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Mathematics

ON A PROPERTY OF NORMING CONSTANTS OF STURM–LIOUVILLE PROBLEM

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A connection, which shows the dependence of norming constants on boundary conditions, was found using the Gelfand–Levitan method for the solution of inverse Sturm–Liouville problem.

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1. Introduction. Let $L(q,\alpha,\beta)$ denote the Sturm–Liouville boundary value problem

$$\ell y \equiv -y'' + q(x)y = \mu y, \quad x \in (0,\pi),\ \mu \in \mathbb{C}, \tag{1}$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \quad \alpha \in (0,\pi), \tag{2}$$

$$y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \quad \beta \in (0,\pi), \tag{3}$$

where q is a real-valued, summable on $[0,\pi]$ function (we write $q \in L^1_{\mathbb{R}}[0,\pi]$). By $L(q,\alpha,\ \beta)$ we also denote the self-adjoint operator, generated by the problem (1)–(3) (see [1]). It is known that under these conditions the spectra of the operator $L(q,\alpha,\beta)$ is discrete and consists of real, simple eigenvalues [1], which we denote by $\mu_n = \mu_n(q,\alpha,\beta) = \lambda_n^2(q,\alpha,\beta)$, $n = 0,1,2,\ldots$, emphasizing the dependence of μ_n on q, α and β.

Let $\varphi(x,\mu,\alpha,q)$ and $\psi(x,\mu,\beta,q)$ are the solutions of Eq. (1), which satisfy the initial conditions

$$\varphi(0,\mu,\alpha,q) = \sin\alpha, \ \ \varphi'(0,\mu,\alpha,q) = -\cos\alpha,$$

$$\psi(\pi,\mu,\beta,q) = \sin\beta, \ \ \psi'(\pi,\mu,\beta,q) = -\cos\beta,$$

correspondingly. The eigenvalues $\mu_n = \mu_n(q,\alpha,\beta)$, $n = 0,1,2,\ldots$, of $L(q,\alpha,\beta)$ are the solutions of the equation

$$\Phi(\mu) = \Phi(\mu,\alpha,\beta) \stackrel{def}{=} \varphi(\pi,\mu,\alpha)\cos\beta + \varphi'(\pi,\mu,\alpha)\sin\beta = 0$$

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or the equation

$$\Psi(\mu) = \Psi(\mu, \alpha, \beta) \stackrel{def}{=} \psi(0, \mu, \beta)\cos\alpha + \psi'(0, \mu, \beta)\sin\alpha = 0.$$

According to the well-known Liouville formula, the wronskian

$$W(x) = W(x, \varphi, \psi) = \varphi \cdot \psi' - \varphi' \psi$$

of the solutions φ and ψ is constant. It follows that $W(0) = W(\pi)$ and consequently $\Psi(\mu, \alpha, \beta) = -\Phi(\mu, \alpha, \beta)$. It is easy to see that the functions $\varphi_n(x) = \varphi(x, \mu_n, \alpha)$ and $\psi_n(x) = \psi(x, \mu_n, \beta)$, $n = 0, 1, 2, \ldots$, are the eigenfunctions, corresponding to the eigenvalue μ_n. Since all eigenvalues are simple, there exist constants $c_n = c_n(q, \alpha, \beta)$, $n = 0, 1, 2, \ldots$, such that

$$\varphi(x, \mu_n) = c_n \cdot \psi(x, \mu_n). \tag{4}$$

The squares of the L^2-norm of these eigenfunctions:

$$a_n = a_n(q, \alpha, \beta) = \int_0^\pi |\varphi_n(x)|^2 dx,\ n = 0, 1, 2, \ldots,$$

$$b_n = b_n(q, \alpha, \beta) = \int_0^\pi |\psi_n(x)|^2 dx,\ n = 0, 1, 2, \ldots,$$

are called the norming constants.

In this paper we consider the case $\alpha, \beta \in (0, \pi)$, i.e. we assume that $\sin\alpha \neq 0$ and $\sin\beta \neq 0$. In this case we consider the solution $\tilde{\varphi}(x, \mu, \alpha, q) := \dfrac{\varphi(x, \mu, \alpha, q)}{\sin\alpha}$ of the equation (1), which has the initial values

$$\tilde{\varphi}(0, \mu, \alpha, q) = 1,\ \tilde{\varphi}(x, \mu, \alpha, q) = -\cot\alpha,$$

and also we consider the solution $\tilde{\psi}(x, \mu, \beta, q) := \dfrac{\psi(x, \mu, \beta, q)}{\sin\beta}$. Of course, the functions $\tilde{\varphi}_n(x) := \tilde{\varphi}(x, \mu_n, \alpha, q)$ and $\tilde{\psi}_n(x) := \tilde{\psi}(x, \mu_n, \alpha, q)$, $n = 0, 1, 2, \ldots$, are the eigenfunctions, corresponding to the eigenvalue μ_n. It follows from (4) that for norming constants $\tilde{a}_n := \|\tilde{\varphi}_n\|^2 = \dfrac{a_n}{\sin^2\alpha}$, $\tilde{b}_n := \|\tilde{\psi}_n\|^2 = \dfrac{b_n}{\sin^2\beta}$ the following connections

$$\tilde{b}_n = \frac{b_n}{\sin^2\beta} = \frac{a_n}{c_n^2\sin^2\beta} = \frac{\tilde{a}_n\sin^2\alpha}{c_n^2\sin^2\beta} \tag{5}$$

hold.

