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Mathematics

STABILITY OF FREQUENCY DISTRIBUTION IN FRAME OF NATURAL PARAMETRIZATION. I

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In this paper the stability problem for frequency distribution in frame of natural parameterization is formulated and discussed. The case of finite number of independent parameters is characterized. A corresponding stability problem is investigated in terms of l_p-metric.

Keywords: frequency distribution, l_p-metric, stability by parameters.

Introduction. The sequence $\{p_n\}_0^\infty$ forms a frequency distribution (FD) if $p_n > 0,\ n \ge 0$ and $\sum p_n = 1$. In the bioinformatics (see [1]):

$$p_n = n^{-\rho} L(n),\ \ n \ge 1,\ \ 1 < \rho < +\infty,\ \lim_{n\to\infty} L(n) = L \in R^+ = (0,+\infty),$$
$$\frac{L(n)}{L(n-1)} = 1 + o\left(\frac{1}{n}\right),\ \ n \to +\infty, \tag{1}$$

$$p_n > p_{n+1},\ \frac{p_n}{p_{n+1}} > \frac{p_{n+1}}{p_{n+2}} \ \text{ starting from some } n_0 \ge 0 . \tag{2}$$

Assume that in (1) $\rho \in (2,+\infty)$ and in (2) $n_0 = 0$.

The unknown FDs are approximated by various parametric distributions $\{p_n(\vec c)\}_0^\infty$ with the vector $\vec c$ of parameters, that are referred to also as FDs.

Let $\vec c = \vec c_m = (c_1,\dots,c_m) \in \Omega,\ m < +\infty$, and $K \subseteq \Omega$ be a *bounded, closed, convex* set, and μ be some metric in the set $\left\{\{p_n(\vec c_m)\}_0^\infty : \vec c_m \in \Omega\right\}$.

We say that m-parametric FD $\{p_n(\vec c_m)\}_0^\infty$ with independent parameters is μ-stable (with respect to the parameters) on K, if uniformly on $\vec c_m, \vec c\,'_m \in K$

$$\lim_{|\vec c_m - \vec c\,'_m| \to 0} \mu\left(\{p_n\}_0^\infty, \{p'_n\}_0^\infty\right) = 0 \ . \tag{3}$$

The parameters are *independent*, if not one of these is a function of others. Here

$$|\vec c_m - \vec c\,'_m| = \sum_{i=1}^m |c_i - c'_i|,\ \vec c_m = (c_1,\dots,c_m), \vec c\,'_m = (c'_1,\dots,c'_m),\ p_n = p_n(\vec c_m),\ \ p'_n = p_n(\vec c\,'_m),\ n \ge 0.$$

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Similarly, $\{p_n(\vec{c}_m)\}_0^\infty$ is μ-stable, if (3) holds for any K.

Stability Problem. Introduce the metric μ and find conditions for μ-stability of FD $\{p_n(\vec{c}_m)\}_0^\infty$.

In this paper the stability problem for FD $\{p_n\}_0^\infty$ in bioinformatics is formulated in frame of *natural parameterization* (NP) in case of finite number of independent parameters and is solved in terms of l_p-metric, $p>0$.

Natural Parameterization (NP). Due to [2], $\{p_n\}_0^\infty$ is a FD, iff

$$p_n = p_0\prod_{k=1}^{n}\varepsilon_k,\ n>1,\ \varepsilon_n>0,\ \ p_0=\left(1+\sum_{n\ge1}\prod_{k=1}^{n}\varepsilon_k\right)^{-1}, \tag{4}$$

$$\sum_{n\ge1}\prod_{k=1}^{n}\varepsilon_k<+\infty. \tag{5}$$

Then $\varepsilon_n=(p_n/p_{n-1}),\ n\ge1$. The coefficients $\varepsilon_1,\varepsilon_2,...$ are treated as parameters of $\{p_n\}_0^\infty$ in its NP (4)–(5). Let $T=\{1,2,...\}=\bigcup_{j=1}^{m}T_j,\ T_j=\{k_j,k_j+1,...,k_{j+1}-1\}$, $j=\overline{1,m}$, where $1=k_1<k_2<...<k_m<+\infty(=k_{m+1})$, and any parameter in the set $G_j=\{\varepsilon_k:k\in T_j\}$ is *uniquely* determined by the vector $(c_1,...,c_j)$. Here $c_i=\varepsilon_{k_i}$, $i=\overline{1,j}$. The parameters $c_1,...,c_m$ are *independent.* This is a characterization of m-parametric FD $\{p_n\}_0^\infty$ in its NP (4)–(5).

