EXPLICIT FACTORIZATION OF A \((P,Q)\)-CIRCULANT MATRIX-FUNCTIONS

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The paper considers a factorization problem of a matrix-function, obtained from a circulant by a right and left multiplication by diagonal rational matrix-functions. Formulas for partial indices are obtained by means of ranks of a finite number of explicit type matrices. A factorization construction of this matrix-function based on factorization of finite number of functions is given as well.

**Keywords:** matrix-function, factorization, partial indices.

1. Let \( \Gamma \) be a Carleson contour, which bounds finitely connected bounded domain \( \Omega_0(0 \in \Omega_0), \Omega_\infty = \overline{\mathbb{C}} \setminus \Omega_0. \) It is known (see [1]) that the singular integral operator \( S \), defined by the formula
\[
(S\varphi)(t) = \frac{1}{\pi i} \int_{\Gamma} \frac{\varphi(\tau)}{\tau - t} d\tau, \quad t \in \Gamma,
\]
where the integral is understood in a sense of a principal value, is a bounded operator in the space \( L_p(=L_p(\Gamma)), \ 1 < p < \infty. \) We define projectors \( P_\pm = \frac{1}{2}(I \pm S) \) and classes of functions \( L^+_p = \text{Im}P_+, \quad L^-_p = \text{Im}P_-, \quad L^0_p = L^+_p + L^-_p + C. \)

Everywhere below we denote the space of \( n \)-dimensional vector-columns \((n \times n\)-order matrices) with elements from the linear space \( X \) by \( X^n \) \((X^{n\alpha\alpha})\). The abbreviations m.-f. and v.-f. will be used for matrix-function and vector-function respectively. By \( \tau_k(k \in \mathbb{Z}) \) we denote a function defined by \( \tau_k(t) = t^k. \)

By factorization of a m.-f. \( G \) of order \( n \times n \) in the space \( L_p \) along the contour \( \Gamma \) we mean the representation \( G = G_\Lambda G_\nu^{-1}, \) where a) \( G_\pm \in (L^0_p)^{n\alpha\alpha}, \)
\( G_k^{-1} \in (L^0_q)^{n\alpha\alpha}, \quad q = \frac{p}{p-1}; \) b) \( A = \text{diag}(\kappa_1, \ldots, \kappa_n), \) where \( \kappa_1 \leq \ldots \leq \kappa_n \) are numbers called partial indices. A factorization of m.-f. \( G \) satisfying to the condition \( G^{\pm 1} \in L^0_{p\alpha} \) is called generalized, if the operator \( G \cdot P_k G^{-1} \cdot I \) is bounded in \( L^n_p. \)

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Suppose that $\varphi_j \in L_\infty$, and $p_j, q_j$ ($i = 1, \ldots, n$) are rational functions with poles lying outside the contour $\Gamma$, $P = \text{diag}[p_1, \ldots, p_n]$, $Q = \text{diag}[q_1, \ldots, q_n]$ and $\Phi = (\varphi_j)_{j=1}^n$, where $\varphi_j = \varphi_{j-i+1}, i \leq j$, and $\varphi_j = \varphi_{n+j-i+1}, i > j$. The m.-f. $\Phi$ is a circulant and, therefore, a m.-f. $G$, defined by the equality $G = P\Phi Q$, we will call $(P, Q)$-circulant. The explicit representation of this m.-f. will be:

$$G = \begin{pmatrix}
p_1q_1\varphi_1 & p_1q_2\varphi_2 & \cdots & p_1q_n\varphi_n \\
p_2q_1\varphi_2 & p_2q_2\varphi_1 & \cdots & p_2q_n\varphi_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
p_nq_1\varphi_n & p_nq_2\varphi_1 & \cdots & p_nq_n\varphi_1
\end{pmatrix}.
$$

The present paper suggests a method of an explicit factorization of the m.-f. $G$. By the explicit factorization we mean a reduction of a factorization problem of the m.-f. $G$ to a finite number of factorizations of scalar functions and a finite number of solutions of linear algebraic problems. The suggested approach is based on algebraic properties of the Toeplitz operators’ family (see [2, 3]) and extends the method developed in [5, 6].

