ON AN INTEGRO-DIFFERENTIAL EQUATION OF PSEUDOPARABOLIC-
PSEUODOHYPERBOLIC TYPE WITH DEGENERATE KERNELS

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In the article the questions of solvability of boundary value problem for
a homogeneous pseudoparabolic-pseudohyperbolic type integro-differential
equation with degenerate kernels are considered. The Fourier method based
on separation of variables is used. A criterion for the one-valued solvability
of the considering problem is found. Under this criterion the one-valued
solvability of the problem is proved.

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kernels, integral condition, one valued solvability.

Problem Statement. The partial differential equations of third and fourth
order are important with their physical applications [1-5]. Problems, where the type
of differential equation is changing in the considering domain, have important
applications [6-8]. The mixed type differential equations have been studied by many
authors, in particular in [9-16].

In the present paper we consider the one-valued solvability of nonlocal
problem for a mixed type integro-differential equation with degenerate kernels. So,
in the rectangular domain \( \Omega = \{(t,x)| -T < t < T, \ 0 < x < l \} \) we consider the
following mixed type equation:

\[
\begin{align*}
U_t - U_{txx} - U_{xx} + \nu \int_{0}^{T} K_1(t,s)U(s,x)ds &= 0, \quad t > 0, \\
U_t - U_{txx} - U_{xx} + \nu \int_{-T}^{0} K_2(t,s)U(s,x)ds &= 0, \quad t < 0,
\end{align*}
\]

where \( T \) and \( l \) are given positive real numbers; \( \nu \) is spectral real parameter,
\( K_j(t,s) = a_j(t)b_j(s), \ a_j(t), b_j(s) \in C[-T; T], \ j = 1,2. \)

Problems. Find in the domain \( \Omega \) the function

\[
U(t,x) \in C(\Omega) \cap C^1(\Omega \cup \{x = 0\} \cup \{x = l\}) \cap C^{1,2}(\Omega_+) \cap C^{2,2}(\Omega_-) \cap C^{1,2}_{t,x}(\Omega_+) \cap C^{2,2}_{t,x}(\Omega_-),
\]

satisfying to the Eq. (1) and following conditions:

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\[ \int_0^T U(t,x) \, dt = \varphi(x), \quad 0 \leq x \leq l, \quad (2) \]
\[ U(t,0) = U(t,l) = 0, \quad -T \leq t \leq T, \quad (3) \]

where \( C^r \) is a class of functions having continuous derivatives \( \frac{\partial^r}{\partial t^r}, \frac{\partial^r}{\partial x^r} \); \( C^{r,s} \) is a class of functions having continuous derivatives \( \frac{\partial^{r,s}}{\partial t^r \partial x^s}, \frac{\partial^{r,s}}{\partial t^r \partial x^s} \); \( C^{r,s}_{t,x} \) is a class of functions having continuous derivatives \( \frac{\partial^{r,s}}{\partial t^r \partial x^s} \); \( C^{r,s}_{t,x} \) is a class of functions having continuous derivatives \( \frac{\partial^{r,s}}{\partial t^r \partial x^s} \); \( C^{r,s}_{t,x} \) is a class of functions having continuous derivatives \( \frac{\partial^{r,s}}{\partial t^r \partial x^s} \); \( C^{r,s}_{t,x} \) is a class of functions having continuous derivatives \( \frac{\partial^{r,s}}{\partial t^r \partial x^s} \).

We solve Eqs. (10) by the method of variation of arbitrary constants.

By denoting \( \varphi(t) = 0, \varphi(0) = \varphi(l) = 0, \varphi(t) = \varphi(t) = 0, \varphi(t) = \varphi(t) = 0, \) and \( \varphi(t) = \varphi(t) = 0, \varphi(t) = \varphi(t) = 0, \) form a complete system of orthonormal functions in \( L_2[0; l] \), while \( \mu_n = \frac{\pi n}{l} \) are the corresponding eigenvalues and

\[ u_n(t) = \sqrt{\frac{2}{l}} \int_0^l U(t,x) \sin \mu_n x \, dx, \quad n = 1, 2, \ldots \quad (5) \]

Substituting the series (5) into Eq. (1), we obtain

\[ u_n'(t) + \lambda_n^2 u_n(t) = v \int_0^t a_1(t) b_1(s) u_n(s) \, ds, \quad t > 0, \quad (6) \]
\[ u_n''(t) + \lambda_n^2 u_n(t) = v \int_{-T}^t a_2(t) b_2(s) u_n(s) \, ds, \quad t < 0, \quad (7) \]

where \( \lambda_n^2 = \frac{\mu_n^2}{1 + \mu_n^2}, \mu_n = \frac{\pi n}{l} \).