So, $\{p_n\}_0^\infty=\{p_n(\vec{c}_m)\}_0^\infty$ and $\varepsilon_k=h_k(c_1,...,c_j)$ for given $j,\ j=\overline{1,m}$, and $\varepsilon_k\in G_j$. It is clear that $h_{k_j}(c_1,...,c_j)=c_j$ for $j=\overline{1,m}$. Assume that:

a) h_k for $k\ne k_j, j=\overline{1,m}$, is increased by each parameter separately;

b) partial derivatives $(\partial h_k/\partial c_i)$ for $i=\overline{1,j}$ exist and are uniformly bounded with respect to $k\ge1$.

Now recall the form of l_p-metric: $l_p\left(\{p_n\}_0^\infty,\{p'_n\}_0^\infty\right)=\sum_{n\ge o}|p_n-p'_n|^p$, and formulate in this case the *Stability Problem*: for admissible $p>0$ prove the l_p-stability of FD $\{p_n\}_0^\infty$.

Theorem 1. In our case the FD $\{p_n\}_0^\infty$ is $l_{1/2}$-stable.

For given $K\subseteq\Omega$ and $j=1,m$ denote

$$\underline{c}_j=\{\inf c_j:\vec{c}_m\in K\},\quad \overline{c}_j=\{\sup c_j:\vec{c}_m\in K\}. \tag{6}$$

It is easy to see that for given $j,\ j=\overline{1,m}$, there is $\vec{c}_m\in K$ with $c_j=\overline{c}_j$. Indeed, if for all $\vec{c}_m\in K$ the components c_j are identical, then the statement is obvious. Assume that there are $\vec{c}_m,\vec{c}'_m\in K$ with $c_j<c'_j$ for given $j,\ j=\overline{1,m}$. Due to the

convexity of K, for any $c\in[c_j,c'_j]$ there is $\vec{c}''_m\in K$ such that $c''_j=c$. In this case, as it follows from the definition of $\overline{c}_j$ for given $j,\ j=\overline{1,m}$, there is a sequence $\left\{\vec{c}_m^{(k)}\right\}_1^\infty\in K$ such that $\lim\limits_{k\to+\infty}c_j^{(k)}=\overline{c}_j$. Extracting the convergent subsequence $\left\{\vec{c}_m^{(k_s)}\right\}_1^\infty\in K$ from $\left\{\vec{c}_m^{(k)}\right\}_1^\infty$, the statement is proved due to the closeness of K.

Similarly, for given $j,\ j=\overline{1,m}$, there is $\vec{c}_m\in K$ with $c_j=\underline{c}_j$.

Let us show that in (6)

$$\underline{c}_1<\underline{c}_2<\ldots<\underline{c}_m \quad\text{and}\quad \overline{c}_1<\overline{c}_2<\ldots<\overline{c}_m. \tag{7}$$

In order to prove the second chain of inequalities (7) assume the opposite, i.e. there are indices i and $j,\ i<j$, such that $\overline{c}_i\ge\overline{c}_j$. There are $\vec{c}_m,\vec{c}'_m\in K$ with $c_i=\overline{c}_i$ and $c'_j=\overline{c}_j$. Because of (2) we have $\varepsilon_s=(p_s/p_{s-1})<(p_{s+1}/p_s)$ for $s\ge1$, which implies that $\overline{c}_i<c_j$ and $c'_i<\overline{c}_j$ in $\vec{c}_m$ and $\vec{c}'_m$ respectively. So, we get that in $\vec{c}_m$ the component c_j exceeds the component $c'_j=\overline{c}_j$ in $\vec{c}'_m$. This contradicts to (6).

The first chain of inequalities (7) is proved similarly.

Now, given $K\subseteq\Omega$ introduce a *bounded, closed, convex* set $K^*\subseteq\Omega$ that contains K, satisfies the conditions (6), where K is replaced by K^* and

$$\vec{c}_{m*}=(\underline{c}_1,\ldots,\underline{c}_m)\in K^*,\quad \vec{c}_m^{\,*}=(\overline{c}_1,\ldots,\overline{c}_m)\in K^*. \tag{8}$$

For $K^*\subseteq\Omega$ denote: $\rho(\vec{c}_m^{\,*})$ is the parameter ρ in the presentation (1) of $\left\{\rho(\vec{c}_m^{\,*})\right\}_0^\infty$.

