θ_j $(z_j = r_j e^{i\theta_j})$.

As a consequence, we obtain that the pluriharmonic subspace of $h(p, q, \alpha)$ is a self-conjugate space for all $0 < p, q \le \infty$, $\alpha_j > 0$.

Theorem 5 *Let* $0 < p, q \leq \infty, \alpha = (\alpha_1, \ldots, \alpha_n), \alpha_j > 0, 1 \leq j \leq n$. *If* u *is a pluriharmonic function of* $h(p, q, \alpha)$, *and* v *is its pluriharmonic conjugate normalized so that* $v(0) = 0$, *then* $v \in h(p, q, \alpha)$, *and*

$$\|v\|_{p,q,\alpha} \leq C\|u\|_{p,q,\alpha}. \tag{15}$$

Furthermore, for each $j \in [1, n]$ *the following statements are equivalent:*

$$(1-r)^{\alpha} M_p(u; r) = o(1) \qquad as \qquad r_j \to 1-,$$

$$(1-r)^{\alpha} M_p(v; r) = o(1) \qquad as \qquad r_j \to 1-.$$

Proof In view of 2-subharmonicity of $|f|^p$ $(p > 0,\ f \in H(U^2))$, we have $\|f\|_{p,q,\alpha} \leq C(p, q, \alpha)\|f\|^*_{p,q,\alpha}$, where

$$\|f\|^*_{p,q,\alpha} = \begin{cases} \left(\int\limits_{1/2}^{1}\int\limits_{1/2}^{1}\prod\limits_{j=1}^{2}\left(1-r_j\right)^{\alpha_j q-1} M_p^q(f; r)dr_1dr_2\right)^{1/q}, & 0 < q < \infty, \\ \sup\limits_{1/2<r_1,r_2<1} \prod\limits_{j=1}^{2}\left(1-r_j\right)^{\alpha_j} M_p(f; r), & q = \infty. \end{cases}$$

It follows that

$$\|f\|_{p,q,\alpha} \leq C\|f\|^*_{p,q,\alpha} \leq C\|u\|_{p,q,\alpha} + C\|v\|^*_{p,q,\alpha}.$$

The last term can be estimated as follows

$$\begin{aligned} \|v\|^*_{p,q,\alpha} &\leq C\|v(z_1, z_2) - v(0, z_2)\|^*_{p,q,\alpha} + C\|v(0, z_2)\|^*_{p,q,\alpha} \\ &= \left\|\int_0^{r_1} \frac{\partial v(\rho_1\zeta_1, z_2)}{\partial \rho_1} d\rho_1\right\|^*_{p,q,\alpha} + \left\|\int_0^{r_2} \frac{\partial v(0, \rho_2\zeta_2)}{\partial \rho_2} d\rho_2\right\|^*_{p,q,\alpha_2}. \end{aligned}$$

By Cauchy–Riemann equations and Theorem 4 $(\zeta_j = e^{i\theta_j})$

$$\begin{aligned} \|v\|^*_{p,q,\alpha} &\leq C\left\|r_1\frac{\partial v(r_1\zeta_1, z_2)}{\partial r_1}\right\|^*_{p,q,(\alpha_1+1,\alpha_2)} + C\left\|r_2\frac{\partial v(0, r_2\zeta_2)}{\partial r_2}\right\|^*_{p,q,\alpha_2+1} \\ &= C\left\|\frac{\partial u(r_1\zeta_1, z_2)}{\partial \theta_1}\right\|^*_{p,q,(\alpha_1+1,\alpha_2)} + C\left\|\frac{\partial u(0, r_2\zeta_2)}{\partial \theta_2}\right\|^*_{p,q,\alpha_2+1} \\ &\leq C\|u\|_{p,q,(\alpha_1,\alpha_2)} + C\|u(0, z_2)\|_{p,q,\alpha_2} \leq C\|u\|_{p,q,\alpha}. \end{aligned}$$

