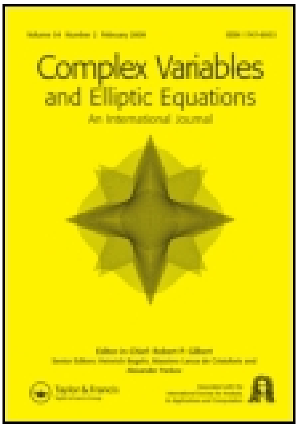

Complex Variables and Elliptic Equations: An International Journal

Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gcov20

Sharp inclusions and lacunary series in mixed-norm spaces on the polydisc

K.L. Avetisyan [a]

[a] Faculty of Physics, Yerevan State University, Alex Manoogian st. 1, Yerevan 0025, Armenia

Version of record first published: 20 Sep 2011.

Complex Variables and Elliptic Equations, 2013
Vol. 58, No. 2, 185–195, http://dx.doi.org/10.1080/17476933.2011.561839

Sharp inclusions and lacunary series in mixed-norm spaces on the polydisc

K.L. Avetisyan*

Faculty of Physics, Yerevan State University, Alex Manoogian st. 1, Yerevan 0025, Armenia

Communicated by R.P. Gilbert

(*Received 14 July 2009; final version received 5 February 2011*)

We establish the sharpness and strictness of continuous inclusions in mixed-norm spaces of n-harmonic functions on the unit polydisc of $\mathbb{C}^n$. To this end, we modify a wellknown counterexample of Hardy and Littlewood and give a characterization of lacunary series with Hadamard gaps in mixed-norm and weighted Hardy spaces.

Keywords: mixed-norm; polydisc; Hadamard gaps; lacunary series

AMS Subject Classifications: 32A37; 32A05

1. Introduction

Let $U^n = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j| < 1, 1 \le j \le n\}$ be the unit polydisc in $\mathbb{C}^n$ and $U^1 = \mathbb{D}$ the unit disc, and let $\mathbb{T}^n = \{w = (w_1, \ldots, w_n) \in \mathbb{C}^n : |w_j| = 1, 1 \le j \le n\}$ be the n-dimensional torus, the distinguished boundary of U^n. We will deal with n-harmonic functions on the polydisc U^n, i.e. functions harmonic in each variable z_j separately. Denote by $h(U^n)$ and $H(U^n)$ the sets of n-harmonic and holomorphic functions in U^n, respectively. The pth integral mean of a measurable function f in U^n is denoted as usual by

$$M_p(f; r) = \left\| f(r\cdot) \right\|_{L^p(\mathbb{T}^n; dm_n)}, \quad r = (r_1, \ldots, r_n) \in [0, 1)^n, \ 0 < p \le \infty,$$

where dm_n is the n-dimensional Lebesgue measure on $\mathbb{T}^n$. The collection of n-harmonic (holomorphic) functions f, for which $\|f\|_{h^p} = \sup_{r \in (0,1)^n} M_p(f; r) < +\infty$, is the usual Hardy space h^p (respectively H^p).

The quasi-normed space $h(p, q, \alpha)(0 < p, \ q \le \infty, \ \alpha = (\alpha_1, \ldots, \alpha_n))$ is the set of those functions f n-harmonic in the polydisc U^n, for which the quasi-norm

$$\|f\|_{p,q,\alpha} = \begin{cases} \left(\displaystyle\int_{(0,1)^n} \prod_{j=1}^{n} (1 - r_j)^{\alpha_j q - 1} M_p^q(f; r) \prod_{j=1}^{n} \mathrm{d}r_j \right)^{1/q}, & 0 < q < \infty, \\ \displaystyle\sup_{r \in (0,1)^n} \prod_{j=1}^{n} (1 - r_j)^{\alpha_j} M_p(f; r), & q = \infty, \end{cases}$$

*Email: avetkaren@ysu.am

is finite. If for each j, $1 \le j \le n$, $(1-r)^{\alpha} M_p(u;r) = o(1)$ as $r_j \to 1^-$, then we say that n-harmonic function u belongs to the little space $h_0(p, \infty, \alpha)$. For the subspaces consisting of holomorphic functions let

$$H(p,q,\alpha) = H(U^n) \cap (p,q,\alpha), \quad H_0(p,\infty,\alpha) = H(U^n) \cap h_0(p,\infty,\alpha).$$

For $p = q < \infty$ the spaces $H(p,q,\alpha)$, $h(p,q,\alpha)$ coincide with the wellknown weighted Bergman spaces, while for $q = \infty$ they are known as weighted Hardy or growth spaces. A lot of work is devoted to the mixed-norm and Bergman spaces consisting of holomorphic or pluriharmonic functions. We refer the reader to [1–4] for n-harmonic mixed-norm spaces on the polydisc.

In [1], among others, the following theorem is proved.