Theorem 1 follows from the next

Theorem 2. In our case the FD $\{p_n\}_0^\infty$ is l_p-stable on K^* with any

$$p>(1/\rho(\vec{c}_m^{\,*})). \tag{9}$$

Indeed, the l_p-stability on K^* implies the l_p-stability on K, which generates K^* with the same p (see (9)). So, in particular, $\{p_n\}_0^\infty$ is $l_{1/2}$-stable on *any* $K\subseteq\Omega$.

Auxiliary Statements. Rewrite (4) in the form

$$p_n(\vec{c}_m)=\frac{g_n(\vec{c}_m)}{g(\vec{c}_m)},\quad n\ge0,\quad g(\vec{c}_m)>0, \tag{10}$$

$$g_n(\vec{c}_m)=\prod_{k=1}^n\varepsilon_k,\ n\ge0,\quad g(\vec{c}_m)=\frac{1}{p_0(\vec{c}_m)}=1+\sum_{n\ge1}\prod_{k=1}^n\varepsilon_k\quad\left(\prod_{k=0}^1\equiv1\right). \tag{11}$$

Let K^* be generated by K. Since $\vec{c}_{m*}=(\underline{c}_1,\ldots,\underline{c}_m)\in K^*$, $\vec{c}_m^{\,*}=(\overline{c}_1,\ldots,\overline{c}_m)\in K^*$, where $\underline{c}_j$ and $\overline{c}_j$ for $j=\overline{1,m}$ are defined in (6) and $\underline{c}_1<\underline{c}_2<\ldots<\underline{c}_m$, $\overline{c}_1<\overline{c}_2<\ldots<\overline{c}_m$ (see (7)), therefore, due to condition (a),

$$g_n(\vec{c}_m^{\,*})=\max_{\vec{c}_m\in K}g_n(\vec{c}_m)\quad\text{for all}\quad n\ge0, \tag{12}$$

$$g_n(\vec{c}_{m*})=\min_{\vec{c}_m\in K^*}g(\vec{c}_m). \tag{13}$$

Lemma 1. The function $g(\vec{c}_m)$ is continuous on K^*.

Proof. The conditions (a) and (b) on K^* imply the existence of constant $A \in [1, +\infty)$ (depending only on K^*) such that for any $j = \overline{1, m}$, $k \in T_j$ and $\vec{c}_m \in K^*$

$$0 \le \frac{\partial h_k(c_1, ..., c_j)}{\partial c_i} \le A, \quad i = \overline{1, j}. \tag{14}$$

With the help of (14) and *Mean Value Theorem* we obtain: for given $j = \overline{1, m}$, $k \in T_j$ and any $\vec{c}_m, \vec{c}'_m \in K$

$$\left|\varepsilon_k - \varepsilon'_k\right| = \left|h_k(c_1, ..., c_j) - h_k(c'_1, ..., c'j)\right| \le$$
$$\le \sum_{i=1}^{j} \left|h_k(c_1, ..., c_i, c'_{i+1}, ..., c'_j) - h_k(c_1, ..., c_{i-1}, c'_i, ..., c'_j)\right| \le A \sum_{i=1}^{j} \left|c_i - c'_i\right| \le A\left|\vec{c}_m - \vec{c}'_m\right|, \tag{15}$$

where $\varepsilon_k = h_k(c_1, ..., c_j)$, $\varepsilon'_k = h_k(c'_1, ..., c'_j)$. For given $j = \overline{1, m}$ and $k \in T_j$ denote $r_k(\overline{c}_1, ... \overline{c}_j) = \left(\prod_{i=1}^{j-1} \prod_{s \in T_i} h_s(\overline{c}_1, ..., \overline{c}_i) \right) \prod_{s=k_j}^{k} h_s(\overline{c}_1, ..., \overline{c}_j)$, and continue the estimations using (15)

$$\left| \prod_{s=1}^{k} \varepsilon_s - \prod_{s=1}^{k} \varepsilon'_s \right| \le \sum_{i=1}^{k} \left| \prod_{s=1}^{i} \varepsilon_s \prod_{s=i=1}^{k} \varepsilon'_s - \prod_{s=1}^{i-1} \varepsilon_s \prod_{s=i}^{k} \varepsilon'_s \right| \le$$
$$\le \sum_{i=1}^{k} \left(\prod_{s=1}^{i-1} \varepsilon_s \prod_{s=i+1}^{k} \varepsilon'_s \right) \left|\varepsilon_i - \varepsilon'_i\right| \le \frac{r_k(\vec{c}_1, ..., \vec{c}_j)}{\underline{c}_1} \sum_{i=1}^{k} \left|\varepsilon_i - \varepsilon'_i\right| \le \frac{r_k(\vec{c}_1, ..., \vec{c}_j)}{\underline{c}_1} A\left|\vec{c}_m - \vec{c}'_m\right| k \ .$$

