In this work we construct a weighted space \( L^p_\mu[0,1]^2 \), in which functions with the norm of that space are presented by Walsh double series, which coefficients are monotone in all ways.

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**Introduction.** Let \( |E| \) be the Lebesgue measure of a measurable set \( E \subseteq [0,1] \) (or \( E \subseteq [0,1] \times [0,1] = [0,1]^2 \)), and let \( L^p[0,1] \), \( p \geq 1 \), be the class of all those measurable functions \( f(x) \) on \([0,1]\) such that
\[
\int_0^1 |f(x)|^p dx < \infty.
\]
(1)

Let \( \mu(x,y) \) be a positive Lebesgue-measurable function (weight function) defined on \([0,1]^2\). We denote by \( L^p_\mu[0,1]^2 \) the space of all measurable functions on \([0,1]^2\) with the norm
\[
\|f\|_{L^p_\mu} = \left( \int_0^1 \int_0^1 |f(x,y)|^p \mu(x,y) dx dy \right)^{1/p} < \infty : \ p \in [1,\infty).
\]
(2)

In the sequel we will accept the terms “measure” and “measurable” in the sense of Lebesgue.

**Definition 1.** The nonzero members of a double sequence \( \{b_{k,s}\}_{k,s=0}^\infty \) are said to be in a monotonically decreasing order over all rays, if \( b_{k_2,s_2} < b_{k_1,s_1} \) when \( k_2 \geq k_1, s_2 \geq s_1, k_2 + s_2 > k_1 + s_1 \) \( (b_{k,s}, \neq 0, \ i = 1,2) \).

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Let \( f(x,y) \in L^p[0,1]^2 \), \( p \geq 1 \), and let
\[
\sum_{k,s=0}^{\infty} c_{k,s} \varphi_k(x) \varphi_s(y)
\tag{3}
\]
be series with double Walsh system.

The spherical and rectangular partial sums of the series (3) will be denoted by
\[
S_R(x,y) = \sum_{k+s \leq R^2} c_{k,s} \varphi_k(x) \varphi_s(y) \quad \text{and} \quad S_{N,M}(x,y) = \sum_{k=0}^{N} \sum_{s=0}^{M} c_{k,s} \varphi_k(x) \varphi_s(y),
\]
respectively.

**Definition 2.** Let \( f(x,y) \in L^p_{\mu}[0,1]^2 \). We will say that the series (3) converges to the function \( f(x,y) \) in \( L^p_{\mu}[0,1]^2 \)-norm with respect to spheres, if
\[
\lim_{R \to \infty} \left( \int_0^1 \int_0^1 |S_R(x,y) - f(x,y)|^p \mu(x,y) \, dx \, dy \right)^{1/p} = 0.
\]
The convergence with respect to rectangles is defined in the same way. More general statements of these definitions can be found in [1][12].

**Definition 3.** A series \( \sum_{k,s=0}^{\infty} b_{k,s} \varphi_k(x) \varphi_s(y) \) is called universal in \( L^p_{\mu}[0,1]^2 \) with respect to the subseries, if for every function \( f(x,y) \in L^p_{\mu}[0,1]^2 \) there exists a subseries \( \sum_{i,j=0}^{\infty} b_{i,j} \varphi_i(x) \varphi_j(y) \), which converges to \( f \) in \( L^p_{\mu}[0,1]^2 \)-norm.

In this work we will discuss the existence of Walsh universal double series with respect to the subseries in weighted \( L^p_{\mu}[0,1]^2 \)-spaces.

Note that different kind of partial sums (e.g. spherical, rectangular, square) behave differently in the concepts of convergence in \( L^p[0,1]^2 \), \( p \geq 1 \), and convergence almost everywhere. Also, many classical results (for instance, Carleson’s [2], Riesz’s [13] and Kolmogorov’s [14] theorems) cannot be extended from the one-dimensional case to the two-dimensional (see [3][15], [16]).

In [14] Harris constructed a function \( f \in L^p[0,1]^2 \) with \( 1 \leq p < 2 \) such that the Fourier–Walsh series of \( f(x,y) \) in the Walsh double system diverges almost everywhere and in \( L^p[0,1]^2 \)-norm with respect to spheres.

