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Mathematics

ON SOME SINGULAR INTEGRAL EQUATIONS ON THE SEMI-AXIS

V. V. SIMONYAN*

Chair of Differential Equations and Functional Analysis, YSU

In this paper the non-characteristic singular integral equations on the semi-axis are discussed. The solution of these equations is reduced to the solution of one-dimensional pseudo-differential equations. Some examples of singular equations for which explicit solutions exist are provided.

***Keywords*:** singular equation, pseudo-differential equation, matrix coupling.

1⁰. Let $\overline{\mathbb{R}}$ be a two-point compactification of $\mathbb{R}=(-\infty,+\infty)$ and $C(\overline{\mathbb{R}})$ – the Banach algebra of all complex valued continuous function on $\overline{\mathbb{R}}$. What follows, $\rho(z)=z^{\beta}$ $(-1<\beta<1)$ is understood as the branch of this function that is analytical in $\mathbb{C}\setminus(-\infty,0)$ and assumes positive values on the positive semi-axis $\mathbb{R}_+=(0,+\infty)$. Let Σ_β be a subalgebra of all linear bounded operators acting from and to the weighted space $L_2(\mathbb{R}_+,\rho)$, generated by operators I and $S_{\mathbb{R}_+}$, where $S_{\mathbb{R}_+}$ is a singular integral operator acting along $\mathbb{R}_+$:

$$\left(S_{\mathbb{R}_+}y\right)(x)=\frac{1}{\pi i}\int_0^{+\infty}\frac{y(\tau)}{\tau-x}d\tau .$$

Here the integral is perceived in terms of the main value. The notation F means the following Fourier transform:

$$(Fy)(\xi)=\hat{y}(\xi)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{ix\xi}y(x)dx \quad \left(y\in L_2(\mathbb{R}),\xi\in\mathbb{R}\right),$$

and γ is the operator continuously reflecting $L_2(\mathbb{R}_+,\rho)$ in $L_2(\mathbb{R})$ according to formula $(\gamma y)(x)=e^{sx}y(e^{\alpha x})$, where $\alpha>0$, $\sigma=\alpha(\beta+1)/2$ and $s=\sigma+i\zeta$. It is not difficult to ascertain that the operator $\gamma^{-1}F$ is the Mellin transform.

Below the multiplicative operator by function u is denoted as Λ_u (i.e. $\Lambda_u y=uy$). For any function $a\in C(\overline{\mathbb{R}})$ the operator $K_a=\gamma^{-1}F\Lambda_a F^{-1}\gamma$ belongs to algebra Σ_β (see, e.g., [1, 2]).

* E-mail: vahan@simonyans.com

As is well known, the characteristic equation $\left(\Lambda_a + S_{\mathbb{R}_+}\Lambda_b\right)y = f$ $\left(y, f \in L_2\left(\mathbb{R}_+, \rho\right)\right)$ admits the explicit solution [3]. The present paper is devoted to the investigation of "difficult" singular integral operators of a type $\widetilde{K} = \sum_{m=1}^{N} K_{\varphi_m}\Lambda_{\psi_m}$ $\left(\varphi_m \in C\left(\overline{\mathbb{R}}\right), \psi_m \in L_\infty\left(\mathbb{R}_+\right)\right)$, acting from $L_2\left(\mathbb{R}_+, \rho\right)$ to itself.

Let $\mathbb{H}_r(\mathbb{R})$ $(r \ge 0)$ be Sobolev–Slobodetski spaces of the generalized functions u, the Fourier transform $\hat{u}$ of which belongs to $L_2\left(\mathbb{R}, (1+|x|)^r\right)$ space. Following [4], the class of locally integrable functions A on $\mathbb{R}$, complying to condition $\left|A(\xi)\right| \le c\left(1+|\xi|\right)^r$, is denoted as $\mathbb{S}_r^0$.