2. The Main Result. The aim of this paper is to prove the following assertion.

Theorem. For the norming constants $\tilde{a}_n$ and $\tilde{b}_n$ the following connections hold:

$$\frac{1}{\tilde{a}_0} - \frac{1}{\pi} + \sum_{n=1}^{\infty}\left(\frac{1}{\tilde{a}_n} - \frac{2}{\pi}\right) = \cot\alpha, \tag{6}$$

$$\frac{1}{\tilde{b}_0} - \frac{1}{\pi} + \sum_{n=1}^{\infty}\left(\frac{1}{\tilde{b}_n} - \frac{2}{\pi}\right) = -\cot\beta. \tag{7}$$

For the solution $\tilde{\varphi}$ it is well known the representation (see [2, 3])

$$\tilde{\varphi}(x,\lambda,\alpha,q) = \cos\lambda x + \int_0^x G(x,t)\cos\lambda t dt, \tag{8}$$

where for the kernel $G(x,t)$ we have (in particular) (see [3])

$$G(x,x) = -\cot\alpha + \frac{1}{2}\int_0^x q(s)ds. \tag{9}$$

Besides, it is known that $G(x,t)$ satisfies to the Gelfand–Levitan integral equation

$$G(x,t) + F(x,t) + \int_0^x G(x,s)F(s,t)ds = 0, \ \ 0 \le t \le x, \tag{10}$$

where the function $F(x,t)$ is defined by the formula (see [3])

$$F(x,t) = \sum_{n=0}^{\infty}\left(\frac{\cos\lambda_n x\cos\lambda_n t}{\tilde{a}_n} - \frac{\cos nx\cos nt}{a_n^0}\right), \tag{11}$$

where $a_0^0 = \pi$ and $a_n^0 = \pi/2$ for $n = 1, 2, \ldots$ It easily follows from (9)–(11) that

$$G(0,0) = -F(0,0) = -\sum_{n=0}^{\infty}\left(\frac{1}{\tilde{a}_n} - \frac{1}{a_n^0}\right) =$$

$$= -\left(\frac{1}{\tilde{a}_0} - \frac{1}{\pi}\right) - \sum_{n=1}^{\infty}\left(\frac{1}{\tilde{a}_n} - \frac{2}{\pi}\right) = -\cot\alpha. \tag{12}$$

Thus, (6) is proved.

Let us now consider the functions ($n = 0, 1, 2, \ldots$)

$$p(x,\mu_n) = \frac{\varphi(\pi - x,\mu_n,\alpha,q)}{\varphi(\pi,\mu_n,\alpha,q)} = \frac{\varphi(\pi - x,\mu_n)}{\varphi(\pi,\mu_n)}. \tag{13}$$

Since $\varphi(x,\mu,\alpha,q)$ satisfies the Eq. (1) and

$$p'(x,\mu_n) = -\frac{\varphi'(\pi - x,\mu_n)}{\varphi(\pi,\mu_n)}, \ \ p''(x,\mu_n) = \frac{\varphi''(\pi - x,\mu_n)}{\varphi(\pi,\mu_n)},$$

we can see that $p(x,\mu_n)$ satisfies the equation

$$-p''(x,\mu_n) + q(\pi - x)p(x,\mu_n) = \mu_n p(x,\mu_n)$$

and the initial conditions

$$p(0,\mu_n) = 1, \ \ p'(0,\mu_n) = -\frac{\varphi'(\pi,\mu_n)}{\varphi(\pi,\mu_n)} = -(-\cot\beta) = \cot\beta = -\cot(\pi - \beta). \tag{14}$$

We also have

$$p(\pi,\mu_n) = \frac{\varphi(0,\mu_n)}{\varphi(\pi,\mu_n)} = \frac{\sin\alpha}{\varphi(\pi,\mu_n)} = \frac{\sin(\pi - \alpha)}{\varphi(\pi,\mu_n)},$$

$$p'(\pi,\mu_n) = -\frac{\varphi'(0,\mu_n)}{\varphi(\pi,\mu_n)} = -\frac{-\cos\alpha}{\varphi(\pi,\mu_n)} = \frac{-\cos(\pi - \alpha)}{\varphi(\pi,\mu_n)}.$$

From this it follows that $p_n(x) := p(x, \mu_n)$ satisfies the boundary condition

$$p_n(\pi)\cos(\pi-\alpha) + p_n'(\pi)\sin(\pi-\alpha) = 0,\ n = 0, 1, 2, \ldots$$

