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The Arithmetics of Par, Spot and Forward Curves

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Abstract

The understanding of the dynamics and shape of term structure of interest rates has been of a big interest to practitioners and academics. It is well documented that if the spot curve is monotonically upward-sloping (with decreasing growth rate) the par curve shows the same pattern, while the forward rate curve is first upward-sloping and then it inverts. Spot curve lies above the par curve, and the forward rate curve lies above the spot curve. If the spot curve is inverted, the pattern and ordering of the curves revert and the three curves are identical only if they are flat. This article aims to present mathematically the joint behavior of par, spot, and forward curves in discrete time setting.

Keywords: yield curve, spot curve, forward curve, par curve, implied spot curve

1. Introduction

Yield curve is a graphical representation of interest rates of similar (credit quality, coupon payment frequency, yield calculation convention, etc.) bonds against their maturities at a given time. Curves that plot par yields, spot rates and forward rates are respectively known as par curve, spot curve (term structure) and forward curve (Martellini et al., 2003, CFA Institute Investment Series, 2015). The understanding of the shape and dynamics of yield curves is interesting for fixed-income professionals, regulators and monetary policy makers.

The academic literature focuses on the yield curve analysis representing a number of theories that were developed and tested to address the question of why yield curves show a specific pattern over time or at a given point in time (Smith, 2011, Martellini et al., 2003, Choudhry, 2001). The classic theories of the term structure of interest rates are well known and still direct attention to the drivers of bond yields (Smith, 2011) (Note 1).

It is also well documented that if the spot curve is monotonically upward-sloping (with decreasing growth rate) the par curve has the same pattern while the forward rate curve is first upward-sloping and then inverts (Figure 1). Spot curve lies above the par curve, and the forward rate curve lies above the spot curve. If the spot curve is inverted, the pattern and ordering of the curves revert. The three curves are identical only if they are flat.

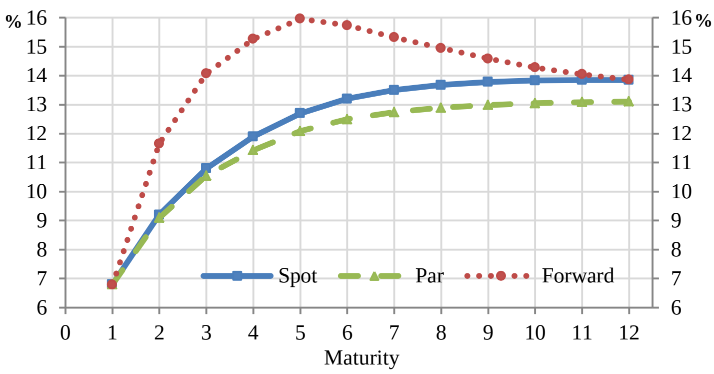

Figure 1. Par, spot and one-year forward rate curves

Source: Author's calculations (hypothetic spot curve).

The pattern and ordering of the par, spot, and forward curves is usually explained in terms of averages ((Fabozzi, 2005, Tuckman et al., 2012, CFA Institute Investment Series, 2015 (Note 2, Note 3)). Nevertheless, we think that the pattern and ordering of the par, spot and forward curves requires more attention: a detailed mathematical exhibition of the above-mentioned relationships may be valuable.

The aim of this article is to present mathematically the joint behavior of par, spot, and forward curves and to develop a tool to extend the analysis to any shape of a yield curve.

Firstly, it is useful to review the concepts of yield-to-maturity, par yield, spot rate, and forward rate.

2. Definitions and Denotations

2.1 Spot Rate

Spot rate (s_j) is the discount rate of a single future cash flow. A coupon bond can be viewed as a bundle of zero-coupon bonds and the unbundled cash flows can be valued separately. All cash flows discounted with respective spot rates will sum-up to the bond's price.

$$P = \frac{c_{1,n}}{(1+s_1)^1} + \cdots + \frac{c_{j,n}}{(1+s_j)^j} + \cdots + \frac{c_{n,n}}{(1+s_n)^n} \quad (1)$$

Where $c_n = c_{1,n} = \cdots = c_{n-1,n}, j < n, c_{n,n} = c_n + 100$ for an n-year to maturity bond assuming annual coupon payments (for simplicity), s_j is the respective spot rate for the j-th cash flow, and P is the price.

2.2 Yield-to-Maturity

Yield-to-maturity (y_n) is the single discount rate that equates the present value of a bond's cash flows to its market price. The par yield is the yield-to-maturity of a theoretical bond whose price equals par.

$$P = \frac{c_{1,n}}{(1+s_1)^1} + \cdots + \frac{c_{j,n}}{(1+s_j)^j} + \cdots + \frac{c_{n,n}}{(1+s_n)^n} = \frac{c_{1,n}}{(1+y_n)^1} + \cdots + \frac{c_{j,n}}{(1+y_n)^j} + \cdots + \frac{c_{n,n}}{(1+y_n)^n} \quad (2)$$

2.3 Forward Rate

Forward rate $(f_{m,n})$ is the annualized rate between m-year and n-year spot rates $(m < n)$. For $m = 1$, forward rates such as $f_{1,2}, f_{1,3}, \dots, f_{1,n}$ are known as implied spot rates and, for $m = n - 1$, forward rates such as $f_{1,2}, f_{2,3}, \dots, f_{n-1,n}$ are known as one-year forward rates.

$$(1 + f_{m,n})^{n-m} = \frac{(1+s_n)^n}{(1+s_m)^m} \quad (3)$$

Log-linearizing the above expression and applying Taylor's first order approximation to both sides of equation (3), yields the following relationship between forward and spot rates (Ilmanen, 1995):

$$f_{m,n} \approx \frac{n \times s_n - m \times s_m}{n - m} \quad (4)$$

3. Mutual Positions of Par, Spot and Forward Curves

3.1 The Patterns and Ordering of Forward and Spot Curves

In order to show how a given shape of a spot curve translates into a specific shape of a forward curve, without loss of generality one-year forward rate curve is considered for analysis. Equation 4 implies the following relation between one-year forward rate and spot rates:

$$f_{n-1,n} = s_n + (n-1) \times (s_n - s_{n-1}) \quad (5)$$

From equation 5, one can make the following inferences:

1) If spot curve is flat implying that $s_n = s_{n-1} = \cdots = s_1$, forward curve is flat as well. Furthermore, forward and spot curves are identical $(f_{n-1,n} = s_n)$.