2. We introduce functions $\psi_j = \sum_{s=0}^{\varepsilon} e^{(j-1)(s-1)}\varphi_s$, $j = 1, \ldots, n$, where $\varepsilon = \exp\frac{2\pi i}{n}$ and an m.-f. $\Psi = \text{diag}[\psi_1, \ldots, \psi_n]$. It is known that $\Phi = E\Psi E^{-1}$, where $E = (e^{(k-1)(s-1)})_{k,s=1}^n$, $E^{-1} = \frac{1}{n}(e^{-(k-1)(s-1)})_{k,s=1}^n$. Consequently, $G = P\Psi E^{-1}Q$. By the Theorem 3.10 from [4], we have that the m.-f. $G$ admits a generalized factorization, iff the functions $\psi_j (j = 1, \ldots, n)$ admit generalized factorization.

Below, without loss of generality, we will assume that $q_1(0), \ldots, q_n(0) \in \overline{\mathbb{C}} \setminus \{0\}$.

Let the functions $\psi_j (j = 1, \ldots, n)$ admit the generalized factorization $\psi_j = \psi_j^\tau \psi_j^{-1}(\psi_j^\tau)^{-1}$. Then we have $G = P\Psi E^{-1}A^1\tau^\tau E^{-1}Q$, where $\Psi^\tau = \text{diag}[\psi_1^\tau, \ldots, \psi_n^\tau]$, $A^1\tau = \text{diag}[\tau_{k_1}, \ldots, \tau_{k_n}]$ and $\Psi^\tau = \text{diag}[\psi_1^\tau, \ldots, \psi_n^\tau]$.

We define m.-f. $\tilde{A} = \tau^{-\tau}PE^1\Psi A^1\tau$ and $\tilde{B} = (\Psi^\tau)^{-1}E^{-1}\tilde{A}$, where $\chi_{\text{max}} = \max \{\chi_1, \ldots, \chi_n\}$. Then $G = \tau^{-\tau}PE^1\Psi A^1\tau \tilde{B}$.

Let $p_i = \frac{p_{i1}}{p_{i2}}$, $q_i = \frac{q_{i1}}{q_{i2}}$. We write the polynomials $p_{i1}, p_{i2}, q_{i1}, q_{i2}$ as follows: $p_{i1} = p_{i1}^1p_{i1}^2$, $p_{i2} = p_{i2}^1p_{i2}^2$, $q_{i1} = q_{i1}^1q_{i1}^2$, $q_{i2} = q_{i2}^1q_{i2}^2$, where $p_{ik}, q_{ik}$ ($k = 1, 2$) are polynomials, whose zeros lie in $\Omega_k$ respectively. We denote by $p_{ik}^\pm (q_{ik}^\pm)$ ($k = 1, 2$) polynomials, which are the least common multiples of $p_{ik}^\pm, q_{ik}^\pm (q_{ik}^\pm, \ldots, q_{ik}^\pm)$. Let $A = p_{02}^{-1}A$, $B = \frac{q_{02}}{q_{02}}\tilde{B}$, where

$\tilde{A} = \tau^{-\tau}PE^1\Psi A^1\tau \tilde{B}$.
ν₀ = max deg p_i + deg p_{02}^i. Then \( G = τ_{ξ_6} A B \), where \( A ∈ (L_p^\mathbb{X})^{αn} \), \( B ∈ (L_q^\mathbb{X})^{αn} \), \( A^{-1} ∈ (M_q^-)^{αn} \), \( B^{-1} ∈ (M_p^-)^{αn} \), \( X_0 = X_{max} + ν_0 \). We define a number \( ν_- = ν_0 + X_0 - x_{min} + deg q_{02} + deg p_{01}^i + max deg p_{12} \), where \( x_{min} = min \{x_1, x_2, \ldots, x_n \} \), and polynomial \( q_- = q_{01}^i q_{02} \) (we denote its degree by \( ν_- \)). Further, we define a v.-f. \( q_+ = p_{01}^i p_{02}^i \) and \( n_- = deg q_- \). Consider families of Hankel and Toeplitz operators

\[
H_j^+ : D_p(A^{-1}) → L_p^+, H_j^- : D_p(B^{-1}) → L_p^-, j ≥ 1, \text{ defined by formulas}
\]

\[
H_j^+ φ = P_+ (τ_j^{-1} φ), H_j^- φ = P_-(τ_j B^{-1} φ), T_j φ = P_+(τ_j A B φ), \text{ where } \frac{j^2}{2} = \frac{j± |j|}{2}.
\]

We denote by \( \mathcal{Z}_j \) the space of vector polynomials \( \sum_k k z^k, \phi_k ∈ \mathbb{C}^n \), in the case when \( j > 0 \) (\( j ∈ \mathbb{Z} \)), and the space of vector polynomials in \( z^{-1} \) of type \( \sum_k k z^{-k} \) in the case when \( j < 0 \) (\( j ∈ \mathbb{Z} \)). We will suppose that \( \mathcal{Z}_0 = \{0\} \). We define a family of finite-dimensional operators \( K_j = H_j^+ H_j^- \mid_{L_p^+} \), \( j ∈ \mathbb{Z} \). Denote \( N_j = ker T_j \).