By denoting

\[ \alpha_n = \int_0^t b_1(s) u_n(s) \, ds, \quad (8) \]
\[ \beta_n = \int_{-T}^t b_2(s) u_n(s) \, ds, \quad (9) \]

the Eqs. (6) and (7) can be written by

\[ u_n'(t) + \lambda_n^2 u_n(t) = v \alpha_1(t) \alpha_n, \quad t > 0, \quad u_n'(t) + \lambda_n^2 u_n(t) = v \alpha_2(t) \beta_n, \quad t < 0. \]

We solve Eqs. (10) by the method of variation of arbitrary constants

\[ u_n(t) = A_n \exp \left\{ -\lambda_n^2 t \right\} + \eta_{1n}(t), \quad t > 0, \quad (11) \]
\[ u_n(t) = C_n \cos \lambda_n t + D_n \sin \lambda_n t + \eta_{2n}(t), \quad t < 0, \quad (12) \]

where \( A_n, B_n, C_n, D_n \) are while arbitrary constants to be determined and \( \eta_{1n}(t) = v \alpha_n h_n(t), \quad \eta_{2n}(t) = v \beta_n \delta_n(t) \).
\[ h_n(t) = \int_0^t \exp \{-\lambda_n^2(t-s)\} a_1(s)ds, \quad \delta_n(t) = \frac{1}{\lambda_n^2} \int_0^t \sin \lambda_n(t-s)a_2(s)ds. \]

From the statement of the problem it follows that \( U(0+0,x) = U(0-0,x); U_t(0+0,x) = U_t(0-0,x). \) Hence, taking into account (5), we obtain

\[
u_n(0+0) = \sqrt{\frac{T}{\pi}} \int_0^t U(0+0,x) \sin \mu_n xdx = \]
\[
= \sqrt{\frac{T}{\pi}} \int_0^t U(0-0,x) \sin \mu_n xdx = u_n(0-0). \tag{13} \]

Differentiating (5) one times with respect to \( t \) similarly (13), we derive

\[
\dot{u}_n(0+0) = \sqrt{\frac{T}{\pi}} \int_0^t U_t(0+0,x) \sin \mu_n xdx = \]
\[
= \sqrt{\frac{T}{\pi}} \int_0^t U_t(0-0,x) \sin \mu_n xdx = \dot{u}_n(0-0). \tag{14} \]

From (11) and (12), taking into account (13) and (14), we obtain that \( B_n = A_n, C_n = -\lambda_n A_n. \) Then functions (11) and (12) take the form

\[
u_n(t) = A_n \exp \{-\lambda_n^2 t\} + 1n(t), \quad t > 0, \tag{15} \]
\[
u_n(t) = A_n \cos \lambda_n t - \lambda_n A_n \sin \lambda_n t + 1n(t), \quad t < 0. \tag{16} \]

Taking into account (5), the condition (2) takes the following form

\[
\int_0^T u_n(t)dt = \sqrt{\frac{T}{\pi}} \int_0^t \int_0^T U(t,x) dt \sin \mu_n xdx = \sqrt{\frac{T}{\pi}} \int_0^t \phi(x) \sin \mu_n xdx = \phi_n, \tag{17} \]

where \( \phi_n = \sqrt{\frac{T}{\pi}} \int_0^t \phi(x) \sin \mu_n xdx, \quad n = 1, 2, \ldots \)

To find the unknown coefficients \( A_n \) in (15) and (16), we use the condition (17)

\[
\int_0^T u_n(t)dt = \int_0^T A_n \exp \{-\lambda_n^2 t\} + 1n(t) \int_0^t A_n \exp \{-\lambda_n^2 t\} + 1n(t) -1 \left[ \exp \{-\lambda_n^2 T\} - 1 \right] + 1n(t) = \phi_n, \tag{18} \]

where \( 1n(t) = \int_0^T 1n(t)dt. \)

Since \( 0 < T < \infty, 0 < \lambda_n^2 < 1, \) we have \( \exp \{-\lambda_n^2 T\} \neq 1. \) So from Eq. (18) \( A_n \) is uniquely determined by

\[
A_n = \frac{\lambda_n^2}{\sigma_n} \left[ \phi_n - 1n + \lambda_n^2 T \right], \quad \sigma_n = 1 - \exp \{-\lambda_n^2 T\}. \]