Theorem A *Let* $0 < p$, $q \le \infty$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$, α_j, $\beta_j \in \mathbb{R}$, $1 \le j \le n$. *Then the following inclusions are continuous*:

$$\begin{aligned}
&\text{(i) } h(p,q,\alpha) \subset h(p,q,\beta), \quad \beta_j \ge \alpha_j \ (1 \le j \le n),\\
&\text{(ii) } h(p,q,\alpha) \subset h(p_0,q,\alpha), \quad 0 < p_0 < p \le \infty,\\
&\text{(iii) } h(p,q,\alpha) \subset h(p,q_0,\alpha), \quad 0 < q < q_0 \le \infty,\\
&\text{(iv) } h(p,q,\alpha) \subset h(p_0,q,\beta), \quad \beta_j \ge \alpha_j + 1/p - 1/p_0,\ p \le p_0 \le \infty,\\
&\text{(v) } h(p,q,\alpha) \subset h(\infty,q_0,\beta), \quad \beta_j > \alpha_j + 1/p,\ 0 < q_0 \le \infty,\\
&\text{(vi) } h(p,q,\alpha) \subset h(p,q_0,\beta), \quad \beta_j > \alpha_j,\ 0 < q_0 \le \infty,\\
&\text{(vii) } H^p \subset H(p_0,q,1/p - 1/p_0), \quad 0 < p < p_0 \le \infty,\ p \le q \le \infty,\\
&\text{(viii) } h^p \subset h(p_0,q,1/p - 1/p_0), \quad 1 < p < p_0 \le \infty,\ p \le q \le \infty,\\
&\text{(ix) } h^p \subset h(p_0,q,\beta), \quad \beta_j > 1/p - 1/p_0,\ 0 < p < p_0 \le \infty,\\
&\text{(x) } \textit{If } u(p,q,\alpha),\ 0 < q < \infty, \quad \textit{then} \quad u \in h_0(p,\infty,\alpha).
\end{aligned}$$

It is natural to ask whether these inclusions are strict and sharp. The main purpose of this article is to prove the strictness and sharpness of the inclusions (i)–(x) in an appropriate sense.

Theorem 1 *Let* $0 < p$, $q \le \infty$, $\alpha_j > 0$, $1 \le j \le n$. *Then all the inclusions* (i)–(x) *are strict and best possible in a certain sense.*

2. Notation and preliminaries

We will use the conventional multi-index notations: $r\zeta = (r_1\zeta_1, \dots, r_n\zeta_n)$, $\zeta^{\alpha} = \zeta_1^{\alpha_1} \cdots \zeta_n^{\alpha_n}$, $dr = dr_1 \cdots dr_n$ for $\zeta \in \mathbb{C}^n$, $r \in [0,1)^n$, $\alpha = (\alpha_1, \dots, \alpha_n)$. Let $\mathbb{N}^n$, $\mathbb{Z}_+^n$ denote the sets of all n-tuples of positive integers and nonnegative integers, respectively.

Throughout this article, the letters $C(\alpha, \beta, \dots), C_{\alpha}$, etc., stand for positive different constants depending only on the parameters indicated. For A, $B > 0$ the notation $A \approx B$ denotes the two-sided estimate $c_1 A \le B \le c_2 A$ with some inessential positive constants c_1 and c_2 independent of the variable involved.

Define the following test function:

$$F_{b,c}(z) := \prod_{j=1}^{n} (1 - z_j)^{-b_j} \left(\log \frac{e}{1 - z_j} \right)^{-c_j}, \quad z \in U^n,$$

where $b=(b_1,\dots,b_n)$, $c=(c_1,\dots,c_n)$, b_j, $c_j\in\mathbb{R}$. The following lemmas can be proved by a direct estimation, for the proof see [4, Section 2.3] or [5].

LEMMA 1 *Suppose that* $n=1$, b, $c\in\mathbb{R}$, $0<p\le\infty$, $0<q<\infty$, $\alpha>0$. *Then*

(a) $F_{b,c}$ *is in* $H(p,q,\alpha)$ *if and only if* $b<\alpha+\frac{1}{p}, c\in\mathbb{R}$ *or* $b=\alpha+\frac{1}{p}, c>\frac{1}{q}$.
(b) $F_{b,c}$ *is in* $H(p,\infty,\alpha)$ *if and only if* $b<\alpha+\frac{1}{p}, c\in\mathbb{R}$ *or* $b=\alpha+\frac{1}{p}, c\ge 0$.
(c) $F_{b,c}$ *is in* $H_0(p,\infty,\alpha)$ *if and only if* $b<\alpha+\frac{1}{p}, c\in\mathbb{R}$ *or* $b=\alpha+\frac{1}{p}, c>0$.

LEMMA 2 *Suppose* $\alpha>0, p>0, a_k\ge 0, I_k=\{j\in\mathbb{N}; 2^k\le j<2^{k+1}\}, k=1,2,\dots$. *Then*

$$\int_0^1 (1-r)^{\alpha-1}\left(\sum_{k=1}^{\infty} a_k r^k\right)^p \mathrm{d}r \approx \sum_{k=0}^{\infty}\frac{1}{2^{\alpha k}}\left(\sum_{j\in I_k} a_j\right)^p,$$

where the involved constants $C=C(p,\alpha)$ *depend only on* p *and* α.

LEMMA 3 *Let* $p>0$, $a_k\ge 0$, $N\in\mathbb{N}$. *Then*

$$\min\{1,N^{p-1}\}\left(\sum_{k=1}^{N} a_k^p\right)\le\left(\sum_{k=1}^{N} a_k\right)^p\le\max\{1,N^{p-1}\}\left(\sum_{k=1}^{N} a_k^p\right).$$

Lemma 2 is due to Mateljević and Pavlović [6], while Lemma 3 is an easy consequence of Hölder's inequality.