**Lemma 1.** A v.-f. \( φ \) belongs to \( N_j \), iff there exists a v.-f. \( ψ ∈ ker K_j \), such that \( φ = τ_j B^{-1} H_j^+ (ψ) \). Besides, the following equality is true:

\[
\dim N_j = ν_- + nj^+ - \dim \ker K_j, j ∈ \mathbb{Z}.
\]  

**Proof.** It is known that (see [7]) \( Im H_j^+ = Im H_j^- \mid_{L_p^+} \), \( j ∈ \mathbb{Z} \). Hence, we have the equality \( H_j^+ (ker K_j) = ker (H_j^+) \mid_{Im H_j^-} \). Therefore, to prove the first part of Lemma, it is enough to see that \( φ ∈ N_j \), iff the v.-f. \( ϕ_0 = τ_j B φ ∈ ker (H_j^+) \mid_{Im H_j^-} \).

Assume that \( φ ∈ N_j \), then \( φ ∈ (L_p^\mathbb{X})^n \) and \( P_+ (τ_j A B φ) = 0 \), i.e. \( τ_j A B φ = f ∈ (L_p^\mathbb{X})^n \). We write the last equality in the form \( τ_j A^{-1} f = τ_j B φ = φ_0 \).

We have \( B ∈ (L_q^-)^n \) and \( φ ∈ (L_p^\mathbb{X})^n \), hence, \( τ_j A^{-1} f ∈ (L_q^-)^n \). Since \( A^{-1} ∈ (M_q^-)^{αn} \), \( f ∈ (L_p^\mathbb{X})^n \), then \( P_+ (τ_j A^{-1} f) \) is a rational function, and, therefore, \( τ_j A^{-1} f ∈ (L_p^\mathbb{X})^n \), i.e. \( f ∈ D_p^\mathbb{X} (A^{-1}) \). It is easy to see that \( H_j^+ f = φ_0 \) and \( B^{-1} φ_0 = τ_j φ ∈ (L_p^\mathbb{X})^n \), i.e. \( φ_0 ∈ D_p^\mathbb{X} (B^{-1}) \) and \( H_j^+ φ_0 = P_+ (τ_j B^{-1} φ_0) = P_+ (φ) = 0 \).

Thus, \( φ_0 ∈ ker (H_j^+) \mid_{Im H_j^-} \).
Conversely, assume that \( \varphi_0 = \tau_{-j} B \varphi \in \ker (H_j^\dagger |_{\text{Im}H_j}) \). The equality \( \text{Im} H_j^\dagger = \text{Im} H_j^\dagger |_{\mathcal{Z}_{(\nu_-,j^-)}} \) implies the existence of \( f \in \mathcal{Z}_{(\nu_-,j^-)} \) such that \( \varphi_0 = \tau_{j} f \in (L_p^p)^n \). Since \( H_j^\dagger \varphi_0 = 0 \), then \( 0 = P_-(\tau_{j} B^{-1} \varphi_0) = P_-(\tau_{j} B^{-1} \tau_{-j} B \varphi) = P_-(\varphi) \) i.e. \( \varphi \in (L_p^p)^n \).

According to the definition of \( \varphi_0 \), \( \tau_{j} B \varphi = H_j^\dagger f = P_-(\tau_{j} A^{-1} f) = \tau_{j} A^{-1} f - P_-(\tau_{j} A^{-1} f) \). Consequently, \( f \tau_{j} A (\tau_{j} A^{-1} f) = \tau_{j} A B \varphi \). Taking into account that \( f \in \mathcal{Z}_{(\nu_-,j^-)} \) and \( A^{-1} \in (M_q^q)^{nq} \), we get \( \tau_{j} A (\tau_{j} A^{-1} f) \in \left( \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \right) \), i.e. \( T_j \varphi = P_+(\tau_{j} A (\tau_{j} A^{-1} f)) = 0 \), which proves the first part of our Lemma.