Substituting the founded values \( A_n \) into formulas (15) and (16), we get

\[
u_n(t) = \frac{\lambda_n^2}{\sigma_n} \left( \phi_o - 1n + \lambda_n^2 T \right) + 1n(t), \quad t > 0, \tag{19} \]
\[
u_n(t) = \frac{\lambda_n^2}{\sigma_n} \left( \phi_o - 1n \right) \left( \cos \lambda_n t - \lambda_n \sin \lambda_n t + 1n(t) \right), \quad t < 0. \tag{20} \]
Taking into account that $\xi_{1n} = \int_0^T \eta_{1n}(t)dt$, $\eta_{1n}(t) = \nu \alpha_n h_n(t)$ and $\eta_{2n}(t) = \nu \beta_n \delta_n(t)$, we rewrite the formulas (19) and (20) as follows

$$u_n(t) = \varphi_n M_{1n}(t) + \nu \alpha_n M_{2n}(t), \quad t > 0,$$

$$u_n(t) = \varphi_n N_{1n}(t) - \nu \alpha_n N_{2n}(t) + \nu \beta_n \delta_n(t), \quad t < 0,$$

where

$$M_{1n}(t) = \frac{\lambda^2_n}{\sigma_n} \exp \{- \lambda^2_n t\}, \quad M_{2n}(t) = h_n(t) - M_{1n}(t) \int_0^T h_n(t)dt,$$

$$N_{1n}(t) = \frac{\lambda^2_n}{\sigma_n} \left[ \cos \lambda_n t - \lambda_n \sin \lambda_n t \right], \quad N_{2n}(t) = N_{1n}(t) \int_0^T h_n(t)dt,$$

$$0 < \lambda_n = \sqrt{\frac{\mu^2_n}{1 + \mu^2_n}} < 1, \quad \mu_n = \frac{\pi n}{T}.$$

Substituting (21) into (9) and (22) into (10), we obtain the countable system of two algebraic equations (CSTAE) of variables $\alpha_n$ and $\beta_n$

$$\begin{cases}
\alpha_n (1 - \nu P_{2n}) = \varphi_n P_{1n}, \\
\alpha_n \nu Q_{2n} + \beta_n (1 - \nu Q_{3n}) = \varphi_n Q_{1n},
\end{cases}$$

(23)

where $P_{1n} = \int_0^T b_1(s) M_{1n}(s)ds$, $P_{2n} = \int_0^T b_1(s) M_{2n}(s)ds$, $Q_{1n} = \int_0^T b_2(s) N_{1n}(s)ds$, $Q_{2n} = \int_{T}^0 b_2(s) N_{2n}(s)ds$, $Q_{3n} = \int_{T}^0 b_2(s) \delta_n(s)ds$.

For solvability of CSTAE (23) we impose the following condition

$$\nu = \nu_n \neq \frac{1}{P_{2n}}, \quad \nu = \nu_n \neq \frac{1}{Q_{3n}}.$$ 

(24)

We subtract the values $\nu_1n = \frac{1}{P_{2n}}$ and $\nu_2n = \frac{1}{Q_{3n}}$ of spectral parameter $\nu$ from the set of real numbers $R = (-\infty; \infty)$. The obtained set $\Lambda = R \setminus \{\nu_1, \nu_2\}$ is called the set of regular values of the parameter $\nu$. For all values $\nu \in \Lambda$ the condition (24) is fulfilled. For regular values of the kernel of the mixed integro-differential Eq. (1), first we solve CSTAE (23), and then problem (1)–(3). Substituting the solution of CSTAE (23)

$$\alpha_n = \varphi_n \frac{P_{1n}}{1 - \nu P_{2n}}, \quad \beta_n = \varphi_n \left[ \frac{Q_{1n}}{1 - \nu Q_{3n}} - \frac{Q_{2n}}{1 - \nu Q_{3n}} \right],$$

in (21) and (22), we derive

$$u_n(t, \nu) = \Phi_n(t, \nu), \quad t > 0,$$

$$u_n(t, \nu) = \varphi_n \Psi_n(t, \nu), \quad t < 0,$$

(25)

(26)

where

$$\Phi_n(t, \nu) = M_{1n}(t) + \nu M_{2n}(t) \frac{P_{1n}}{1 - \nu P_{2n}},$$

$$\Psi_n(t, \nu) = N_{1n}(t) - \nu N_{2n}(t) + \nu \beta_n \delta_n(t).$$
\[ \Psi_n(t, v) = N_1n(t) - vN_2n(t) \frac{P_{1n}}{1 - vP_{2n}} + v\delta_n(t) \left[ \frac{Q_{1n}}{1 - vQ_{3n}} - \frac{Q_{2n}}{1 - vQ_{3n}} \cdot \frac{P_{1n}}{1 - vP_{2n}} \right]. \]