Thus, due to (11), for any $k \ge 1$ on K^* we have

$$\left|g_k(\vec{c}_m) - g_k(\vec{c}'_m)\right| = \frac{A}{\underline{c}_1} k g_k(\vec{c}^{\,*}_m) \left|\vec{c}_m - \vec{c}'_m\right| \ . \tag{16}$$

Since (1) holds with $\rho \in (2, +\infty)$, therefore,

$$B = \sum_{k \ge 1} k\, g_k(\vec{c}^{\,*}_m) < +\infty \ . \tag{17}$$

Taking into account (11), (16) and (17), for $\vec{c}_m, \vec{c}'_m \in K^*$ we obtain an inequality

$$\left|g(\vec{c}_m) - g(\vec{c}'_m)\right| = \left| \sum_{k \ge 1} \left(g_k(\vec{c}_m) - g_k(\vec{c}'_m) \right) \right| \le \frac{A}{\underline{c}_1} \left|\vec{c}_m - \vec{c}'_m\right| \sum_{k \ge 1} k\, g_k(\vec{c}_m) = D\left|\vec{c}_m - \vec{c}'_m\right|, \tag{18}$$

where the constant $D = (AB / \underline{c}_1) \in R^+$ depends only on K^*.

With the help of inequality (18) the continuity of $g(\vec{c}_m)$ on K^* is proved.

During the proof of Lemma 1 the following statement was established (see (16)).

Lemma 2. The functions $g(\vec{c}_m)$ for all $n \ge 0$ are continuous on K^*.

Stability Criterion. Let the FD $\{p_n(\vec{c}_m)\}_0^{\infty}$, where $\vec{c}_m = (c_1, ..., c_m) \in \Omega$, is a vector of independent parameters that satisfies conditions (1), (2), and has the form (10). In [3] under the following additional conditions on $K \in \Omega$:

1. There is $\vec{c}_m^{\,+} \in K$ such that $g_n(\vec{c}_m^{\,+}) = \max_{\vec{c}_m \in K} g_n(\vec{c}_m)$ for all $n \ge 0$;

2. $g(\vec{c}_m)$ is continuous with respect to $\vec{c}_m \in K$,

it was established

Criterion. FD $\{p_n(\vec{c}_m)\}_0^\infty$ is l_p-stable on K, iff uniformly on $\vec{c}_m, \vec{c}\,'_m \in K$

$$\lim_{|\vec{c}_m - \vec{c}\,'_m| \to 0} \left| g_n(\vec{c}_m) - g_n(\vec{c}\,'_m) \right| = 0 \quad \text{separately for } n \ge 0. \tag{19}$$

Here $p > \left(1/\rho(\vec{c}_m^{\,+})\right)$ and $\rho(\vec{c}_m^{\,+})$ is the parameter ρ in (1) for $\left\{p_n(\vec{c}_m^{\,+})\right\}_0^\infty$.

In our case, when $g_n(\vec{c}_m)$, $n \ge 1$, and $g(\vec{c}_m)$ have the form (11), the Condition 1 is fulfilled on K^* with $\vec{c}_m^{\,+} = \vec{c}_m^{\,*}$, the fulfillment Condition 2 on K^* follows from Lemma 1. Since, due to Lemma 2, $g_n(\vec{c}_m)$ for $n \ge 1$ are continuous on K^*, therefore, they are *uniformly continuous* on K^*, which implies (19) on K^* in our case. Thus, applying the Criterion in our case, one obtains Theorem 2.

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Է. Ա. Դանիելյան, Ա. Կ. Արզումանյան

Բնական պարամետրացման շրջանակներում
հաճախականային բաշխումների կայունությունը I

Աշխատանքում ձևակերպվում և ուսումնասիրվում է հաճախականային բաշխումների կայունության խնդիրը բնական պարամետրացման շրջանակներում: Նկարագրված է վերջավոր թվով անկախ պարամետրերի դեպքը: Համապատասխան կայունության խնդիրը հետազոտված է l_p-մետրիկայի տերմիններով:

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Устойчивость частотных распределений в рамках
естественной параметризации. I

В работе формулируется и изучается задача устойчивости частотных распределений в рамках естественной параметризации. Охарактеризирован случай конечного числа независимых параметров. Соответствующая задача устойчивости исследована в терминах l_p-метрики.