It remains to observe that \( \dim N_j = \dim(\ker (H_j^\dagger |_{\text{Im}H_j})) = \dim \text{Im} H_j^\dagger - \dim \text{Im} H_j^\dagger |_{\text{Im}H_j} = \dim \text{Im} H_j^\dagger - \dim \text{Im} \text{Im} H_j \) and \( \dim \text{Im} H_j^\dagger = \nu_+ + nj^- \) (see [7]) to complete the proof.

We define the m.f. \( U = \nu_+ P_-(\tau_{-\nu_+} B^{-1}) P_-(\tau_{-\nu_+} A^{-1}) \) and square matrices \( b^{(j)}_{m,k}, a^{(j)}_{m,k}, u^{(j)}_{m,k} \) \( (m,k,j \in \mathbb{Z}) \), given by

\[
b^{(j)}_{m,k} = \begin{array}{cl} <B^{-1} & \nu_- + j^- \rangle, \\
A^{(j)}_{m,k} &= \begin{cases} A^{(j)}_{m,k} = <A^{-1} & \nu_- + j^- \rangle, \quad \text{when } m < k, \\
u^{(j)}_{m,k} &= <U & \nu_- + j^- \rangle, \quad \text{when } m \geq k, \\
\text{where for a m.f. } \Phi \text{ by } \Phi >_k \text{ we mean the following matrix:} \\
\Phi >_k = \frac{1}{2\pi i} \int e^{-k \Phi(\tau)} d\tau. \end{cases}
\]

For \( j > -\nu_+ \) we define the block matrices \( A_j, B_j, U_j, \) \( K_j \) by:

\[
B_j = \begin{array}{c} b^{(j)}_{m,k} \\
A_j = a^{(j)}_{m,k} \\
u^{(j)}_{m,k} &= <U & \nu_- + j^- \rangle,
\end{array} \quad \text{where } K_j = U_j + B_j A_j, \quad \text{and } j^j = \max \{ j, \nu_+ \}. \]

For \( j \in \mathbb{Z} \) we also define mappings \( \psi_j : C^{\alpha(\nu_-,j^-)}(\mathcal{Z}_{(\nu_-,j^-)}) \to \mathcal{Z}_{(\nu_-,j^-)} \) by the formula \( \psi_j q = \sum_{k=-(\nu_+ + j^-)}^{n-1} q_k t^k \), where \( q = [q_{-(\nu_+ + j^-)}, \ldots, q_{-1}] \), \( q_k \in \mathbb{C}^n \) \((k = -(\nu_+ + j^-), \ldots, -1)\).

The following statement is true:

**Lemma 2.** If \( j < -\nu_+ \), then \( \dim N_j = 0 \). If \( j > -\nu_+ \), then \( \dim N_j = \nu_+ + nj^- - r_j \), where \( r_j = \text{rang } K_j \). Besides, if \( j \geq \nu_+ \), then \( \dim N_j = \nu_+ + nj - n \deg q_0 - \sum_{i=1}^{n-1} (\deg q_i - \deg q_{i+1}) \).

**Proof.** Since \( p_{01}^+ : p_{02}^+ : \tau_{-\nu_+} A^{-1} \in (L_q^\text{ext}) \) and \( q_+ B^{-1} \in (L_p^\text{ext}) \), the following equalities are true:

\[
\sum_{k=0}^{n} A^{-1} <_{m-k} q_+ >_k = 0, \quad m = \nu_+ + 1, \ldots, \tag{2}
\]
\[ \sum_{k=0}^{n} \langle B^{-1} \rangle_{m+k} < q_+ >_k = 0, \ m = -1, -2, \ldots \]  
(3)

Let \( j \leq -n_\circ \), \( q \in \ker K_j \) and \( \varphi = H_j q \). Then it is obvious, that \( \varphi \in \mathcal{R} \setminus (L_0^p)^{n} \). According to Lemma 1, a v.-f. \( \psi = \tau_j B^{-1} \varphi \in N_j \) and \( \psi \in (L_p^0)^{n} \). Since \( \varphi = \tau_j B \psi \), then \( \langle \varphi >_0 = \ldots = < \varphi >_{-j-1} = 0 \). The last equalities mean that 
\[ \sum_{k=0}^{n} \langle A^{-1} \rangle_{m+k} < q_+ >_k = 0 \ (m = 0, \ldots, n_\circ - 1), \] since \( \varphi = \sum_{m=-j}^{n} z^n \sum_{k=1}^{n} \langle A^{-1} \rangle_{m+k} < q_+ >_k \).