Thus for a given function \( f(x,y) \in L^p[0,1]^2 \) it is impossible to find a double series in the Walsh double system converging to \( f(x,y) \) either in \( L^p[0,1]^2 \)-norm or almost everywhere with respect to spheres.

In the present work we prove that for any \( \varepsilon > 0 \) there exists a measurable set \( E \subset [0,1]^2 \) with \( |E| > 1 - \varepsilon \) such that for any function \( f(x,y) \in L^p(E) \), \( p \geq 1 \), one can find a series \( \sum_{k,s=0}^{\infty} b_{k,s} \varphi_k(x) \varphi_s(y) \) with respect to the Walsh double system, which converges to the function \( f(x,y) \) in the \( L^p(E) \)-norm with respect to spheres, that is
\[
\lim_{K \to \infty} \left( \int_E \left( \sum_{k+s \leq K^2} b_{k,s} \varphi_k(x) \varphi_s(y) - f(x,y) \right)^p \, dx \, dy \right)^{1/p} = 0.
\]
The following theorem is true:

**Theorem 1.** ∀ε > 0 there exist a set $E \subset [0,1)^2$ with $|E| > 1 - ε$ and a measurable (weight) function $μ(x,y) : 0 < μ(x,y) \leq 1, (x,y) \in [0,1)^2$, with $μ(x,y) = 1$ on $E$ such that for each $p \in [1,∞)$ and for every function $f(x,y) \in L^p_μ[0,1)^2$ there exists a series with the following property:

$$\lim_{R \to \infty} \int_0^1 \int_0^1 \left| \sum_{k^2 + x^2 \leq R^2} b_{k,x} \phi_k(x) \phi(x,y) - f(x,y) \right|^p μ(x,y)dx dy = 0.$$  

This stronger theorem follows from Theorem 1:

**Theorem 2.** For ∀ε > 0 there exist a set $E \subset [0,1)^2$, $|E| > 1 - ε$, a measurable (weight) function $μ(x,y) : 0 < μ(x,y) \leq 1, (x,y) \in [0,1)^2$, with $μ(x,y) = 1$ on $E$, a series of the form $\sum_{k,s = 0}^∞ d_{k,s} \phi_k(x) \phi_s(y)$, where $\sum_{k,s = 0}^∞ |d_{k,s}|^r < \infty$ for all $r > 2$ and non-zero terms in $\{d_{k,s}\}_{k,s = 0}^∞$ are in the decreasing order over all rays, such that for each $p \in [1,∞)$ and for every function $f(x,y) \in L^p_μ[0,1)^2$ one can find numbers $δ_{k,s} = 0$ or 1 such that

$$\lim_{R \to \infty} \int_0^1 \int_0^1 \left| \sum_{k^2 + x^2 \leq R^2} δ_{k,x} d_{k,x} \phi_k(x) \phi(x,y) - f(x,y) \right|^p μ(x,y)dx dy = 0.$$  

**Remark.** Observe that one cannot claim $μ(x,y) \equiv 1$ in Theorem 2. It can be easily shown that the assumption of the existence of such universal series $\sum_{k,s = 0}^∞ c_{k,s} \phi_k(x) \phi_s(y)$ with respect to the subseries for the space $L^p[0,1)^2, p \geq 1$, simply leads to contradiction. Indeed, if that assumption is true, then for the function $f(x,y) = 5c_{k_0,s_0} \phi_{k_0}(x) \phi_{s_0}(y)$, where $k_0,s_0 > 1$ are any natural numbers and $c_{k_0,s_0} \neq 0$, one can find numbers $δ_k = 0$ or 1 such that

$$\lim_{m \to \infty} \int_0^1 \int_0^1 \left| \sum_{k,s = 0}^m δ_{k,s} c_{k,s} \phi_k(x) \phi_s(y) - 5c_{k_0,s_0} \phi_{k_0}(x) \phi_{s_0}(y) \right| dx dy = 0.$$  

Hence, we will simply get $δ_{k_0,s_0} = 5 > 1$.