Let us denote $q^*(x) := q(\pi - x)$. Since $\mu_n(q^*, \pi-\beta, \pi-\alpha) = \mu_n(q, \alpha, \beta)$ (it is easy to prove and is well known, see for example [4]), it follows, that $p_n(x),\ n = 0, 1, 2, \ldots,$ are the eigenfunctions of the problem $L(q^*, \pi-\beta, \pi-\alpha)$, which have the initial conditions (14), i.e. $p_n(x) = \tilde{\varphi}(x, \mu_n, \pi-\beta, q^*),\ n = 0, 1, 2, \ldots$

Thus, as in (12), for the norming constants $\hat{a}_n = \|p(\cdot, \mu_n)\|^2$ we have

$$\left(\frac{1}{\hat{a}_0} - \frac{1}{\pi}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{\hat{a}_n} - \frac{2}{\pi}\right) = \cot(\pi-\beta) = -\cot\beta. \tag{15}$$

On the other hand, for the norming constants $\hat{a}_n$, according to (4), (5) and (13), we have

$$\hat{a}_n = \int_0^{\pi} p^2(x, \mu_n)dx = \int_0^{\pi} \frac{\varphi^2(\pi - x, \mu_n)}{\varphi^2(\pi, \mu_n)}dx =$$

$$= -\frac{1}{\varphi^2(\pi, \mu_n)}\int_{\pi}^{0} \varphi^2(s, \mu_n)ds = \frac{1}{\varphi^2(\pi, \mu_n)}\int_0^{\pi} \varphi^2(s, \mu_n)ds =$$

$$= \frac{a_n(q, \alpha, \beta)}{\varphi^2(\pi, \mu_n)} = \frac{\tilde{a}_n \sin^2\alpha}{c_n^2 \sin^2\beta} = \tilde{b}_n.$$

Therefore, we can rewrite (15) in the form

$$\left(\frac{1}{\tilde{b}_0} - \frac{1}{\pi}\right) - \sum_{n=1}^{\infty}\left(\frac{1}{\tilde{b}_n} - \frac{2}{\pi}\right) = -\cot(\pi-\beta) = \cot\beta.$$

Thus, (7) is true and Theorem is proved. □

3. Remark. It is known from the inverse Sturm–Liouville problems, that the set of eigenvalues $\left\{\mu_n\right\}_{n=0}^{\infty}$ and the norming constants $\left\{\tilde{a}_n\right\}_{n=0}^{\infty}$ uniquely determine the problem $L(q, \alpha, \beta)$. That means, in particular, that we can determine $\left\{\tilde{b}_n\right\}_{n=0}^{\infty}$ by these two sequences. Now we will derive the precise formulae for these connections.

It is known that the specification of the spectra $\left\{\mu_n(q, \alpha, \beta)\right\}_{n=0}^{\infty}$ uniquely determines the characteristic function $\Phi(\mu)$ (see [4], Lemma 1(*iii*); [5], Lemma 2.2) and also its derivative $\partial\Phi(\mu)/\partial\mu = \dot{\Phi}(\mu)$ ([5], Lemma 2.3).

In particular, if $\alpha, \beta \in (0, \pi)$ the following formulas hold:

$$\dot{\Phi}(\mu_0) = -\pi \sin\alpha \sin\beta \prod_{k=1}^{\infty} \frac{\mu_k - \mu_0}{k^2} \tag{16}$$

and (if $n \neq 0$, i.e. $n = 1, 2, \ldots$)

$$\dot{\Phi}(\mu_n) = -\frac{\pi}{n^2}\left[\mu_0 - \mu_n\right] \sin\alpha \sin\beta \prod_{k=1, k\neq n}^{\infty} \frac{\mu_k - \mu_n}{k^2}. \tag{17}$$

On the other hand, it is easy to prove the relation (see [5], Eq. (2.16) in Lemma 2.2 and see [4], Lemma 1 (*iii*))

$$a_n = -c_n \cdot \dot{\Phi}(\mu_n). \tag{18}$$

Taking into account the connections (5) and (16)–(18), we can find formulae for $\frac{1}{\tilde{b}_0}$ and $\frac{1}{\tilde{b}_n}$, $n = 1, 2, \ldots$:

$$\frac{1}{\tilde{b}_0} = \frac{\tilde{a}_0}{\pi^2 \left(\prod_{k=1}^{\infty} \frac{\mu_k - \mu_n}{k^2} \right)^2},$$

$$\frac{1}{\tilde{b}_n} = \frac{\tilde{a}_n n^4}{\pi^2 [\mu_0 - \mu_n]^2 \left(\prod_{k=1, k\neq n}^{\infty} \frac{\mu_k - \mu_n}{k^2} \right)^2}.$$

So, we can change the second assertion in Theorem by the following equation

$$\frac{\tilde{a}_0}{\pi^2 \left(\prod_{k=1}^{\infty} \frac{\mu_k - \mu_n}{k^2} \right)^2} - \frac{1}{\pi} +$$

$$+ \sum_{n=1}^{\infty} \left(\frac{\tilde{a}_n n^4}{\pi^2 [\mu_0 - \mu_n]^2 \left(\prod_{k=1, k\neq n}^{\infty} \frac{\mu_k - \mu_n}{k^2} \right)^2} - \frac{2}{\pi} \right) = -\cot\beta.$$

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