Next, by Theorems 1, 3 and 4, the following statements are equivalent

$$(1-r_1)^{\alpha_1}(1-r_2)^{\alpha_2} M_p(u;r) = o(1) \qquad \text{as} \qquad r_1 \to 1-,$$

$$(1-r_1)^{\alpha_1+1}(1-r_2)^{\alpha_2} M_p\left(\frac{\partial u}{\partial \theta_1};r\right) = o(1) \qquad \text{as} \qquad r_1 \to 1-,$$

$$(1-r_1)^{\alpha_1+1}(1-r_2)^{\alpha_2} M_p\left(\frac{\partial v}{\partial r_1};r\right) = o(1) \qquad \text{as} \qquad r_1 \to 1-,$$

$$(1-r_1)^{\alpha_1}(1-r_2)^{\alpha_2} M_p(v;r) = o(1) \qquad \text{as} \qquad r_1 \to 1-.$$

This completes the proof of Theorem 5. □

Remark 3 Inequality Eq. 15 is well known for the unit ball of $\mathbb{C}^n$, see [12, 19, 23]. For more general bounded symmetric domains see [18] ($1 \le p \le \infty$, $0 < q < \infty$) and [13] ($0 < p = q < \infty$), while for Bergman spaces (i.e. for $p = q$) with general weights in the polydisc see [17].

4 Bloch Spaces on the Polydisc

In this section, we discuss two different Bloch spaces $\mathcal{B}$ and $\mathcal{B}h$ of functions n-harmonic in U^n. The first space $\mathcal{B}$ corresponds to that introduced by Timoney [24] for holomorphic functions in U^n (see also [5]), while the second space $\mathcal{B}h$ agrees with Definition 1 for $p = q = \infty$ (see also [2, 4, 5, 25]).

Definition 2 A function $u(z)$ n-harmonic in U^n, is said to be in the Bloch space $\mathcal{B} = \mathcal{B}(U^n)$ or $\mathcal{B}h = \mathcal{B}h(U^n)$ if

$$\|u\|_{\mathcal{B}} = |u(0)| + \max_{1\le j\le n} \sup_{z\in U^n} (1-|z_j|)\left|\frac{\partial}{\partial r_j}u(z)\right| < +\infty,$$

$$\|u\|_{\mathcal{B}h} = \sup_{z\in U^n} (1-|z|)|\mathcal{D}^1 u(z)| < +\infty,$$

respectively. Here $\mathcal{D}^1 u(z) = D^1\{ru(z)\} = \dfrac{\partial^n}{\partial r_1\cdots\partial r_n}\{r_1\cdots r_n u(r\zeta)\}$.

It is easy to see that

$$\|u\|_{\mathcal{B}} \approx |u(0)| + \sum_{j=1}^n \sup_{z\in U^n} (1-|z_j|)\left|\frac{\partial}{\partial r_j}u(z)\right|,$$

$$\|u\|_{\mathcal{B}h} = \|\mathcal{D}^1 u\|_{\infty,\infty,1} \approx \sup_{1/2<r_1,\cdots,r_n<1} \prod_{j=1}^n \left(1-r_j\right) M_\infty(\mathcal{D}^1 u; r).$$

The following theorem asserts that $\mathcal{B}h$ is strictly wider than $\mathcal{B}$.

Theorem 6 *The inclusion $\mathcal{B} \subset \mathcal{B}h$ is continuous and strict.*

Proof Suppose that $u \in \mathcal{B}(U^2)$. Since $\mathcal{D}^1 = 1 + r_1 \frac{\partial}{\partial r_1} + r_2 \frac{\partial}{\partial r_2} + r_1 r_2 \frac{\partial^2}{\partial r_1 \partial r_2}$,

$$\begin{aligned}
\|u\|_{\mathcal{B}h} &= \sup_{z\in U^2} (1-|z_1|)(1-|z_2|)|\mathcal{D}^1 u(z_1, z_2)| \\
&\le \sup_{z\in U^2} (1-r_1)(1-r_2)\left|\frac{\partial u}{\partial r_1}\right| + \sup_{z\in U^2} (1-r_1)(1-r_2)\left|\frac{\partial u}{\partial r_2}\right| \\
&\quad + \sup_{z\in U^2} (1-r_1)(1-r_2)|u(z_1, z_2)| + \sup_{z\in U^2} (1-r_1)(1-r_2)\left|\frac{\partial^2 u}{\partial r_1 \partial r_2}\right| \\
&\equiv I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