**The Main Lemma.** The Walsh system is defined as follows. Let $r(x)$ be a 1-periodic function on [0,1) defined by $r = \chi_{[0,1/2)} - \chi_{[1/2,1)}$, where $\chi_E(x)$ denotes the characteristic function of the set $E$, that is,

$$\chi_E(x) = \begin{cases} 
1, & \text{if } x \in E, \\
0, & \text{if } x \notin E.
\end{cases}$$

The Rademacher system $R = \{r_n : n = 0,1,\ldots\}$ is defined by

$$r_n(x) = r(2^n x) \text{ for all } x \in R, n = 0,1,\ldots \quad (4)$$

Recall the definition of the Walsh system $\{ϕ_n\}(x)$ in Paley order (see [13]). Define

$$ϕ_n(x) = \prod_{k=0}^∞ ϕ_k^n(x), \quad (5)$$
where $\sum_{k=0}^{\infty} n_k 2^k$ is the unique binary expansion of $n$ with $n_k$ 0 or 1.

The following lemma, which immediately follows from Lemma 4 from [17], plays a central role in the proof of our Theorem:

**Lemma 1.** Let $\{\phi_k\}$ be the Walsh system. Then for each $0 < \delta < 1$ there exists a measurable positive function $\mu(x,y)$ with $|\{(x,y) \in [0,1)^2; \mu(x,y) = 1\}| > 1 - \delta$ such that for any numbers $\varepsilon \in (0,1)$, $N \in \mathbb{N}$, $p_0 > 1$ and for each function $f \in L^{p_0}[0,1)^2$, $\|f\|_{p_0} > 0$, one can find a polynomial $Q(x,y)$ of the form

$$Q(x,y) = \sum_{k,s=N}^{M} c_{k,s} \phi_k(x) \phi_s(y),$$

satisfying the following conditions:

1) the nonzero coefficients in $\{|c_{k,n}|, k,n = N,\ldots,M\}$ are in decreasing order over all rays;

2) $\sum_{k,n=N}^{M} |c_{k,n}|^{2+\varepsilon} < \varepsilon$;

3) $\int_0^1 \int_0^1 |Q(x,y) - f(x,y)|^{p_0} \mu(x,y) dx dy < \varepsilon^{p_0};$

4) $\max_{\sqrt{2N} \leq R \leq \sqrt{M}} \left( \int_0^1 \int_0^1 \sum_{2N^2 \leq k^2 + s^2 \leq R^2} c_{k,s} \phi_k(x) \phi_s(y) \mu(x,y) dx dy \right)^{1/p} \leq \left( \int_0^1 \int_0^1 |f(x,y)|^{p} \mu(x,y) dx dy \right)^{1/p} + \varepsilon$ for all $p \in [1,p_0]$.

**Proof of Theorem 2.**

**Proof.** Let $0 < \varepsilon < 1$, $p_n \nearrow \infty$ ($p_1 > 1$) and let

$$\left\{ f_k(x,y) \right\}_{k=1}^{\infty}$$

be a sequence of all polynomials in the Walsh system with rational coefficients.

Successively applying Lemma 1, we can find a measurable weight function $\mu(x,y)$, a set $E \subset [0,1)^2$ such that

$$\mu(x,y) = 1 \text{ on } E, \ |E| > 1 - \varepsilon,$$

and polynomials

$$Q_n(x,y) = \sum_{k,s=m_n-1}^{m_n} b_{k,s}^{(n)} \phi_k(x) \phi_s(y), m_n \nearrow,$$

which satisfy the following conditions for every $n \geq 1$:
\[
\int_0^1 \int_0^1 \left| \overline{\Omega}_n(x,y) - f_n(x,y) \right|^{p_n} \mu(x,y) dxdy \leq 2^{-8p_n(n+1)}. \tag{9}
\]

