A NOTE ON MIXED NORM SPACES OF ANALYTIC FUNCTIONS

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ABSTRACT. A direct and elementary proof of an estimate of Littlewood is given together with an application concerning the sharpness and strictness of some inclusions in mixed norm spaces of analytic functions.

Key words and phrases: Analytic function; Mixed norm space; Sharp inclusions.

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1. Introduction

Let \( \mathbb{D} \) be the unit disc in the complex plane and \( \mathbb{T} \) its boundary. If \( f(z) = f(re^{i\theta}) \) is a measurable function in \( \mathbb{D} \), then we write as usual
\[
M_p(f; r) = \|f(r\cdot)\|_{L^p(\mathbb{T}; dm)}, \quad 0 \leq r < 1, \quad 0 < p \leq \infty,
\]
where \( dm \) is the Lebesgue measure on \( \mathbb{T} \). The collection of analytic functions \( f(z) \), for which
\[
\|f\|_{H^p} = \sup_{0 \leq r < 1} M_p(f; r) < +\infty,
\]
is the usual Hardy space \( H^p \). The quasi-normed space
\[
H(p, q, \alpha)(0 < p, q \leq \infty, \alpha > 0)
\]
is the set of those functions \( f(z) \) analytic in the unit disc \( \mathbb{D} \), for which the quasi-norm
\[
\|f\|_{p,q,\alpha} = \left\{ \left( \int_0^1 (1 - r)^{\alpha q - 1} M_p^q(f; r) dr \right)^{1/q}, \quad 0 < q < \infty, \right. \\
\left. \sup_{0 \leq r < 1} (1 - r)^\alpha M_p(f; r), \quad q = \infty, \right.
\]
is finite. If \((1 - r)^\alpha M_p(f; r) = o(1)\) as \( r \to 1^-\), then we write \( f \in H_0(p, q, \alpha) \). For \( p = q < \infty \) the spaces \( H(p, q, \alpha) \) coincide with the well-known weighted Bergman spaces, while \( q = \infty \) they are known as growth spaces, and \( H_0(p, \infty, \alpha) \) corresponding "little" space.

The mixed norm spaces consisting of harmonic functions will be denoted by \( h(p, q, \alpha) \). In \( \mathbb{R}^n \) among others, some continuous inclusions of Hardy–Littlewood–Flett type in \( h(p, q, \alpha) \) are proved in the context of functions \( n \)-harmonic in the unit polydisc of \( \mathbb{C}^n \).

**Theorem 1.** The following inclusions are continuous for any \( \alpha, \beta \in \mathbb{R}, 0 < p, q \leq \infty \):

(i) \( h(p, q, \alpha) \subset h(p, q, \beta), \quad \beta > \alpha, \)

(ii) \( h(p, q, \alpha) \subset h(p_0, q, \alpha), \quad 0 < p_0 < p \leq \infty, \)

(iii) \( h(p, q, \alpha) \subset h(p, q_0, \alpha), \quad 0 < q < q_0 \leq \infty, \)

(iv) \( h(p, q, \alpha) \subset h(p_0, q, \beta), \quad \beta \geq \alpha + 1/p - 1/p_0, \quad 0 < p < p_0 \leq \infty, \)

(v) \( h(p, q, \alpha) \subset h(p_0, q_0, \beta), \quad \beta > \alpha + 1/p, \quad 0 < p_0, q_0 \leq \infty, \)

(vi) \( h(p, q, \alpha) \subset h(p, q_0, \beta), \quad \beta > \alpha, \quad 0 < q_0 \leq \infty, \)

(vii) \( H^p \subset H \left( p_0, q, \frac{1}{p} - \frac{1}{p_0} \right), \quad 0 < p < p_0 \leq \infty, \quad 0 < p < q \leq \infty. \)

Of course, the inclusions (i), (ii) are obvious, while some others are much deeper, for instance, (iii), (iv) and (vii) which were originally proved by Hardy and Littlewood \([8, \text{Th.31}]\) and Flett \([5, \text{pp.755-756}]\) for functions analytic in the unit disc, see also \([4, \text{Th.5.11}], [6, \text{Th.3.1}], [9]\).

The purpose of this note is to prove that the inclusions (i)-(vii) for analytic functions are strict and best possible in a certain sense. See Theorem \([2]\) below for the precise formulation.