Now we substitute (25) and (26) into series (4) and we get

\[ U(t, x, v) = \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \phi_n \Phi_n(t, v) \sin \nu x, \quad t > 0, \tag{27} \]

\[ U(t, x, v) = \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \phi_n \Psi_n(t, v) \sin \nu x, \quad t < 0. \tag{28} \]

**The Justification of the Solvability of the Boundary Value Problem (1)–(3).** We show that under certain conditions with respect to the function \( \phi(x) \) the series (27) and (28) converge absolutely and uniformly. Indeed, in the formulation of the problem the functions \( M_{1n}(t), M_{2n}(t) \) are uniformly bounded on the segment \([0; T]\), and the functions \( N_{1n}(t), N_{2n}(t) \) and \( \delta_n(t) \) are uniformly bounded on the segment \([-T; 0]\). We consider such regular values of the spectral parameter \( v \in \Lambda \), for which \( |\Phi_n(t, v)| < \infty \) for all \( t \in [0; T] \) and \( |\Psi_n(t, v)| < \infty \) for all \( t \in [-T; 0] \). We note that \( 0 < \lambda_n < 1 \). So for any natural \( n \) from the (25) and (26) we have estimates

\[ |u_n(t)| \leq C_{1n} |\phi_n|, \tag{29} \]

\[ |u''_n(t)| \leq C_{2n} |\phi_n|, \tag{30} \]

where

\[ C_{1n} = \max \left\{ \max_{t \in [0, T]} |\Phi_n(t, v)|; \max_{t \in [-T, 0]} |\Psi_n(t, v)| \right\}, \]

\[ C_{2n} = \max \left\{ \max_{t \in [0, T]} |\Phi''_n(t, v)|; \max_{t \in [-T, 0]} |\Psi''_n(t, v)| \right\}. \]

**Condition A.** Suppose that the following condition is satisfied:

\[ \left( \sum_{n=1}^{\infty} |C_{1n}|^2 \right)^{1/2} < \infty, \quad i = 1, 2. \]

**Condition B.** Suppose that the function \( \phi(x) \in C^2[0; l] \) on the segment \([0; l]\) has piecewise-continuous third-order derivatives and \( \phi(0) = \phi(l) = \phi_x(0) = \phi_x(l) = 0 \).

Then by integrating by parts 3 times with respect to the variable \( x \) in the integral

\[ \phi_n = \left( \frac{2}{l} \right)^{1/2} \int_0^l \phi(x) \sin \frac{\pi n}{l} x dx \]

we get

\[ \phi_n = -\left( \frac{l}{\pi} \right)^3 p_n \frac{n}{n^2}, \tag{31} \]

\[ \sum_{n=1}^{\infty} p_n^2 \leq \frac{4}{l^2} \int_0^l |\phi_{xxx}(x)|^2 dx < \infty. \tag{32} \]

We note that formula (32) represents the Bessel inequality. Using (31) and (32), taking into account (29) and (30), now we can show that the series (27) and (28) converge absolutely and uniformly in the domain \( \Omega \). In this case termwise
differentiation of these series (27) and (28) with respect to variables \( t \) and \( x \) is possible and the obtaining series will converge absolutely and uniformly in the domain \( \Omega \).

Indeed, using (29), (31) and (32) and applying the Minkowski and Hölder inequalities, for series (27) and (28) in the domain \( \Omega \) the following estimate is obtained:

\[
|U(t,x)| = \left| \sum_{n=1}^{\infty} u_n(t) \sin \mu_n x \right| \leq \sqrt{\sum_{n=1}^{\infty} |C_{1n}|^2} \sqrt{\sum_{n=1}^{\infty} |\phi_n|^2} \leq \\
\leq \gamma \sum_{n=1}^{\infty} \frac{1}{n^\theta} |p_n| \leq \gamma \sum_{n=1}^{\infty} \frac{1}{n^\theta} \sqrt{\sum_{n=1}^{\infty} |p_n|^2} \leq \\
\leq \frac{2\gamma}{l} \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^\theta}} \sqrt{\int_0^l \left| (\phi_{xxx}(x)) \right|^2 dx} < \infty,
\]

where \( \gamma = \frac{\sum_{n=1}^{\infty} |C_{1n}|^2}{l^{3/2}} \).