Hence, using (2) and observing that \( < q_+ >_0 \neq 0 \) and \( n_\circ \leq n_\circ \), we obtain
\[ < \varphi >_{-j} = \sum_{k=1}^{n} \langle A^{-1} \rangle_{-j+k} < q_+ >_{-k} = \sum_{m=-j}^{n} \langle A^{-1} \rangle_{-j+k} < q_+ >_{-k} = 0. \]

Similarly we get \( < \varphi >_{-j+1} = < \varphi >_{-j+2} = \ldots = 0 \), i.e. \( \varphi = 0 \). The Lemma 1 implies that \( N_j = \{ 0 \} \).

Let now \( j > -n_\circ \) while \( q \in \mathcal{R}_{-n_\circ + j} \). Using \( \tau_j \cdot P_j (\tau_j A^{-1}) q \in \bigg( \frac{0}{L_q} \bigg)^n \), \( \tau_j \cdot P_j (\tau_j A^{-1}) q \in (L_q)^{n} \) and equality \( A^j = \tau_j \cdot P_j (\tau_j A^{-1}) q + \tau_j \cdot P_j (\tau_j A^{-1}) q + \tau_j \cdot P_j (\tau_j A^{-1}) q + \cdots \), we obtain that \( H_j \cdot q = \tau_j \cdot P_j (\tau_j A^{-1}) q + g \), where
\[ g(t) = \tau_j \cdot P_j (\tau_j \cdot P_j (\tau_j A^{-1}) q) = \sum_{k=0}^{n} < g >_k t^k, \]
\( < g >_k = \sum_{m=-j+k}^{n} \langle A^{-1} \rangle_{m-j+k} < q >_m \). Hence, taking into account that \( \tau_j \cdot P_j (\tau_j B^{-1}) \tau_j \cdot P_j (\tau_j A^{-1}) q \in (L_q)^{n} \), we get that \( H_j \cdot H_j \cdot q = P_j (\tau_j U q) + H_j \cdot g \), where \( h = \tau_{j+1} \cdot q \). Consequently, the condition \( K_j q = 0 \) is equivalent to the following infinite system of equalities:
\[ \sum_{k=0}^{n} \langle U >_{m-k-j} < h >_k + \sum_{k=0}^{n} \langle B^{-1} >_{m-k-j} < g >_k = 0, \ m = -1, -2, \ldots \]

It is easy to see that \( \langle h >_{0}, \ldots, < h >_{j} >_{j+1} \rangle^T = \psi_j^{-1} \cdot q \) and \( \langle g >_{0}, \ldots, < g >_{j+1} \rangle^T = A \psi_j^{-1} \cdot q \). The remark above implies that the condition \( q \in \ker K_j \) is equivalent to the equality \( U \psi_j^{-1} \cdot q + B \psi_j^{-1} \cdot q = 0 \). By writing the last equality in the following form \( \mathcal{K} \psi_j^{-1} \cdot q = 0 \), we finally obtain that \( K_j q = 0 \), iff \( K_j \psi_j^{-1} \cdot q = 0 \). Consequently, \( \dim \ker K_j = \dim \ker K_j \). In view of (1) the following equality is true:
\[ \dim N_j = \nu_+ + n_j^+ - \dim \operatorname{Im} K_j = \nu_+ + n_j^+ - (n(\nu_+ + j^+)) - \dim \ker K_j = \nu_+ + n_j^+ - r_j. \]

It remains to prove the last statement of our Lemma 2. Let \( j \geq \nu_+ \) and \( q \in \mathcal{Z}_j \), then \( \varphi = \epsilon_j Aq \in (L_p^0)^n \), \( \varphi \in D_p^r(A^{-1}) \) and \( H_j^* \varphi = q \). Consequently, \( \mathcal{Z}_j \subset \operatorname{Im} H_j^* \). A v.-f. \( y \in D_q^r(B^{-1}) \) we write as follows \( y = \tilde{q} + q_y \), where \( y_0 \in D_q^r(B^{-1}) \), while \( \tilde{q} \) is a vector polynomial, whose degree does not exceed \( \nu_+ - 1 \). We have \( q.B^{-1}y_0 \in (L_q^0) \cap L_q^0 \) (i.e. \( q.B^{-1}y_0 \in (L_q^0)^n \)), and, therefore, the equality \( B^{-1}y = B^{-1}\tilde{q} + q.B^{-1}y_0 \) implies that \( H_q^*y = H_q^*\tilde{q} \), i.e. \( \operatorname{Im} H_q^* = \operatorname{Im} \left( H_q^* |_{\mathcal{Z}_q} \right) \).