2. Estimates

Throughout the paper, the capital letters \( C(\alpha, \beta, \ldots), C_\alpha \) stand for different positive constants depending only on the parameters indicated. For \( A, B > 0 \) the notation \( A \approx B \) denotes the two-sided estimate \( C_1 A \leq B \leq C_2 A \) with some inessential positive constants \( C_1 \) and \( C_2 \) independent of the variable involved.

The estimates appearing in the next lemma were essentially proved by Littlewood in \([10, \text{pp.93-96}]\), see also in \([2], [3, \text{p.14}]\). Such type inequalities are usually proved by means of growth estimates for Taylor coefficients, which were due to Faber and Littlewood \([10, \text{pp.93–96}], [11, \text{Ch.5, Th.2.31}]\). Below we give a direct and elementary proof of the estimates avoiding growth estimates for Taylor coefficients.
Lemma 1. Suppose that \( \alpha, \beta \in \mathbb{R} \) and
\[
J_{\alpha,\beta} = J_{\alpha,\beta}(r) := \int_{-\pi}^{\pi} \left| 1 - re^{i\theta} \right|^{-\alpha - 1} \left| \log \frac{e}{1 - re^{i\theta}} \right|^{-\beta} d\theta.
\]
Then for all \( 0 \leq r < 1 \)
\[
J_{\alpha,\beta} \approx \begin{cases} 
(1 - r)^{-\alpha} \left( \log \frac{e}{1 - r} \right)^{-\beta}, & \alpha > 0, \beta \in \mathbb{R}, \\
1, & \alpha < 0, \beta \in \mathbb{R},
\end{cases}
\]
(2.1)
\[
J_{0,\beta} \approx \begin{cases} 
\left( \log \frac{e}{1 - r} \right)^{1-\beta}, & \beta < 1, \\
1, & \beta > 1, \\
\log \left( e \log \frac{e}{1 - r} \right), & \beta = 1,
\end{cases}
\]
(2.2)
where the involved constants \( C = C(\alpha,\beta) > 0 \) depend only on \( \alpha, \beta \).

Proof. It suffices to prove all the estimates only for all \( r \) close enough to 1, and moreover for all \( z \in \mathbb{D} \) lying in a small neighborhood of 1. For the expression \( \left| 1 - re^{i\theta} \right| = \sqrt{(1 - r)^2 + 4r \sin^2 \frac{\theta}{2}} \), we have the simple estimate
\[
\frac{1}{\sqrt{2}} \left( 1 - r + 2\sqrt{r} \left| \theta \right| \right) \leq 1 - re^{i\theta} \leq 1 - r + \left| \theta \right|,
\]
in particular,
\[
\frac{1}{\pi} (1 - r + |\theta|) \leq 1 - re^{i\theta} \leq 1 - r + |\theta|, \quad \frac{1}{2} \leq r < 1.
\]
Define the ring sector \( E := \{ z = re^{i\theta} \in \mathbb{D} : \frac{9}{10} < r < 1, \left| \theta \right| < \frac{1}{2} \} \), so that \( |1 - z| < \frac{1}{2} (z \in E) \), and the following inequalities are valid:
\[
\log \frac{1}{1 - z} \leq \log \frac{1}{|1 - z|} + \frac{\pi}{2} \leq 5 \log \frac{1}{|1 - z|}, \quad z \in E,
\]
\[
\log \frac{1}{1 - z} \geq \log \frac{1}{|1 - z|} \geq \log \frac{1}{1 - r + |\theta|} \geq \log \frac{5}{3} > \frac{1}{2}, \quad z \in E.
\]
Assuming that \( \frac{9}{10} < r < 1 \) everywhere below and \( \alpha > 0 \), we begin with the proof of the first estimate in (2.1).

By the estimates (2.3) and (2.4), we obtain
\[
J_{\alpha,\beta} = \left( \int_{|\theta| > 1/2} + \int_{|\theta| < 1/2} \right) \frac{d\theta}{|1 - re^{i\theta}|^{\alpha + 1} \left| \log \frac{e}{1 - re^{i\theta}} \right|^\beta}
\approx C(\alpha, \beta) + C(\alpha, \beta) \int_0^{1/2} \frac{d\theta}{(1 - r + \theta)^{\alpha + 1} \left( \log \frac{1}{1 - r + \theta} \right)^\beta}
\]
\[
= C(\alpha, \beta) + C(\alpha, \beta) \int_{\log \frac{1}{1 - r} \theta}^{e^{\alpha t}} \frac{dt}{t^\beta} \approx \int_1^{e^{\alpha t}} \frac{dt}{t^\beta}.
\]
Here we have used the inequalities
\[ 0 < \log \frac{5}{3} < \log \frac{1}{3/2 - r} < \log 2, \quad \frac{9}{10} < r < 1. \]

