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REAL AND COMPLEX ANALYSIS

Divergent Triangular Sums of Double Trigonometric Fourier Series

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Abstract—In this paper we consider some problems on divergence of triangular and sectoral sums for double trigonometric Fourier series. An example of a function from $\cap_{1\le p<\infty} L^p$ with almost everywhere divergence triangular partial sums of double trigonometric Fourier series is constructed.

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1. INTRODUCTION

L. Carleson [3] has proved that the partial sums of a trigonometric Fourier series of any function $f \in L^2(\mathbb{T})$ converge almost everywhere (a. e.). Developing Carleson's method, A. Hunt [6], P. Sjölin [10] and N. Antonov [1] have established that the a.e. convergence property of Fourier series remains true for wider classes of functions. For instance, Antonov [1] has proved a.e. convergence of Fourier series of functions from the class $L\log(L)\log\log\log(L)$, which is the largest known Orlicz class, whose functions possess this property. Similar problems have been studied also for multiple Fourier series. If a function $f \in L^1(\mathbb{T}^2)$ has the following double Fourier series

$$f(x,y) \sim \sum_{n,m=-\infty}^{+\infty} c_{nm} e^{i(nx+my)}, \quad c_{nm} = c_{nm}(f),$$

and $G \subset \mathbb{R}^2$ is some domain, then we denote

$$S_G(x,y,f) = \sum_{(n,m)\in G} c_{nm} e^{i(nx+my)}. \tag{1.1}$$

Let $P \subset \mathbb{R}^2$ be an arbitrary open polygonal domain containing the origin, and let $\lambda P = \{(\lambda x, \lambda y) : (x,y) \in P\}$, $\lambda > 0$. Ch. Fefferman [4] has proved that for any function $f \in L^p(\mathbb{T}^2)$ the partial sums $S_{\lambda P}(x,y,f)$ converge a.e. as $\lambda \to \infty$. In [10], P. Sjölin has proved a similar result for more broad class of functions $L(\log L)^3 \log\log L$ in the case where P is a rectangle, and if P is a square, then the property of a.e. convergence remains true also in $L(\log L)^2 \log\log L$. Improving the Sjölin's second result, in [1] N. Antonov has established a.e. convergence of the square partial sums of Fourier series in the class $L(\log L)^2 \log\log\log L$. N. Tevzadze [11] has proved that for any sequence of rectangles $R_1 \subset R_2 \subset R_3 \subset \ldots$ in $\mathbb{R}^2$, with sides parallel to coordinate axes and for any function $f \in L^2(\mathbb{T}^2)$, the partial sums $S_{R_k}(x,y,f)$ converge a.e..

Note that in all above mentioned results the sequences of partial sums depend on a single parameter, and their proofs are based on Carleson's theorem and on other one-dimensional results. The properties of the following partial sums are somewhat different

$$S_{NM}(x,y,f) = \sum_{|n|\le N, |m|\le M} c_{nm} e^{i(nx+my)}, \tag{1.2}$$

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where the sum is over all possible rectangles with sides parallel to coordinate axes. Ch. Fefferman [5] has constructed an example of continuous function $f \in C(\mathbb{T}^2)$, for which the sums (1.2) diverge everywhere as $N, M \to \infty$.

Note also that in the above mentioned convergence theorems for double Fourier series were considered polygons with sides parallel to fixed directions. It turns out that, this requirement is crucial for these results. In this paper we show that even slight freedom of sides directions of summing polygon makes worse the situation. We consider domains of the following forms:

$$\Delta(a,b) = \left\{(x,y) \in \mathbb{R}^2 : \, a|x| + b|y| \le 1\right\}, \quad a,b > 0, \tag{1.3}$$

$$W(\alpha,\beta) = \{(x,y) \in \mathbb{R}^2 : \, |x| = r\cos\theta, |y| = r\sin\theta, \, r \ge 0, \, \alpha \le \theta \le \beta\}, \quad 0 \le \alpha < \beta \le \frac{\pi}{2}. \tag{1.4}$$

Domains of the form (1.3) represent rhombus. If $\Delta = \Delta(a,b)$, then we denote

$$\rho(\Delta) = \frac{\max\{a,b\}}{\min\{a,b\}}.$$

It is clear that Δ becomes a square if $a = b$, or equivalently, if $\rho(\Delta) = 1$. Notice also that each domain (1.4) can be represented in the form of union of four sectors of the form

$$V(\alpha,\beta) = \{(x,y) \in \mathbb{R}^2_+ : x = r\cos\theta, y = r\sin\theta, \, r \ge 0, \, \alpha \le \theta \le \beta\}, \quad 0 \le \alpha < \beta \le 2\pi, \tag{1.5}$$

where $\mathbb{R}_+$ stands for the set of real nonnegative numbers. A sequence of domains G_k is called complete if $\cup_{k=1}^{\infty} G_k = \mathbb{R}^2$. The theorem that follows is a consequence of the aforementioned general result by Fefferman [4], and as it was noted in [4], is equivalent to the general theorem.

Theorem A.[Ch. Fefferman] *If Δ_k, $k = 1, 2, \ldots$, is a complete increasing sequence of squares of the form* (1.3) *(i.e. $\rho(\Delta_k) = 1$), then for any function $f \in L^p(\mathbb{T}^2)$, $p > 1$, the following relation holds:*

$$\lim_{k\to\infty} S_{\Delta_k}(x,y,f) = f(x,y) \text{ a. e. on } \mathbb{T}^2. \tag{1.6}$$

The next theorem shows that for arbitrary domains of the form (1.3) Theorem A is not valid. More precisely, we have the following result.

Theorem 1.1. *There exist a function $f \in \cap_{1\le p<\infty} L^p(\mathbb{T}^2)$ and a complete increasing sequence of domains Δ_k, $k = 1, 2, \ldots$, of the form* (1.3)*, such that $\rho(\Delta_k) \to 1$ and we have*

$$\limsup_{k\to\infty} |S_{\Delta_k}(x,y,f)| = \infty \textit{ a. e. on } \mathbb{T}^2.$$

Observe that the condition $\rho(\Delta_k) \to 1$ means that at infinity the rhombus Δ_k become squares. Theorem 1.1 shows that even this is not enough for validity of the convergence property (1.6). A similar result also is proved for sectoral sums. Note that although for an unbounded domain G the sum in (1.1) is not finite, it is a.e. determined in the case $f \in L^2(\mathbb{T}^2)$.

Theorem 1.2. *For any increasing sequence of domains W_k, $k = 1, 2, \ldots$, of the form* (1.3) *there exists a function $f \in \cap_{1\le p<\infty} L^p(\mathbb{T}^2)$, such that*

$$\limsup_{k\to\infty} |S_{W_k}(x,y,f)| = \infty \textit{ a. e. on } \mathbb{T}^2.$$

It is clear that domains of the form

$$\Delta(a,b) \cap \mathbb{R}^2_+, \tag{1.7}$$

$$W(\alpha,\beta) \cap \mathbb{R}^2_+, \tag{1.8}$$

are respectively triangles and sectors with vertices at the origin. It is well-known that the double Fourier series (1.1) of a real-valued function $f \in L(\mathbb{T}^2)$ can be written in the real form (by sines and cosines), and the sum (1.1), corresponding to some rhombus domain (1.3), coincides with the sum of terms of double real Fourier series with indices from the triangle (1.7). Similarly, the sum of the series (1.1) over the domains (1.4) becomes to the sum over sectors (1.8) in the real case.