Let $\psi_0(x) = \left(1+\left|\alpha^{-1}\ln x\right|\right)^r$ $(x \in \mathbb{R}_+)$ and functions $\psi_i \in L_\infty(\mathbb{R}_+)$ $(i = 1,\dots,N)$. Now determine the functions $A_0 = \psi_0 \circ l,\ A_m = (\psi_m \circ l)A_0$ $(m = 1,\dots,N)$ on $\mathbb{R}$, where $l(x) = e^{\alpha x}$ $(x \in \mathbb{R})$. It follows from here that $\psi_0 = A_0 \circ h$ and $\psi_m = (A_m \circ h)\psi_0^{-1}$ $(m = 1,\dots,N)$, where $h(x) = \alpha^{-1}\ln x$ when $x \in \mathbb{R}_+$. It is obvious that $A_m \in \mathbb{S}_r^0$ for all $m = 0,\dots,N$.

Let us consider a operator $A(x,D)$ with a symbol $A(x,t) = \sum_{m=1}^{N}\varphi_m(x)A_m(t)$:

$$A(x,D)u = \int_{-\infty}^{\infty} e^{-ixt}A(x,t)\hat{u}(t)\,dt.$$

Since $A(x,D)u = \sum_{m=1}^{N}\Lambda_{\varphi_m}A_m(D)u$, then for $r \ge 0$ the reflection $A(x,D)$ may be prolonged till the continuous reflection from $\mathbb{H}_r(\mathbb{R})$ to $L_2(\mathbb{R})$(see [4]).

Below the solution of "difficult" equation $\widetilde{K}z = g$ is reduced to a pseudo-differential equation $A(x,D)y = f$ for some f. The examples of "difficult" singular integral equations investigated based on this relationship are provided.

2⁰. Let $X_i\,(i = 1,\dots,4)$ be linear spaces on the field of complex numbers, $\omega_1 : X_1 \to X_2,\ \omega_2 : X_2 \to X_1,\ \omega_3 : X_1 \to X_3,\ \omega_4 : X_4 \to X_1$ are linear reflections and ω_4 is an invertible reflection. Let us consider the linear reflections

$$\begin{aligned}
&T : X_2 \to X_2 \oplus X_3, && K : X_4 \to X_1 \oplus X_3,\\
&A_{12} : X_1 \oplus X_3 \to X_2 \oplus X_3, && A_{21} : X_2 \to X_4,\\
&A_{22} : X_1 \oplus X_3 \to X_4, && B_{11} : X_2 \oplus X_3 \to X_2,\\
&B_{12} : X_4 \to X_2, && B_{21} : X_2 \oplus X_3 \to X_1 \oplus X_3,
\end{aligned}$$

determined by equations

$$T = \begin{bmatrix} I_{X_2} + \omega_1\omega_2 \\ \omega_3\omega_2 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -\omega_1 & 0 \\ -\omega_3 & I_{X_3} \end{bmatrix},$$

$$A_{21} = \left[-\omega_4^{-1}\omega_2 \right], \quad A_{22} = \left[\omega_4^{-1} \quad 0 \right], \quad B_{11} = \left[I_{X_2} \quad 0 \right], \quad B_{12} = \left[\omega_1\omega_4 \right],$$

$$B_{21} = \begin{bmatrix} \omega_2 & 0 \\ 0 & I_{X_3} \end{bmatrix}, \quad K = \begin{bmatrix} \omega_4 + \omega_2\omega_1\omega_4 \\ \omega_3\omega_4 \end{bmatrix}.$$

By means of direct calculations it is easy to make sure of the validity of equality

$$\begin{bmatrix} T & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & K \end{bmatrix}.$$

Based on this equality and results of [5] the following statement is derived.

Assignment 1. If $K^{(-1)}$ is pseudo-inverse of K, then the reflection

$$T^{(-1)} = B_{11} - B_{12}K^{(-1)}B_{21}$$

is pseudo-inverse of T. Similarly, if $T^{(-1)}$ is pseudo-inverse of T, then

$$K^{(-1)} = A_{22} - A_{21}T^{(-1)}A_{12}$$

is pseudo-inverse of K. Spaces $\mathrm{Ker}T$ and $\mathrm{Ker}K$ are isomorphic. Besides, the following equalities are valid:

$$\mathrm{Ker}T = B_{12}\mathrm{Ker}K, \quad \mathrm{Ker}K = A_{21}\mathrm{Ker}T,$$

$$\mathrm{Im}\,T = B_{21}^{-1}\,\mathrm{Im}\,K, \quad \mathrm{Im}\,K = A_{12}^{-1}\,\mathrm{Im}\,T.$$