3. Lacunary series in $H(p,q,\alpha)$

This section can be viewed as a continuation of [2] where lacunary series in growth spaces $H(p,\infty,\alpha)$ are studied. Lacunary series in classical function spaces such as Bloch, Bergman, Besov, Dirichlet, Q-type spaces, have been extensively studied recently [7–21]. Recall that a sequence $\{m_k\}_{k=0}^{\infty}$ of positive integers is said to be lacunary (or Hadamard) if there exists a constant $\lambda>1$ such that $\frac{m_{k+1}}{m_k}\ge\lambda$ for all $k=0,1,2,\dots$. A corresponding power series is called a lacunary series. For the polydisc we will consider the lacunary series of the form

$$f(z)=\sum_{k\in\mathbb{Z}_+^n} a_{k_1\dots k_n} z_1^{m_{1k_1}}\cdots z_n^{m_{nk_n}},\quad z\in U^n. \tag{1}$$

The following theorem is an extension of classical Paley–Kahane–Khintchine inequalities to the polydisc.

THEOREM *B*. ([2]) *Let* $\{m_{j,k_j}\}_{k_j=0}^{\infty}$, $j=1,2,\dots,n$ *be arbitrary lacunary sequences and* f *be a holomorphic function in* U^n *given by a convergent lacunary series* (1). *Then for any* p, $0<p<\infty$, f *is in Hardy space* H^p *if and only if* $\{a_k\}\in\ell^2$. *Moreover, the corresponding norms are equivalent*: $\|f\|_{H^p}\approx(\sum_{k\in\mathbb{Z}_+^n}|a_{k_1\dots k_n}|^2)^{1/2}$, *where the involved constants are independent of* f.

Let $\mathcal{R}^\beta$ be the Hadamard operator of fractional integro-differentiation of order $\beta=(\beta_1,\dots,\beta_n)$, $\beta_j\in\mathbb{R}$,

$$\mathcal{R}^\beta f(z)=\sum_{k\in\mathbb{Z}_+^n}(1+k_1)^{\beta_1}\cdots(1+k_n)^{\beta_n} a_{k_1\dots k_n} z_1^{k_1}\cdots z_n^{k_n}.$$

The following two theorems characterize lacunary series in weighted Hardy spaces $H(p,\infty,\alpha)$ and are essentially proved in [2].

Theorem 2 *Let* $\{m_{j,k_j}\}_{k_j=0}^{\infty}$, $j=1,2,\dots,n$ *be arbitrary lacunary sequences,* $\alpha=(\alpha_1,\dots,\alpha_n)$, $\alpha_j>0$, $\beta=(\beta_1,\dots,\beta_n)$, $\beta_j\in\mathbb{R}$, *and f be a holomorphic function in* U^n *given by a convergent lacunary series* (1). *Then the following statements are equivalent*:

(a) $\mathcal{R}^\beta f\in H(\infty,\infty,\alpha)$;

(b) $\mathcal{R}^\beta f\in H(p,\infty,\alpha)$ *for some* $p\in(0,\infty)$;

(c) $\mathcal{R}^\beta f\in H(p,\infty,\alpha)$ *for all* $p\in(0,\infty)$;

(d) $\displaystyle\sup_{k\in\mathbb{Z}_+^n}\frac{|a_k|}{m_{1,k_1}^{\alpha_1-\beta_1}\cdots m_{n,k_n}^{\alpha_n-\beta_n}}<+\infty.$

Also, corresponding norms are equivalent:

$$\|\mathcal{R}^\beta f\|_{\infty,\infty,\alpha}\approx\|\mathcal{R}^\beta f\|_{p,\infty,\alpha}\approx\sup_{k\in\mathbb{Z}_+^n}\frac{|a_k|}{m_{1,k_1}^{\alpha_1-\beta_1}\cdots m_{n,k_n}^{\alpha_n-\beta_n}}.$$

The next assertion is a 'little oh' version of Theorem 2.

Theorem 3 *Let* $\{m_{j,k_j}\}_{k_j=0}^{\infty}$, $j=1,2,\dots,n$ *be arbitrary lacunary sequences,* $\alpha=(\alpha_1,\dots,\alpha_n)$, $\alpha_j>0$, $\beta=(\beta_1,\dots,\beta_n)$, $\beta_j\in\mathbb{R}$, *and f be a holomorphic function in* U^n *given by a convergent lacunary series* (1). *Then the following statements are equivalent*:

(a) $\mathcal{R}^\beta f\in H_0(\infty,\infty,\alpha)$;

(b) $\mathcal{R}^\beta f\in H_0(p,\infty,\alpha)$ *for some* $p\in(0,\infty)$;

(c) $\mathcal{R}^\beta f\in H_0(p,\infty,\alpha)$ *for all* $p\in(0,\infty)$;

(d) $\displaystyle\lim_{k_j\to\infty}\frac{a_k}{m_{1,k_1}^{\alpha_1-\beta_1}\cdots m_{n,k_n}^{\alpha_n-\beta_n}}=0$ *for each* $1\le j\le n$.

Proof of Theorems 2 and 3 The series expansion of $\mathcal{R}^\beta f$ is lacunary, too,

$$\mathcal{R}^\beta f(z)=\sum_{k\in\mathbb{Z}_+^n}(1+m_{k_1})^{\beta_1}\cdots(1+m_{k_n})^{\beta_n}a_k\,z_1^{m_{k_1}}\cdots z_n^{m_{k_n}}.$$

So, it suffices to apply Theorems 3 and 4 in [2] to the function $\mathcal{R}^\beta f$. ■

Remark 1 It is easily seen that Theorems 2 and 3 cover all (weighted) Bloch and little Bloch spaces and generalize and improve the corresponding results in [7,8,11,16,17,21,22]. Versions of Theorems 2 and 3 for the unit ball in $\mathbb{C}^n$ are given in [15].