Since by l’Hôpital rule,
\[ \int _{1}^{x} \frac{e^{\alpha t}}{t^\beta} dt \sim \frac{e^x}{\alpha x^\beta} \quad \text{as} \quad x \to +\infty \ (\alpha > 0), \]
we conclude that
\[ J_{\alpha, \beta} \approx \frac{e^{\alpha \log \frac{1}{1-r}}}{(\log \frac{1}{1-r})^\beta} = \frac{1}{(1 - r)^\alpha (\log \frac{1}{1-r})^\beta} \]
for all \( r \) sufficiently close to 1. It proves the first inequality in (2.1). The second inequality in (2.1) when \( \alpha < 0 \) follows from (2.5).

We now turn to the proof of (2.2) when \( \alpha = 0 \).

**Case** \( \beta < 1 \). Making use of the estimates (2.3) and (2.4), we deduce that
\[ J_{0, \beta} = \int_{|\theta| > 1/2} + \int_{|\theta| < 1/2} \approx C_\beta + C_\beta \int_{0}^{1/2} \frac{d\theta}{(1 - r + \theta)(\log \frac{1}{1-r+\theta})^\beta} \]
\[ = C_\beta + C_\beta \left[ (\log \frac{1}{1-r})^{1-\beta} - (\log \frac{1}{3/2-r})^{1-\beta} \right] \]
\[ \approx \left( \log \frac{1}{1-r} \right)^{1-\beta}, \]
(2.6)
where we have used the inequalities
\[ 0 < \left( \log \frac{5}{3} \right)^{1-\beta} < \left( \log \frac{1}{3/2-r} \right)^{1-\beta} < (2)^{1-\beta} < \left( \frac{1}{2} \log \frac{1}{1-r} \right)^{1-\beta} \]
for all \( \frac{9}{10} < r < 1 \).

**Case** \( \beta = 1 \). In view of (2.3) and (2.4), we obtain for all \( r \) close enough to 1
\[ J_{0, 1} \approx C + C \int_{0}^{1/2} \frac{d\theta}{(1 - r + \theta)(\log \frac{1}{1-r+\theta})} \]
\[ = C + C \left[ \log \left( \log \frac{1}{1-r} \right) - \log \left( \log \frac{1}{3/2-r} \right) \right] \]
\[ \approx \log \left( \log \frac{1}{1-r} \right), \]
(2.7)
where
\[ \log \log \frac{5}{3} < \log \log \frac{1}{3/2-r} < \log \log 2 < 0, \quad \frac{9}{10} < r < 1. \]

**Case** \( \beta > 1 \). Similarly to (2.5), we have
\[ J_{0, \beta} \approx C_\beta + C \int_{\log \frac{1}{1-r}}^{\log \frac{1}{1-r+\theta}} \frac{1}{t^\beta} dt \approx C_\beta + C \int_{1}^{\log \frac{1}{1-r}} \frac{1}{t^\beta} dt \approx 1. \]
(2.8)

Combining (2.6)–(2.8), we obtain (2.2). This completes the proof. \( \blacksquare \)
3. AN APPLICATION IN MIXED NORM SPACES

Define the following test function

\[ F_{b,c}(z) := (1 - z)^{-b} \left( \log \frac{e}{1 - z} \right)^{-c}, \quad z \in \mathbb{D}, \]

where \( b, c \in \mathbb{R} \). The functions \( F_{b,c} \) are very useful as typical functions in many function spaces, see, for example, [2]-[7]. The next lemma gives exact information on \( F_{b,c} \) to be in \( H(p, q, \alpha) \) or \( H_0(p, \infty, \alpha) \).

**Lemma 2.** Suppose that \( b, c \in \mathbb{R} \), \( 0 < p \leq \infty, 0 < q < \infty, \alpha > 0 \). Then

(a) \( F_{b,c} \) is in \( H(p, q, \alpha) \) if and only if \( b < \alpha + \frac{1}{p}, c \in \mathbb{R} \) or \( b = \alpha + \frac{1}{p}, c > \frac{1}{q} \).