3⁰. Let $\mathbb{M}_r(\mathbb{R})$ be some direct addition to the linear space $\mathbb{H}_r(\mathbb{R})$ in $L_2(\mathbb{R})$ space, and

$$\pi_1 : L_2(\mathbb{R}) \to \mathbb{H}_r(\mathbb{R}), \ \pi_2 : L_2(\mathbb{R}) \to \mathbb{M}_r(\mathbb{R})$$

are projection operators that are related by the ratio

$$\pi_1 y + \pi_2 y = y, \ y \in L_2(\mathbb{R}).$$

Let us determine a space $W = \left\{ f : \psi_0^{-1} f \in L_2(\mathbb{R}_+, \rho) \right\}$. Then, apply the Assignment 1 for spaces

$$X_1 = L_2(\mathbb{R}_+, \rho), \ X_2 = \mathbb{H}_r(\mathbb{R}), \ X_3 = \mathbb{M}_r(\mathbb{R}), \ X_4 = W$$

and operators

$$\omega_1 = \pi_1 F^{-1}\gamma, \ \omega_2 = \sum_{m=1}^{N} K_{\varphi_m}\gamma^{-1}FA_m(D) - \gamma^{-1}F, \ \omega_3 = \pi_2 F^{-1}\gamma, \ \omega_4 = \Lambda_{\psi_0^{-1}}.$$

By using the equality $F^{-1}\gamma K_{\varphi_m}\gamma^{-1}F = \Lambda_{\varphi_m}$ $(m = 1,\ldots,N)$ it is easy to ascertain that the operator T acting from $\mathbb{H}_r(\mathbb{R})$ to $L_2(\mathbb{R})$ coincides with $A(x,D)$.

Lemma 1. Let

$$f = (\tilde{f}, 0) \in L_2(\mathbb{R}_+, \rho) \oplus \mathbb{M}_r(\mathbb{R}).$$

The function z is the solution of equation $Kz = f$, iff $z \in L_2(\mathbb{R}_+, \rho)$ and $\tilde{K}z = \tilde{f}$.

Proof. Let z be the solution of equation $Kz = f$, that is

$$\omega_4 z + \omega_2\omega_1\omega_4 z = \tilde{f}, \quad \omega_3\omega_4 z = 0. \tag{1}$$

First prove that $z \in L_2(\mathbb{R}_+, \rho)$. Really, from the second equation of (1) it follows that $g = F^{-1}\gamma\Lambda_{\psi_0^{-1}} z \in \mathbb{H}_r(\mathbb{R})$. Consequently, $\gamma z = FA_0(D)g \in L_2(\mathbb{R})$, that is $z \in L_2(\mathbb{R}_+, \rho)$. Note now that from (1) it also follows that $\pi_1 F^{-1}\gamma\Lambda_{\psi_0^{-1}} z = F^{-1}\gamma\Lambda_{\psi_0^{-1}} z$. From the first equation of (1) it follows that $\tilde{K}z = \tilde{f}$.

Let $z \in L_2(\mathbb{R}_+, \rho)$ and $\tilde{K}z = \tilde{f}$. From equality $F^{-1}\gamma\Lambda_{\psi_0^{-1}} z = A_0^{-1}(D)F^{-1}\gamma z$ it follows that

$$\omega_3\omega_4 z = 0 .$$

Since $\tilde{K}z = \tilde{f}$, we have

$$\left(\sum_{m=1}^{N} K_{\varphi_m}\Lambda_{\psi_m}\right) z = \left(\sum_{m=1}^{N} K_{\varphi_m}\Lambda_{A_m\circ h}\Lambda_{\psi_0^{-1}}\right) z = \left(\Lambda_{\psi_0^{-1}} + \left(\sum_{m=1}^{N} K_{\varphi_m}\Lambda_{A_m\circ h} - I\right)\Lambda_{\psi_0^{-1}}\right) z =$$

$$= \left(\Lambda_{\psi_0^{-1}} + \left(\sum_{m=1}^{N} K_{\varphi_m}\gamma^{-1}FA_m(D) - \gamma^{-1}F\right)F^{-1}\gamma\Lambda_{\psi_0^{-1}}\right) z =$$

$$= \left(\Lambda_{\psi_0^{-1}} + \left(\sum_{m=1}^{N} K_{\varphi_m}\gamma^{-1}FA_m(D) - \gamma^{-1}F\right)\pi_1 F^{-1}\gamma\Lambda_{\psi_0^{-1}}\right) z =$$

$$= (\omega_4 + \omega_2\omega_1\omega_4) z = \tilde{f}.$$

Thus, the equalities are correct, that is $Kz = f$. The proof is completed.