The main result of this section is the following theorem extending Theorem 2 to all $q\in(0,\infty)$.

THEOREM 4 *Let* $\{m_{j,k_j}\}_{k_j=0}^{\infty}$, $j=1,2,\dots,n$ *be arbitrary lacunary sequences*, $0<q<\infty$, $\alpha=(\alpha_1,\dots,\alpha_n)$, $\alpha_j>0$, *and* f *be a holomorphic function in* U^n *given by a convergent lacunary series* (1). *Then the following statements are equivalent*:

$$(a)\ f\in H(\infty,q,\alpha);$$
$$(b)\ f\in H(p,q,\alpha)\quad \textit{for some } p\in(0,\infty);$$
$$(c)\ f\in H(p,q,\alpha)\quad \textit{for all } p\in(0,\infty);$$
$$(d)\ \sum_{k\in\mathbb{Z}_+^n}\frac{|a_{k_1\dots k_n}|^q}{m_{k_1}^{\alpha_1 q}\cdots m_{k_n}^{\alpha_n q}}<+\infty.$$

Also, corresponding norms are equivalent:

$$\|f\|_{\infty,q,\alpha}\approx\|f\|_{p,q,\alpha}\approx\left(\sum_{k\in\mathbb{Z}_+^n}\frac{|a_{k_1\dots k_n}|^q}{m_{k_1}^{\alpha_1 q}\cdots m_{k_n}^{\alpha_n q}}\right)^{1/q}.$$

Proof We may assume that $n=2$. Let $f(z_1,z_2)=\sum_{j,k=0}^{\infty}a_{jk}\,z_1^{m_j}z_2^{n_k}$.

The implication (a) $\Rightarrow$ (b) is obvious because of the elementary inclusion $H(\infty,q,\alpha)\subset H(p,q,\alpha)$.

The implication (b) $\Rightarrow$ (c) follows from Theorem *B* which asserts that $M_p(f;r_1,r_2)\approx M_s(f;r_1,r_2)$ for any s, $0<s<\infty$.

For proving the implication (c) $\Rightarrow$ (d), assume that $f\in H(2,q,\alpha)$. Then, by Theorem B, we have

$$\begin{aligned}\|f\|_{2,q,\alpha}^q&=\int_0^1\int_0^1(1-r)^{\alpha q-1}\left(\int_{\mathbb{T}^2}\left|\sum_{j,k=0}^{\infty}a_{jk}\,r_1^{m_j}\zeta_1^{m_j}r_2^{n_k}\zeta_2^{n_k}\right|^2\mathrm{d}m_2(\zeta)\right)^{q/2}\mathrm{d}r_1\,\mathrm{d}r_2\\&\ge C\int_0^1\int_0^1(1-r)^{\alpha q-1}\left(\sum_{j,k=0}^{\infty}|a_{jk}|^2\,r_1^{m_j}r_2^{n_k}\right)^{q/2}\mathrm{d}r_1\,\mathrm{d}r_2\\&=C\int_0^1(1-r_2)^{\alpha_2 q-1}\int_0^1(1-r_1)^{\alpha_1 q-1}\left(\sum_{j=0}^{\infty}G_j(r_2)r_1^{m_j}\right)^{q/2}\mathrm{d}r_1\,\mathrm{d}r_2,\end{aligned}$$

where $G_j(r_2):=\sum_{k=0}^{\infty}|a_{jk}|^2\,r_2^{n_k}$. Applying Lemmas 2 and 3, and then Fubini's theorem, we obtain

$$\begin{aligned}\|f\|_{p,q,\alpha}^q&\ge C\int_0^1(1-r_2)^{\alpha_2 q-1}\sum_{m=0}^{\infty}\frac{1}{2^{m\alpha_1 q}}\left(\sum_{m_j\in I_m}G_j(r_2)\right)^{q/2}\mathrm{d}r_2\\&\ge C\int_0^1(1-r_2)^{\alpha_2 q-1}\sum_{m=0}^{\infty}\sum_{m_j\in I_m}\frac{1}{m_j^{\alpha_1 q}}\left(G_j(r_2)\right)^{q/2}\mathrm{d}r_2\\&\ge C\int_0^1(1-r_2)^{\alpha_2 q-1}\sum_{j=0}^{\infty}\frac{1}{m_j^{\alpha_1 q}}\left(G_j(r_2)\right)^{q/2}\mathrm{d}r_2\end{aligned}$$

$$= C\sum_{j=0}^{\infty}\frac{1}{m_j^{\alpha_1 q}}\int_0^1 (1-r_2)^{\alpha_2 q-1}\left(\sum_{k=0}^{\infty}|a_{jk}|^2\, r_2^{n_k}\right)^{q/2} \mathrm{d}r_2$$
$$\geq C\sum_{j=0}^{\infty}\frac{1}{m_j^{\alpha_1 q}}\sum_{m=0}^{\infty}\frac{1}{2^{m\alpha_2 q}}\left(\sum_{n_k\in I_m}|a_{jk}|^2\right)^{q/2}$$
$$\geq C\sum_{j=0}^{\infty}\frac{1}{m_j^{\alpha_1 q}}\sum_{m=0}^{\infty}\left(\sum_{n_k\in I_m}\frac{|a_{jk}|^q}{n_k^{\alpha_2 q}}\right) = C\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{|a_{jk}|^q}{m_j^{\alpha_1 q}\, n_k^{\alpha_2 q}},$$

where $C = C(p, q, \alpha_1, \alpha_2, \lambda_1, \lambda_2)$.