(b) \( F_{b,c} \) is in \( H(p, \infty, \alpha) \) if and only if \( b < \alpha + \frac{1}{p}, c \in \mathbb{R} \) or \( b = \alpha + \frac{1}{p}, c \geq 0 \).

(c) \( F_{b,c} \) is in \( H_0(p, \infty, \alpha) \) if and only if \( b < \alpha + \frac{1}{p}, c \in \mathbb{R} \) or \( b = \alpha + \frac{1}{p}, c > 0 \).

**Proof.** The results follow from corresponding estimates of Lemma 1.

\[ M_p(F_{b,c}; r) \approx (1 - r)^{-b+1/p} \left( \log \frac{e}{1 - r} \right)^{-c}, \quad 0 \leq r < 1, \]

if \( 1/p < b \leq \alpha + 1/p \).

Lemma 2 enables us to prove the sharpness and strictness of the inclusions (i)-(vii) in Theorem 1.

**Theorem 2.** Suppose that \( 0 < p, q, p_0, q_0 \leq \infty, \alpha, \beta > 0 \) are arbitrary. Then:

(i) \( H(p, q, \alpha) \subset H(p, q, \beta), \beta > \alpha, \) is strict.

(ii) \( H(p, q, \alpha) \subset H(p_0, q, \alpha), p_0 < p, \) is strict.

(iii) \( H(p, q, \alpha) \subset H(p, q_0, \alpha), q < q_0, \) is strict, and the inclusion \( H(p, q, \alpha) \subset H_0(p, \infty, \alpha) \) is sharp in the sense that \( \alpha \) on the right cannot be decreased.

(iv) \( H(p, q, \alpha) \subset H(p_0, q, \beta), p \leq p_0, \) holds if and only if \( \beta \geq \alpha + \frac{1}{p} - \frac{1}{p_0} \).

(v) \( H(p, q, \alpha) \subset H(\infty, q_0, \beta), \beta > \alpha + 1/p, q_0 < q, \) is strict and sharp in the sense that \( \beta \) cannot be decreased.

(vi) \( H(p, q, \alpha) \subset H(p, q_0, \beta), \beta > \alpha, q_0 < q, \) is strict and sharp in the sense that \( \beta \) cannot be decreased.

(vii) \( H^p \subset H(p_0, q, 1/p - 1/p_0), p < p_0, p \leq q, \) is sharp in the sense that it fails for \( p > q \).

**Proof.** (i) The inclusion (i) is strict because of the function \( F_{\alpha+1/p,0} \) for \( q < \infty \), and the function \( F_{\beta+1/p,0} \) for \( q = \infty \).

(ii) The strictness of the inclusion (ii) is proved by the examples \( F_{\alpha+1/p,0} \) for \( 0 < q < \infty \), and \( F_{\alpha+1/p_0,0} \) for \( q = \infty \).

(iii) The strictness of the inclusion (iii) is proved by the examples \( F_{\alpha+1/p,0} \) for \( q_0 = \infty \), and \( F_{\alpha+1/p_1/q} \) for \( q_0 < \infty \). The sharpness of the second inclusion in (iii) means that the inclusion \( H(p, q, \alpha) \subset H_0(p, \infty, \alpha - \varepsilon) \) is false for any \( 0 < p \leq \infty, 0 < q < \infty, 0 < \varepsilon < \alpha \). The function \( F_{\alpha+1/p-\varepsilon/2,0}(z) \) gives a corresponding example.

(iv) The statement (iv) is proved in [11, p.733].

(v)-(vi) The inclusions (v) and (vi) are strict because of the example \( F_{\alpha+1/p,0} \). On the other hand, the inclusions (v) and (vi) are sharp for \( q_0 < q \) in the sense that \( \beta \) cannot be decreased. The function \( F_{\alpha+1/p,1/q_0}(z) \) gives a suitable example for both inclusions.
The inclusion (vii) is sharp in the sense that the condition \( p \leq q \) is essential, that is for \( p > q \) the inclusion (vii) fails. A corresponding example can be provided by the function \( F_{1/p, \lambda}(z) \), where \( 1/p < \lambda < 1/q \). Indeed, \( F_{1/p, \lambda}(z) \in H^p \), but \( F_{1/p, \lambda}(z) \) is not in \( H(p_0, q, 1/p - 1/p_0) \), by Lemma 2.

**REFERENCES**