The operator \( T^* : D_q(B) \rightarrow (L_q^0)^n \) is defined by formula \( T^*y = P_y(By) \). We prove that \( \ker T^* = \operatorname{Im} H_q^* \). Assume that \( \varphi \in \operatorname{Im} H_q^* \). Then there exists \( y \in \mathcal{Z}_q \), such that \( \varphi = H_q^*y \in P_y(B^{-1}y) \). Now \( B\varphi = y - Bp_y(B^{-1}y) \), \( Bp_y(B^{-1}y) \in (L_q^0)^n \) implies \( 0 = (L_q^0) \cap L_q^0 \), i.e. \( \varphi \in \ker T^* \). Conversely, if \( \varphi \in \ker T^* \), \( \varphi \in (L_q^0)^n \) and \( B\varphi = \varphi \in (L_q^0)^n \), i.e. \( \varphi = B^{-1}\varphi = P_y(B^{-1}y) = H_q^*\varphi \). \( B \) admits a left factorization in \( L_q \) (see [4]), and, as is known, \( \dim \ker T^* \) coincides with the sum of positive partial indices of \( B \). Since \( B \) is analytic inside the circle, then its partial indices are nonnegative (see [4]). Therefore, \( \dim \ker T^* \) coincides with total index of \( B \). On the other hand, total index of \( B \) is equal to the number of zeros inside \( \Omega_+ \) (by taking into account their multiplicities) of function \( \det B \). Thus, \( \dim \operatorname{Im} H_q^* = n \deg q_0 + \sum_{i=1}^n (\deg q_{i1} - \deg q_{i2}) \). For \( j \geq 0 \) we obtain

\[ \operatorname{Im} H_q^* \supset \operatorname{Im} K_j = \operatorname{Im} \left( H_q^* |_{\mathcal{Z}_q \cap \operatorname{Im} K_j} \right) \supset \operatorname{Im} \left( H_q^* |_{\mathcal{Z}_q \cap \operatorname{Im} K_j} \right). \]

and for \( j \geq \nu_+ \) we get \( \operatorname{Im} H_q^* \supset \operatorname{Im} K_j \supset \operatorname{Im} H_q^* \). Consequently, we have \( \dim \operatorname{Im} K_j = \dim \operatorname{Im} H_q^* \), \( j \geq \nu_+ \), and the Lemma is proved.

Note that, particularly, the following statement is proved:

**Corollary 1.** For \( j > -\nu_+ \) the following equality is true: \( \ker K_j = \psi \ker K_j \).

**Theorem 1.** The partial indices of m.-f. \( G \) can be calculated by formulas:

\[ \kappa_i = -\nu_+ + \chi_0 + \operatorname{card} \{ j : n\theta_j - r_j + r_{j-1} < i, \quad j = -\nu_+ + 1, \ldots, \nu_+ + 2 \}, \quad (4) \]

where \( r_{-\nu_+} = \nu_+ \), \( r_j = \operatorname{rang} K_j \) (\( j > -\nu_+ \)) and \( \theta_j = 1, j > 0, \theta_j = 0, j \leq 0 \).

**Proof.** It is known that \( \dim N_j \) is equal to the sum of negative partial indices of the m.-f. \( \tau_{\chi_0^+}G \) with the minus sign. The partial indices of the m.-f. \( \tau_{\chi_0^+}G \) are equal to \( \kappa_i - \chi_0 - j (i = 1, \ldots, n) \). As we have \( j > \kappa_i - \chi_0 \), then \( \dim N_j > 0 \). Consequently, \( -\nu_+ \leq \kappa_i - \chi_0 \). Similarly (see [5]), it is not difficult to see that
where \( \eta_-, \eta_+ \) are arbitrary integer numbers satisfying to \( \eta_- \leq \kappa_i - \chi_0 \leq \eta_+ \). We can take \( \eta_- \) to be equal to \(-\nu_-\), while by Lemma 2 we can choose \( \eta_+ \) to be the number \( \nu_+ + 2 \). Taking into account also the equality \( \dim N_j - \dim N_{j-1} = n\theta_j + r_{j-1} - r_j \), we get (4).