It is clear that the quantities I_1 and I_2 are dominated by $\|u\|_{\mathcal{B}}$. To estimate I_3 and I_4 we use Theorem 4 and obtain

$$\begin{aligned}
I_3 &= \sup_{0<r_2<1} (1-r_2)\Big[\sup_{0<r_1<1} (1-r_1)|u(z_1, z_2)|\Big] \\
&\le C \sup_{0<r_2<1} (1-r_2)\left(|u(0, z_2)| + \sup_{0<r_1<1} (1-r_1)^2 \left|\frac{\partial u(z_1, z_2)}{\partial r_1}\right|\right) \\
&\le C|u(0,0)| + C \sup_{0<r_2<1} (1-r_2)^2 \left|\frac{\partial u(0, z_2)}{\partial r_2}\right| + C \sup_{0<r_1, r_2<1} (1-r_1)\left|\frac{\partial u(z_1, z_2)}{\partial r_1}\right| \\
&\le C\|u\|_{\mathcal{B}}, \\
I_4 &= \sup_{z\in U^2} (1-r_1)(1-r_2)\left|\frac{\partial^2 u}{\partial r_1 \partial r_2}\right| \le C \sup_{z\in U^2} (1-r_1)\left|\frac{\partial u}{\partial r_1}\right| \le C\|u\|_{\mathcal{B}}.
\end{aligned}$$

Thus, $\|u\|_{\mathcal{B}h} \le C\|u\|_{\mathcal{B}}$. The converse inclusion is false because of the example $f_0(z_1, z_2) = \prod_{j=1}^2 \log \frac{e}{1-z_j}$ which is in $\mathcal{B}h(U^2)$, but not in $\mathcal{B}(U^2)$. Thus, the proof is complete. □

The wider Bloch space $\mathcal{B}h$ possesses some advantages. Unlike $\mathcal{B}$, the space $\mathcal{B}h$ is the image of $L^\infty(U^n)$ under the Bergman type operator

$$T_{\beta,\gamma}(u)(z) = \frac{(1-|z|^2)^\gamma}{\Gamma(\beta+\gamma)} \int_{U^n} \left(1-|\zeta|^2\right)^{\beta-1} P_{\beta+\gamma}(z,\zeta)u(\zeta)dm_{2n}(\zeta),$$

where $P_{\beta+\gamma} = \mathcal{D}^{\beta+\gamma} P$ is the Poisson–Bergman kernel [2, p.735]. Namely, the map $T_{\beta,0} : L^\infty(U^n) \longrightarrow \mathcal{B}h$ is bounded and onto, and also the map $T_{\beta,\gamma} : \mathcal{B}h \longrightarrow L^\infty(U^n)$ is bounded for $\beta_j, \gamma_j > 0\, (1 \le j \le n)$, see [2].

Theorem 7 *If $\alpha_j > 0\, (1 \le j \le n)$, then $\mathcal{D}^{-\alpha}(h(\infty, \infty, \alpha)) = \mathcal{B}h$, with equivalent norms.*

Proof We only prove the typical case $0 < \alpha_1 < 1 < \alpha_2$. By semigroup type formulas of Lemma 4

$$\mathcal{D}^{(1,1)}\mathcal{D}^{-(\alpha_1,\alpha_2)}u = \mathcal{D}_{r_1}^{-\alpha_1}\mathcal{D}_{r_2}^{-\alpha_2}\mathcal{D}_{r_1}^{1}\mathcal{D}_{r_2}^{1}u = r_1^{-\alpha_1}\widetilde{\mathcal{D}}_{r_2}^{-(\alpha_2-1)}\mathcal{D}_{r_1}^{1-\alpha_1}\left\{r_1^{\alpha_1}u\right\}.$$

Hence by Theorems 1 and 4

$$\begin{aligned}\left\|\mathcal{D}^{-(\alpha_1,\alpha_2)}u\right\|_{\mathcal{B}h} &= \left\|\mathcal{D}^{(1,1)}\mathcal{D}^{-(\alpha_1,\alpha_2)}u\right\|_{\infty,\infty,(1,1)} \\ &\approx \left\|\widetilde{\mathcal{D}}_{r_2}^{-(\alpha_2-1)}\mathcal{D}_{r_1}^{1-\alpha_1}\left\{r_1^{\alpha_1}u\right\}\right\|_{\infty,\infty,(1,1)} \\ &\approx \left\|\mathcal{D}_{r_1}^{1-\alpha_1}\left\{r_1^{\alpha_1}u\right\}\right\|_{\infty,\infty,(1,\alpha_2)} \approx \|u\|_{\infty,\infty,(\alpha_1,\alpha_2)}. \qquad \square\end{aligned}$$

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