All nonzero members in the sequence \( \left\{ b_{k,s}^{(n)} \right\} \), \( k, s \in [m_{n-1}, m_n] \), are in decreasing order over all rays for any fixed \( n \geq 1 \) and

\[
\max_{k,s \in [m_{n-1}, m_n]} |b_{k,s}^{(n)}| < \min_{(k,s) \in \text{spec} \overline{\Omega}_{n-1}} |b_{k,s}^{(n-1)}| \quad \text{for all } n = 1, 2, \ldots, \tag{10}
\]

\[
\sum_{k,s=m_{n-1}}^{m_n-1} \left| b_{k,s}^{(n)} \right|^{2+2^{-n}} < \frac{1}{2^{8(n+1)}}, \quad n \geq 1, \tag{11}
\]

\[
\max_{\sqrt{m_{n-1}} \leq R < \sqrt{m_n}} \left( \int_0^1 \int_0^1 \sum_{2n_1^2 \leq k^2 + s^2 \leq R^2} b_{k,s}^{(n)} \phi_k(x) \phi_s(y) \right)^p \mu(x,y) dxdy \leq \tag{12}
\]

\[
\leq 2 \left( \int_0^1 \int_0^1 |f_n(x,y)|^p \mu(x,y) dxdy \right)^{1/p} + 2^{-2n} \quad \text{for all } p \in [1, p_n].
\]

We put

\[
b_{k,s} = \begin{cases} b_{k,s}^{(n)}, & k, s \in [m_{n-1}, m_n], \ n \geq 1, \\ 0, & \text{in other cases.} \end{cases} \tag{13}
\]

Let \( f(x,y) \in L^p(\mu, (0,1)^2), \forall p \geq 1 \). Now assume that the polynomials

\[
\overline{\Omega}_j(x,y) = \sum_{k,s=m_j-1}^{m_j-1} b_{k,s}^{(j)} \phi_k(x) \phi_s(y), \ 1 \leq j \leq q - 1, \tag{14}
\]

have been defined satisfying the conditions

\[
\int_0^1 \int_0^1 f(x,y) - \sum_{j=1}^{q'} \overline{\Omega}_j(x,y) \right|^{p} \mu(x,y) dxdy < 2^{-2q'}, \quad 1 \leq q' \leq q - 1, \tag{15}
\]

\[
\max_{\sqrt{m_{j-1}} \leq R < \sqrt{m_j}} \int_0^1 \int_0^1 \sum_{2m_{j-1}^2 \leq k^2 + s^2 \leq R^2} b_{k,s}^{(j)} \phi_k(x) \phi_s(y) \right|^{p} \mu(x,y) dxdy < 2^{-1/p}. \tag{16}
\]

Choose the function \( f_{q_j} \) from the sequence \( F \) (see (6)) such that
satisfying conditions (15) and (16) for all \( q \) it follows from (15) and (17) that
\[
\left( \int_0^1 \left| f_q(x,y) - \left[ f(x,y) - \sum_{j=1}^{q-1} \overline{d}_j(x,y) \right] \right|^p \mu(x,y) dx dy \right)^{1/p} < 2^{-2(q+2)}. \tag{17}
\]

Taking into account (12) and (16)–(18), we have
\[
\left( \int_0^1 \left| f_q(x,y) \right|^p \mu(x,y) dx dy \right)^{1/p} < 2^{-2(q-1)} + 2^{-2(q+2)}. \tag{18}
\]

By (19)–(22) we have
\[
\left( \int_0^1 \left| f_q(x,y) - \left[ f(x,y) - \sum_{j=1}^{q-1} \overline{d}_j(x,y) \right] \right|^p \mu(x,y) dx dy \right)^{1/p} \leq 2^{-8q} + 2^{-2(q+2)} < 2^{-2q}, \tag{19}
\]

It is clear that we can define by induction polynomials
\[
\overline{d}_q(x,y) = \sum_{k,s=m_q-1}^{m_{q+1}-1} b_{k,s}^{(l_q)} \varphi_k(x) \varphi_s(y), \tag{21}
\]
satisfying conditions (15) and (16) for all \( q \geq 1 \). We set
\[
\delta_{k,s} = \begin{cases} 1, & k, s \in \bigcup_{q=1}^{m_q-1} [m_q-1, m_q), \\ 0, & \text{in other cases}. \end{cases} \tag{22}
\]

By (19)–(22) we have
\[
\lim_{R \to \infty} \left( \int_0^1 \int_0^1 \left| \sum_{0 \leq k^2 + s^2 \leq R^2} \delta_{k,s} b_{k,s} \varphi_k(x) \varphi_s(y) - f(x,y) \right|^p \mu(x,y) dx dy \right)^{1/p} = 0, \tag{23}
\]
i. e. the Theorem 2 is proved.
REFERENCES