Lemmas 1 and 2 imply that \( N_j = \{ \tau_j B^{-1} P_j(\tau_j, A^{-1} \psi_j q), \ q \in \ker K_j \}, \ j > -\nu_- \).

We denote \( \tilde{N}_j = N_j + \tau N_j = \{ \phi + \tau \psi ; \phi, \psi \in N_j \} \) and \( N_j(0) = \{ \phi(0), \phi \in N_j \} \). It is known (see [3]) that \( \tilde{N}_j \subset N_{j+1}, \ j \in \mathbb{Z} \). We denote by \( M_j \) some direct complement of \( \tilde{N}_j \) in \( N_{j+1} \). Spaces \( M_j \) (see [2]) are called \((p,j)\)-index subspaces. We denote \( \xi_i = \kappa_i - \chi_0 + 1 \) (\( i = 1, \ldots, n \)). It is known that \( \tilde{N}_j = \{ 0 \} \) for \( j \leq \xi_1 - 1 \) and \( N_j = \tilde{N}_{j-1} \) for all \( j \in \mathbb{Z} \\{ \xi_1, \ldots, \xi_n \} \). The following statement follows from [3]:

**Proposition 1.** Assume \( \phi_1, \ldots, \phi_m \ (i = 1, \ldots, n, \ m_i \in \mathbb{Z}) \) are bases in the space \( M_{\xi_i} \). Then \( \tau_{-\nu_-} \) is a factorization of \( \tau \). \( \tau_{-\nu_-} \) is invertible.

**Proof.** Let \( \phi \in \tilde{N}_j \), then there exists a vector \( q = [q_{-\nu_-+1}, \ldots, q_{-1}] \in \ker K_j \), \( q_i \in \mathbb{C}^n, s = (-\nu_- + j^*, \ldots, -1) \), such that \( \phi(t) = t^{- j^*} B^{-1}(t) P_j(\tau_j, A^{-1} \psi_j q) \).

\[
< \tau_j, A^{-1} \psi_j q >_m = \frac{1}{2\pi i} \left[ \tau_j A^{-1}(t) \sum_{k=\nu_-+1}^{\infty} q_k t^{k-m-1} dt = \sum_{k=\nu_-+1}^{\infty} \frac{q_k}{2\pi i} \right] A^{-1}(t) t^{k-m-1} dt = \sum_{k=\nu_-+1}^{\infty} < A^{-1} >_{m-k-j^*} q_k.
\]

The \( \nu_- \) is analytic in \( \Omega_+ \), and hence, it can be de expanded into the series \( \phi(t) = B^{-1}(t) \sum_{m=0}^{\infty} \left( \sum_{k=\nu_-+1}^{\infty} < A^{-1} >_{m-k-j^*} q_k \right) t^{m+j^*} \) in a neighborhood of \( 0 \).

Besides, \( N_j(0) = \{ B^{-1}(0) \mathcal{K} q, q \in \ker K_j \} \), since the \( m \)-f. \( B(0) \) is invertible. The existence of a factorization collection follows now from properties of spaces \( N_j(0) \) (see [3]). Proof is completed.

**Theorem 2.** Let \( q_k(i = 1, \ldots, n) \) be a factorization collection for the \( m \)-f. \( G \) and \( \phi_i = \tau_{\xi_i} B^{-1} H_{\xi_i} \psi_{\xi_i} q_i, \ i = 1, \ldots, n \), then \( G_{\psi} = [\phi_1, \ldots, \phi_n] \), \( \Lambda = \text{diag}[t^u, \ldots, t^{v^*}] \),
$G_\ast = GG_\ast A^{-1}$ is a factorization of m.-f. $G$.

**Proof.** Lemma 1 and Corollary 1 imply that $\phi_i \in N_{\xi_i} (i = 1, \ldots, n)$. Since $\phi_1(0), \ldots, \phi_n(0)$ are linearly independent, then $\phi_i$ does not belong to $\tilde{N}_{\xi_i}$. Consequently, $\phi_i \in M_{\tilde{N}_{\xi_i}} (i = 1, \ldots, n)$. Taking into account linear independence of a v.-f. $\phi_i (i = 1, \ldots, n)$ we deduce the proof of our Theorem from the Proposition 1.

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