Proceeding to the implication (d) $\Rightarrow$ (a), we write

$$\|f\|_{\infty,q,\alpha}^q = \int_0^1\int_0^1 (1-r)^{\alpha q-1}\sup_{\zeta\in\mathbb{T}^2}\left|\sum_{j,k=0}^{\infty} a_{jk}\, r_1^{m_j}\zeta_1^{m_j} r_2^{n_k}\zeta_2^{n_k}\right|^q \mathrm{d}r_1\mathrm{d}r_2$$
$$\leq C\int_0^1\int_0^1 (1-r)^{\alpha q-1}\left(\sum_{j,k=0}^{\infty}|a_{jk}|\, r_1^{m_j} r_2^{n_k}\right)^q \mathrm{d}r_1\mathrm{d}r_2.$$

Estimating as above by using Lemmas 2 and 3 leads to

$$\|f\|_{\infty,q,\alpha}^q \leq C\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{|a_{jk}|^q}{m_j^{\alpha_1 q}\, n_k^{\alpha_2 q}},$$

as desired. This completes the proof of Theorem 4. ■

The following is a generalization and an immediate consequence of Theorem 4.

Corollary 1 *Let* $\{m_{j,k_j}\}_{k_j=0}^{\infty}$, $j = 1, 2, \ldots, n$, *be arbitrary lacunary sequences*, $0 < q < \infty$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j > 0$, $\beta = (\beta_1, \ldots, \beta_n)$, $\beta_j \in \mathbb{R}$, *and* f *be a holomorphic function in* U^n *given by a convergent lacunary series* (1). *Then the following statements are equivalent*:

(a) $\mathcal{R}^\beta f \in H(\infty, q, \alpha)$;

(b) $\mathcal{R}^\beta f \in H(p, q, \alpha)$ *for some* $p \in (0, \infty)$;

(c) $\mathcal{R}^\beta f \in H(p, q, \alpha)$ *for all* $p \in (0, \infty)$;

(d) $\displaystyle\sum_{k\in\mathbb{Z}_+^n}\frac{|a_{k_1\ldots k_n}|^q}{m_{k_1}^{(\alpha_1-\beta_1)q}\cdots m_{k_n}^{(\alpha_n-\beta_n)q}} < +\infty.$

Also, corresponding norms are equivalent:

$$\|\mathcal{R}^\beta f\|_{\infty,q,\alpha} \approx \|\mathcal{R}^\beta f\|_{p,q,\alpha} \approx \left(\sum_{k\in\mathbb{Z}_+^n}\frac{|a_{k_1\ldots k_n}|^q}{m_{k_1}^{(\alpha_1-\beta_1)q}\cdots m_{k_n}^{(\alpha_n-\beta_n)q}}\right)^{1/q}.$$

Remark 2 In [16,21], versions of Theorem 4 are proved for weighted Bergman spaces in the unit disc, ball and polydisc. In [17], the equivalence of (b) and (d) in Corollary 1 is proved for ordinary derivatives of functions holomorphic in the unit disc.

Remark 3 Substituting $\beta-\alpha$ ($\beta_j > \alpha_j$) in place of α, we see that Corollary 1 covers all Besov spaces and generalizes the previous similar results in [7,8,21,23].

4. Pointwise estimates in $H(p, q, \alpha)$

We now turn to some pointwise estimates for lacunary series. It is well known that arbitrary function $f \in H(p, q, \alpha)$ satisfies the pointwise estimate

$$|f(z)| \le C(p, q, \alpha, n) \frac{\|f\|_{p,q,\alpha}}{(1-|z|)^{\alpha+1/p}}, \quad z \in U^n, \tag{2}$$

where the exponent $\alpha + 1/p$ in (2) is best possible for general functions. Indeed, the inclusion $H(p, q, \alpha) \subset H(\infty, \infty, \alpha + 1/p - \varepsilon)$ is false for any small $\varepsilon > 0$. The function $F_{\alpha+1/p,2/q}$ is in $H(p, q, \alpha)$, by Lemma 1, but $F_{\alpha+1/p,2/q} \notin H(\infty, \infty, \alpha + 1/p - \varepsilon)$.

The following theorem shows that lacunary series in $H(p, q, \alpha)$ grow more slowly near the distinguished boundary than general functions of $H(p, q, \alpha)$.

THEOREM 5 *Let* $0 < p,\ q \le \infty$, $\{m_{j,k_j}\}_{k_j=0}^{\infty}$, $j = 1, 2, \dots, n$ *be arbitrary lacunary sequences*, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j > 0$, *and* f *be a function of* $H(p, q, \alpha)$ *given by a convergent lacunary series* (1). *Then*

$$|f(z)| \le C(\lambda, p, q, \alpha, n) \frac{\|f\|_{p,q,\alpha}}{(1-|z|)^{\alpha}}, \quad z \in U^n, \tag{3}$$

where the exponents α_j *cannot be decreased.*

Proof By the inclusion (iii) of Theorem A, $H(p, q, \alpha) \subset H(p, \infty, \alpha)$. Since the function $f \in H(p, \infty, \alpha)$ is given by a convergent lacunary series, we obtain by Theorem 2 that

$$(1-|z|)^{\alpha} |f(z)| \le \|f\|_{\infty,\infty,\alpha} \approx \|f\|_{p,\infty,\alpha} \le C \|f\|_{p,q,\alpha}, \quad z \in U^n,$$

as desired.