From (33) it follows that series (27) and (28) converge absolutely and uniformly in domain \( \Omega \). Formally differentiating functions (27) and (28) we obtain

\[
U_{tt}(t,x,v) = \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \phi_n \Phi_n''(t,v) \sin \mu_n x, \quad t > 0, \tag{34}
\]

\[
U_{tt}(t,x,v) = \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \phi_n \Psi_n''(t,v) \sin \mu_n x, \quad t < 0, \tag{35}
\]

\[
U_{xx}(t,x,v) = -\sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \mu_n^2 \phi_n \Phi_n(t,v) \sin \mu_n x, \quad t > 0, \tag{36}
\]

\[
U_{xx}(t,x,v) = -\sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \mu_n^2 \phi_n \Psi_n(t,v) \sin \mu_n x, \quad t < 0, \tag{37}
\]

\[
U_{txxx}(t,x,v) = -\sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \mu_n^2 \phi_n \Phi_n''(t,v) \sin \mu_n x, \quad t > 0, \tag{38}
\]

\[
U_{txxx}(t,x,v) = -\sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \mu_n^2 \phi_n \Psi_n''(t,v) \sin \mu_n x, \quad t < 0, \tag{39}
\]

where \( \mu_n = \frac{\pi n}{l} \).

Analogously to (33), taking into account formulas (30)–(32) and applying the Minkowski and Hölder inequalities, for the series (34) and (35) in domain \( \Omega \) we obtain the following estimate:

\[
|U_{tt}(t,x)| = \left| \sum_{n=1}^{\infty} u_n''(t) \sin \mu_n x \right| \leq \sqrt{\sum_{n=1}^{\infty} |C_{2n}|^2} \sqrt{\sum_{n=1}^{\infty} |\phi_n|^2} \leq \\
\leq \gamma \sum_{n=1}^{\infty} \frac{1}{n^\theta} |p_n| \leq \frac{2\gamma}{l} \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^\theta}} \sqrt{\int_0^l \left| (\phi_{xxx}(x)) \right|^2 dx} < \infty,
\]

where \( \gamma = \frac{\sum_{n=1}^{\infty} |C_{2n}|^2}{l^{3/2}} \).
Analogously, for the series (36) and (37) in the domain $\Omega$ we get the estimate

$$
\left| U_{xx}(t, x) \right| = \sqrt{2} \pi^2 \sum_{n=1}^{\infty} n^2 \left| u_n(t) \right| \sin \mu_n x \leq \gamma_3 \sqrt{\sum_{n=1}^{\infty} n^2 \left| \phi_n \right|^2} \leq \gamma_3 \frac{1}{l} \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^l \left| \phi_{xx}(x) \right|^2 dx} < \infty,
$$

(41)

where $\gamma_3 = \sqrt{2} \pi^2 \frac{l}{\sum_{n=1}^{\infty} \left| C_{1n} \right|^2 \pi^2}.$

Similarly, for the series (38) and (39), as in the estimates (40) and (41) in the domain $\Omega$, we easily obtain that

$$
\left| U_{tttx}(t, x) \right| < \infty.
$$

Consequently, the function $U(t, x)$ on $\Omega$, defined by the series (27) and (28), satisfies the conditions of the problem.

To establish the uniqueness of the solution, we show that under the zero integral condition $\int_0^l U(t, x) dx = 0$, $0 \leq x \leq l$, the boundary value problem (1)–(3) has only the trivial solution. We suppose that $\varphi(x) \equiv 0$. Then $\varphi_n = 0$ and from (27) and (28) in domain $\Omega$ we get

$$
\int_0^l U(t, x) \sin \pi n x dx, \ n = 1, 2, \ldots
$$

Hence, by virtue of the completeness of systems of eigenfunctions $\left\{ \sqrt{2} \frac{\pi}{l} \sin(\pi n/l)x \right\}$ in $L_2[0; l]$ we conclude, that $U(t, x) \equiv 0$ for all $x \in [0; l]$ and $t \in [-T; T]$.

Consequently, for regular values of the spectral parameter $\nu \in \Lambda$ the problem (1)–(3) has a unique solution in domain $\Omega$.

Thus it is proved that the following theorem holds.

**Theorem.** Let the conditions A and B are satisfied. Then for regular values of the spectral parameter $\nu \in \Lambda$ the problem (1)–(3) is uniquely solvable in domain $\Omega$.

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