Theorem 1. To ensure that function $z \in L_2(\mathbb{R}_+, \rho)$ satisfies the equation

$$\sum_{m=1}^{N} K_{\varphi_m}\Lambda_{\psi_m} z = g \, (g \in L_2(\mathbb{R}_+, \rho)), \tag{2}$$

it is necessary and sufficient that function $y = F^{-1}\gamma\Lambda_{\psi_0^{-1}} z$ satisfy the equation

$$A(x, D)y = F^{-1}\gamma g \, (F^{-1}\gamma g \in L_2(\mathbb{R})). \tag{3}$$

Proof. Let the function $z \in L_2(\mathbb{R}_+, \rho)$ be a solution of equation (2). Acting by operator $F^{-1}\gamma$ on both the parts of the equation (2) (it is feasible, since the functions ψ_m $(m = 1, \ldots, N)$ are bounded), we obtain

$$\sum_{m=1}^{N} F^{-1}\gamma K_{\varphi_m}\Lambda_{A_m\circ h}\Lambda_{\psi_0^{-1}} z = F^{-1}\gamma g .$$

Taking into account the equalities $\Lambda_{A_m\circ h} = \gamma^{-1}FA_m(D)F^{-1}\gamma$ and $F^{-1}\gamma K_{\varphi_m}\gamma^{-1}F = \Lambda_{\varphi_m}$ $(m = 1, \ldots, N)$, we find

$$\sum_{m=1}^{N} \Lambda_{\varphi_m} A_m(D) y = F^{-1}\gamma g .$$

Earlier, it has been shown that $y = F^{-1}\gamma\Lambda_{\psi_0^{-1}} z \in \mathbb{H}_r(\mathbb{R})$ (see the proof of Lemma 1). Vice versa, now let the function $y \in \mathbb{H}_r(\mathbb{R})$ be the solution of equation (3). Then it is true that $z = \Lambda_{\psi_0}\gamma^{-1}Fy \in L_2(\mathbb{R}_+, \rho)$, the proof of which is provided in Lemma 1. At the insertion of $y = F^{-1}\gamma\Lambda_{\psi_0^{-1}} z$ into (3), we have

$$\sum_{m=1}^{N} \Lambda_{\varphi_m} A_m(D) F^{-1}\gamma\Lambda_{\psi_0^{-1}} z = F^{-1}\gamma g .$$

Taking into account the equality $\Lambda_{\varphi_m} = F^{-1}\gamma K_{\varphi_m}\gamma^{-1}F \quad (m=1,\dots,N)$, we obtain

$$\sum_{m=1}^{N} F^{-1}\gamma K_{\varphi_m}\gamma^{-1}FA_m(D)F^{-1}\gamma\Lambda_{\psi_0^{-1}}z = F^{-1}\gamma g . \qquad (4)$$

Acting by operator $\gamma^{-1}F$ on both the parts of equality (4), we find

$$\sum_{m=1}^{N} K_{\varphi_m}\gamma^{-1}FA_m(D)F^{-1}\gamma\Lambda_{\psi_0^{-1}}z = g .$$

By using equality $\gamma^{-1}FA_m(D)F^{-1}\gamma = \Lambda_{A_m\circ h} \quad (m=1,\dots,N)$, we find that $\sum_{m=1}^{N} K_{\varphi_m}\Lambda_{\psi_m} z = g$, i.e. the function z is the solution of equation (2).

The Theorem is proved.

4⁰. In case of $N=2$, $\varphi_1(x)=1$ and $\varphi_2(x)=\left[\operatorname{ch}\frac{\pi}{\alpha}\left(x-\xi+i\frac{\alpha\beta}{2}\right)\right]^{-2}$ we obtain $K_{\varphi_1}=I$ and $(K_{\varphi_2}y)(x)=\frac{1}{\pi^2}\int_0^{+\infty} y(\xi)\frac{\ln\xi-\ln x}{\xi-x}d\xi$.