Now we will show that no one of the exponents α_j may be decreased in (3). We assume that there exists some β_1, $0 < \beta_1 < \alpha_1$, such that for every lacunary series $f \in H(p, q, \alpha)$ there exists a constant $C > 0$ such that

$$|f(z)| \le \frac{C\|f\|_{p,q,\alpha}}{(1-|z_1|)^{\beta_1}(1-|z_2|)^{\alpha_2} \dots (1-|z_n|)^{\alpha_n}}, \quad z \in U^n,$$

that is $f \in H(\infty, \infty, (\beta_1, \alpha_2, \dots, \alpha_n))$. Then choosing a multiindex $\gamma = (\gamma_1, \dots, \gamma_n)$ such that $\beta_1 < \gamma_1 < \alpha_1$ and $0 < \gamma_j < \alpha_j$ for all $2 \le j \le n$, define the example

$$f_0(z) = \sum_{k \in \mathbb{Z}_+^n} 2^{k_1 \gamma_1} \dots 2^{k_n \gamma_n} z_1^{2^{k_1}} \dots z_n^{2^{k_n}}, \quad z \in U^n.$$

By Theorems 2 and 4, $f_0 \in H(p, q, \alpha)$, but, on the other hand, $f_0 \notin H(\infty, \infty, (\beta_1, \alpha_2, \dots, \alpha_n))$. This contradiction completes the proof of the theorem. ■

Although we cannot decrease the exponents α_j in (3), however we can improve the estimates (3) in the following sense.

THEOREM 6 *Let* $\{m_{j,k_j}\}_{k_j=0}^{\infty}, j=1,2,\dots,n$ *be arbitrary lacunary sequences,* $0<p\le\infty$, $0<q<\infty$, $\alpha=(\alpha_1,\dots,\alpha_n)$, $\alpha_j>0$, *and* f *be a function of* $H(p,q,\alpha)$ *given by a convergent lacunary series* (1). *Then for each* $1\le j\le n$ *we have*

$$f(z)=o\left(\frac{1}{(1-|z|)^{\alpha}}\right) \quad \text{as } |z_j|\to 1^-. \tag{4}$$

Proof By (x) of Theorem A, $f\in H_0(p,\infty,\alpha)$. Then Theorem 3 with $\beta_j=0$, asserts that $f\in H_0(p,\infty,\alpha)$ is equivalent to $f\in H_0(\infty,\infty,\alpha)$ for lacunary power series f, and the relations (4) follow. Of course, the exponents α_j in (4) cannot be decreased because of Theorem 5. ■

5. A Hardy–Littlewood-type counterexample

Hardy and Littlewood [24, p. 416] defined the following important function

$$f(z):=\frac{e^{i\pi m/2}}{(1-z)^{1/p}}, \quad p=\frac{1}{m+1}, \quad m\in\mathbb{N},\ z\in\mathbb{D}, \tag{5}$$

as an example of holomorphic function in $\mathbb{D}$ whose real part is in Hardy space $h^p(\mathbb{D})$, $0<p<1$, but $f\notin H^p(\mathbb{D})$; moreover, $M_p^p(f;r)\approx\log\frac{e}{1-r}$ for all $0\le r<1$. Later, Duren and Shields [25, p. 257] applied the example (5) to prove the falsity of the inclusion

$$h^p\subset h(1,1,1/p-1), \quad 0<p<1, \tag{6}$$

on the unit disc $\mathbb{D}$. For a polydisc version of (6), see [26, p. 140]. Now we are able to improve the result of Duren and Shields.

THEOREM 7 *Let* $0<p<1$, $p<p_0\le\infty$, $\beta_j>\frac{1}{p}-\frac{1}{p_0}$ *for all* $2\le j\le n$. *Then the inclusion*

$$h^p\subset h\left(p_0,\infty,\left(\frac{1}{p}-\frac{1}{p_0},\beta_2,\dots,\beta_n\right)\right) \tag{7}$$

is false at least for $p=\frac{1}{m+1}$, $m=1,2,\dots$. *Hence, the inclusion*

$$h^p\subset h\left(p_0,q,\left(\frac{1}{p}-\frac{1}{p_0},\beta_2,\dots,\beta_n\right)\right) \tag{8}$$

is false for $p=\frac{1}{m+1}$ $(m=1,2,\dots)$ *and each* q, $0<q\le\infty$.

Proof In view of the inclusion (iii) in Theorem A, it suffices to prove only the falsity of (7). Define the functions

$$g(z_1,\dots,z_n):=\frac{e^{i\pi(m+1)/2}}{(1-z_1)^{1/p}}\log\frac{1}{1-z_1}, \quad z\in U^n,$$
$$u(z_1,\dots,z_n):=\operatorname{Re}g(z_1,\dots,z_n), \quad z\in U^n,$$

which are modifications of (5). It is easily seen by Lemma 1 that

$$|g(z)|=|F_{1/p,-1}(z_1)| \quad \text{and} \quad g\notin H\left(p_0,\infty,\left(\frac{1}{p}-\frac{1}{p_0},\beta_2,\dots,\beta_n\right)\right).$$

Then $u \notin h\big(p_0, \infty, (\frac{1}{p} - \frac{1}{p_0}, \beta_2, \ldots, \beta_n)\big)$ since the operator of pluriharmonic conjugation is bounded in mixed-norm spaces $h(p, q, \alpha)$ on the polydisc (see, e.g. [3]). On the other hand, assuming $z_j = r_j e^{i\theta_j}$, we get

$$\left|u(e^{i\theta_1}, \ldots, e^{i\theta_n})\right| = \left|\mathrm{Re}\ \frac{e^{i\pi(m+1)/2}}{(1-e^{i\theta_1})^{1/p}} \log\frac{1}{1-e^{i\theta_1}}\right|$$
$$= \frac{1}{\left|2\sin\frac{\theta_1}{2}\right|^{m+1}}\left|\cos\frac{\theta_1(m+1)}{2}\log|1-e^{i\theta_1}| + \sin\frac{\theta_1(m+1)}{2}\arg(1-e^{i\theta_1})\right|.$$

Consequently,

$$\left|u(e^{i\theta_1}, \ldots, e^{i\theta_n})\right| \le \frac{C_m}{|\theta_1|^m} = \frac{C_p}{|\theta_1|^{1/p-1}}, \quad e^{i\theta} \in \mathbb{T}^n,$$

so $u \in h^p(U^n)$. Thus, the falsity of the inclusion (7) is proved. ■

Remark 4 Note that the example of Hardy and Littlewood (5) is not sufficient for proving the falsity of the inclusion $h^p \subset h\big(p_0, \infty, \frac{1}{p} - \frac{1}{p_0}\big)$. In fact, we have proved that inclusion (ix) in Theorem A is sharp in the sense that no other choice for the components β_j in (ix) is permitted.

Remark 5 The question of the falsity of the inclusions (7) and (8) for values of $p \in (0, 1)$ other than $p = \frac{1}{m+1}$ $(m = 1, 2, \ldots)$ remains as an open question.

6. Proof of Theorem 1

(i) The inclusion (i) is strict if $\beta_j > \alpha_j$ for anyone j, say $\beta_1 > \alpha_1$. Indeed, according to Lemma 1 the holomorphic function $F_{\alpha+1/p,0}$ belongs to $h(p, q, \beta)$, but not to $h(p, q, \alpha)$, $0 < p \le \infty$, $0 < q < \infty$. Also, the holomorphic function $F_{\beta+1/p,0}$ belongs to $h(p, \infty, \beta)$, but not to $h(p, \infty, \alpha)$, $0 < p \le \infty$.

(ii) The strictness of the inclusion (ii) is proved by the examples $F_{\alpha+1/p,0}$ for $0 < q < \infty$ and $F_{\alpha+1/p_0,0}$ for $q = \infty$.

(iii) The strictness of the inclusion (iii) is proved by the examples $F_{\alpha+1/p,0}$ for $q_0 = \infty$, and $F_{\alpha+1/p,1/q}$ for $0 < q < q_0 < \infty$.

(iv) The sharpness of the inclusion (iv) in a strong form is proved in [1, p. 733]. Namely, the condition $\beta_j \ge \alpha_j + 1/p - 1/p_0$ $(1 \le j \le n)$ is necessary and sufficient for the inclusion (iv). The strictness of the inclusion (iv) is proved by the example

$$f_1(z) = \sum_{k \in \mathbb{Z}_+^n} k_1 \cdots k_n 2^{k_1\alpha_1} \cdots 2^{k_n\alpha_n} z_1^{2^{k_1}} \cdots z_n^{2^{k_n}}, \quad z \in U^n,$$

which is in $H(p, q, \alpha)$, but not in $H(p_0, q, \alpha + 1/p - 1/p_0)$, by Theorems 4 and 2.

(v)–(vi) The inclusions (v) and (vi) are strict because of the example f_1 or $F_{\alpha+1/p,0}$. On the other hand, the inclusions (v) and (vi) are sharp for $q_0 < q$ in the sense that no other choice for the components β_j is permitted. The function $F_{\alpha+1/p,1/q_0}$ gives a suitable example.

(vii)–(ix) The strictness of the inclusions (vii)–(ix) can be proved by the example $f_2(z) = \sum_{k \in \mathbb{Z}_+^n} z_1^{2^{k_1}} \cdots z_n^{2^{k_n}}, \quad z \in U^n$.

The inclusions (vii) and (viii) are sharp in the sense that the condition $p \le q$ is essential, that is for $p > q$ the inclusions (vii) and (viii) are false. A corresponding example can be provided by the function $F_{1/p,\lambda}$, where $1/p < \lambda < 1/q$. Indeed, $F_{1/p,\lambda}$ is in H^p but not in $H(p_0, q, 1/p - 1/p_0)$, by Lemma 1.

On the other hand, the inclusions (viii) and (ix) are sharp in the sense that the parameter p in (viii) cannot be decreased, and no other choice for the components β_j in (ix) is permitted, by Theorem 6.

(x) The strictness of the inclusion (x) follows from the example $F_{\alpha+1/p,1/q}$. Indeed, by Lemma 1, $F_{\alpha+1/p,1/q} \in H_0(p, \infty, \alpha)$, but $F_{\alpha+1/p,1/q} \notin H(p, q, \alpha)$.