Now, let us highlight two examples.

Example 1. Let $A_1(\xi)=-\xi^2-\frac{\pi^2}{\alpha^2}$, $A_2(\xi)=2\frac{\pi^2}{\alpha^2}$ and $r=2$, then we have

$$A(x,D)y(x)=y''(x)+\frac{\pi^2}{\alpha^2}\left(2\left[\operatorname{ch}\frac{\pi}{\alpha}\left(x-\xi+i\frac{\alpha\beta}{2}\right)\right]^{-2}-1\right)y(x),$$

$$\widetilde{K}z(x)=\frac{-\left(\alpha^{-1}\ln x\right)^2-\pi^2/\alpha^2}{\left(1+\left|\alpha^{-1}\ln x\right|\right)^2}z(x)+\frac{2}{\alpha^2}\int_0^{+\infty}\frac{z(\xi)}{\left(1+\left|\alpha^{-1}\ln\xi\right|\right)^2}\cdot\frac{\ln\xi-\ln x}{\xi-x}d\xi .$$

Taking into account that functions

$$y_1(x)=e^{\frac{\pi}{\alpha}x}\left[1+e^{2\frac{\pi}{\alpha}\left(x-\xi+i\frac{\alpha\beta}{2}\right)}\right]^{-1}, \quad y_2(x)=e^{-\frac{\pi}{\alpha}x}\left[1+e^{-2\frac{\pi}{\alpha}\left(x-\xi+i\frac{\alpha\beta}{2}\right)}\right]^{-1}$$

are solutions of equation $A(x,D)y=0$ from class $\mathbb{H}_2(\mathbb{R})$, we obtain according to the Assignment 1 that functions $z_k(x)=-\omega_4^{-1}\omega_2 y_k(x) \quad (k=1,2)$ are the basis of $\operatorname{Ker}\widetilde{K}$.

Example 2. Let $A_1(\xi)=-a_1 i\xi+a_2$, $A_2(\xi)=b_1 i\xi-b_2$ and $r=1$, where $a_1,a_2,b_1,b_2\in\mathbb{C}$, then we find

$$A(x,D)y(x)=\left(a_1-b_1\left[\operatorname{ch}\frac{\pi}{\alpha}\left(x-\xi+i\frac{\alpha\beta}{2}\right)\right]^{-2}\right)y'(x)+\left(a_2-b_2\left[\operatorname{ch}\frac{\pi}{\alpha}\left(x-\xi+i\frac{\alpha\beta}{2}\right)\right]^{-2}\right)y(x),$$

$$\widetilde{K}z(x)=\frac{-a_1 i\alpha^{-1}\ln x+a_2}{1+\left|\alpha^{-1}\ln x\right|}z(x)-\frac{1}{\pi^2}\int_0^{+\infty}\frac{-b_1 i\alpha^{-1}\ln\xi+b_2}{1+\left|\alpha^{-1}\ln\xi\right|}\cdot\frac{\ln\xi-\ln x}{\xi-x}z(\xi)d\xi .$$

Taking into account that the only solution of equation $A(x,D)y(x)=0$ does not belong to class $L_2(\mathbb{R})$, we obtain that equation $\widetilde{K}z(x)=0$ does not have any solutions from $L_2(\mathbb{R}_+,\rho)$.

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Վ. Վ. Սիմոնյան

Կիսաառանցքի վրա որոշ սինգուլյար ինտեգրալ հավասարումների մասին

Աշխատանքում դիտարկվում են սինգուլյար ինտեգրալ հավասա-րումներ կիսաառանցքի վրա: Այս հավասարումների լուծումը հանգեցվում է միաչափ պսևդոդիֆերենցիալ հավասարումների լուծմանը: Բերված են բացահայտ տեսքով լուծելի սինգուլյար հավասարումների օրինակներ:

В. В. Симонян

О некоторых сингулярных интегральных уравнениях на полуоси

В работе рассматриваются нехарактеристические сингулярные интегральные уравнения на полуоси. Решение этих уравнений сводится к решению одномерных псевдодифференциальных уравнений. Приведены примеры сингулярных уравнений, разрешимых в явном виде.