The sharpness of the inclusion (x) is understood in the sense that none of the components α_j can be decreased. Namely, the inclusion

$$h\big(p, q, (\alpha_1, \alpha_2)\big) \subset h_0\big(p, \infty, (\alpha_1 - \varepsilon, \alpha_2)\big)$$

is false for any $0 < p \le \infty$, $0 < q < \infty$, $0 < \varepsilon < \alpha$. The function $F_{\alpha+1/p-\varepsilon/2,0}$ gives a corresponding example. ■

This article represents a part of the author's Dc.Sc. thesis [4] written at Yerevan State University.

References

[1] K.L. Avetisyan, *Continuous inclusions and Bergman type operators in n-harmonic mixed norm spaces on the polydisc*, J. Math. Anal. Appl. 291 (2004), pp. 727–740.

[2] K.L. Avetisyan, *Hardy–Bloch-type spaces and lacunary series on the polydisk*, Glasgow Math. J. 49 (2007), pp. 345–356.

[3] K.L. Avetisyan, *Weighted integrals and Bloch spaces of n-harmonic functions on the polydisc*, Potential Anal. 29 (2008), pp. 49–63.

[4] K.L. Avetisyan, *Weighted spaces of harmonic and holomorphic functions*, Armenian J. Math. 2(4) (2009). Available at http://www.flib.sci.am/eng/journal/Math/MV2ISS4.html

[5] K.L. Avetisyan, *A note on mixed norm spaces of analytic functions*, Austral. J. Math. Anal. Appl. to appear.

[6] M. Mateljević and M. Pavlović, *L^p-behaviour of power series with positive coefficients and Hardy spaces*, Proc. Amer. Math. Soc. 87 (1983), pp. 309–316.

[7] R. Aulaskari, J. Xiao, and R. Zhao, *On subspaces and subsets of BMOA and UBC*, Analysis 15 (1995), pp. 101–121.

[8] R. Aulaskari and G. Csordas, *Besov spaces and the $Q_{q,0}$ classes*, Acta Sci. Math. (Szeged) 60 (1995), pp. 31–48.

[9] S.M. Buckley, P. Koskela, and D. Vukotić, *Fractional integration, differentiation, and weighted Bergman spaces*, Math. Proc. Cambridge Philos. Soc. 126 (1999), pp. 369–385.

[10] J.J. Donaire, D. Girela, and D. Vukotić, *On univalent functions in some Möbius invariant spaces*, J. Reine Angew. Math. 553 (2002), pp. 43–72.

[11] D. Girela and J.A. Peláez, *Integral means of analytic functions*, Ann. Acad. Sci. Fenn. 29 (2004), pp. 459–469.

[12] D. Girela and J.A. Peláez, *Growth properties and sequences of zeros of analytic functions in spaces of Dirichlet type*, J. Austral. Math. Soc. 80 (2006), pp. 397–418.
[13] D. Girela and J.A. Peláez, *Carleson measures for spaces of Dirichlet type*, Integral Eqns Oper. Theory 55 (2006), pp. 415–427.
[14] S.G. Krantz and S. Stević, *On the iterated logarithmic Bloch space on the unit ball*, Nonlinear Anal. 71 (2009), pp. 1772–1795.
[15] S. Li and S. Stević, *Weighted-Hardy functions with Hadamard gaps on the unit ball*, Appl. Math. Comput. 212 (2009), pp. 229–233.
[16] S. Stević, *A generalization of a result of Choa on analytic functions with Hadamard gaps*, J. Korean Math. Soc. 43 (2006), pp. 579–591.
[17] S. Stević, *On Bloch-type functions with Hadamard gaps*, Abstract Appl. Anal. 2007 (2007), Article ID 39176. Available at http://www.hindiawi.com/journals/aaa/2007/039176.cta.html
[18] S. Stević, *Bloch-type functions with Hadamard gaps*, Appl. Math. Comput. 208 (2009), pp. 416–422.
[19] S. Stević, *On Carleson measures and $F(p, q, s)$ space on the unit ball*, J. Comput. Anal. Appl. 12 (2010), pp. 313–320.
[20] H. Wulan and K. Zhu, *Bloch and BMO functions in the unit ball*, Complex Var. Elliptic Equ. 53 (2008), pp. 1009–1019.
[21] K. Zhu, *A class of Möbius invariant function spaces*, Illinois J. Math. 51 (2007), pp. 977–1002.
[22] S. Yamashita, *Gap series and α-Bloch functions*, Yokohama Math. J. 28 (1980), pp. 31–36.
[23] J. Miao, *A property of analytic functions with Hadamard gaps*, Bull. Austral. Math. Soc. 45 (1992), pp. 105–112.
[24] G.H. Hardy and J.E. Littlewood, *Some properties of conjugate functions*, J. Reine Angew. Math. 167 (1932), pp. 405–423.
[25] P. Duren and A. Shields, *Properties of $H^p(0 < p < 1)$ and its containing Banach space*, Trans. Amer. Math. Soc. 141 (1969), pp. 255–262.
[26] J. Mitchell, *Lipschitz spaces of holomorphic and pluriharmonic functions on bounded symmetric domains in $\mathbb{C}^N$ $(N > 1)$*, Ann. Pol. Math. 39 (1981), pp. 131–141.