2) If spot curve is upward-sloping meaning that $s_n > s_{n-1} > \cdots > s_1$, forward curve lies above the spot curve $(f_{n-1,n} > s_n)$ for all maturities.

3) If spot curve is downward-sloping suggesting that $s_n < s_{n-1} < \cdots < s_1$, forward curve lies below the forward curve $(f_{n-1,n} < s_n)$ for all maturities.

The pattern of one-year forward curve can be expressed as a difference of two adjacent forward rates (Δf). Let's denote the difference of two adjacent spot rates a_n such that:

$$s_n = s_{n-1} + a_n, \quad a_n > 0, for\ n > 1 \tag{6}$$

Using equations 5 and 6, the difference of forward rates can be presented as follows:

$$\Delta f = f_{n,n+1} - f_{n-1,n} = s_{n+1} - s_{n-1} + n \times (s_{n+1} - 2s_n + s_{n-1}) = a_n + a_{n+1} + n \times (a_{n+1} - a_n) \tag{7}$$

Equation 7 allows to generalize the second and third conclusions, which are presented below:

2.1) If the spot curve is upward-sloping and, as in most cases, the growth rate of spot rates is decreasing ($a_{n+1} < a_n$), the difference of forward rates for longer maturities can become negative ($\Delta f < 0$), and hence, forward curve is upward-sloping for shorter maturities and can be downward sloping for longer maturities. Since the forward curve under above-mentioned assumptions lies above the spot curve, it implies that the forward curve asymptotically converges to the spot curve for longer maturities.

3.1) For a downward-sloping spot curve, the opposite conclusion holds true.

3.2 The Patterns and Ordering of Implied Spot and Spot Curves

For implied spot curve, Equation 4 is expressed as:

$$f_{1,n} = s_n + (s_n - s_1) \times \frac{1}{n-1} \tag{8}$$

Equation 8 leads to the following conclusions:

4) If spot curve is flat, then implied spot curve is also flat ($f_{1,n} = s_n$), and they are identical.

5) If spot curve is upward-sloping, then implied spot curve lies above the spot curve ($f_{1,n} > s_n$).

6) If spot curve is downward-sloping, then implied spot curve lies below the spot curve ($f_{1,n} < s_n$).

The pattern of implied spot curve can be expressed as a difference of two adjacent implied spot rates (Δf^{is}), which is presented below:

$$\Delta f^{is} = f_{1,n+1} - f_{1,n} = \frac{n}{n-1} \times (s_{n+1} - s_n) - \frac{1}{n \times (n-1)} \times (s_{n+1} - s_1) = \frac{n}{n-1} \times a_{n+1} - \frac{a_2 + a_3 + \cdots + a_n + a_{n+1}}{n \times (n-1)} \tag{9}$$

If $a_2 > a_3 > \cdots > a_{n+1}$ implying that the spot curve increases in a decreasing rate, the following inequality holds true:

$$\frac{a_2 + a_3 + \cdots + a_n + a_{n+1}}{n(n-1)} > \frac{a_{n+1} \times n}{n(n-1)}$$

which in turn yields the equality presented below:

$$\frac{a_2 + a_3 + \cdots + a_n + a_{n+1}}{n(n-1)} = \frac{a_{n+1} \times n}{n(n-1)} + A \tag{10}$$

where $A = \frac{a_2 - a_{n+1} + a_3 - a_{n+1} + \cdots + a_n - a_{n+1}}{n(n-1)} > 0.$

This allows to rewrite equation 8 as:

$$\Delta f^{is} = a_{n+1} - A \tag{11}$$

Equation 11 implies that:

5.1) if the spot curve is upward-sloping and, as in most cases, the growth rate of spot rates is decreasing, implied spot curve is upward-sloping for shorter maturities and can be downward-sloping for longer maturities or at least the growth rate for implied spot rates is decreasing.

6.1) for a downward-sloping spot curve, the opposite conclusion holds true.

3.3 The Ordering of Par and Spot Curves

A par bond with n years to maturity and c_n coupon payments has a yield-to-maturity equal to coupon rate ($c_n/100 = y_n$). One can compute the price of this bond using both spot rates and the par yield ($for\ n = 1, s_1 = y_1$).

$$P = \frac{c_{1,n}}{(1+s_1)^1} + \frac{c_{2,n}}{(1+s_2)^2} + \cdots + \frac{c_{n,n}}{(1+s_n)^n} = \frac{c_{1,n}}{(1+y_n)^1} + \frac{c_{2,n}}{(1+y_n)^2} + \cdots + \frac{c_{n,n}}{(1+y_n)^n} \tag{12}$$

It follows from the above equation that:

7) If spot curve is flat, par yield and spot rates are identical.

8) If spot curve is upward-sloping, then $s_1 < y_n < s_n$ meaning that spot curve lies above the par curve.

9) If spot curve is downward-sloping, then $s_1 > y_n > s_n$ implying that par curve lies above spot curve.

One can also show that if spot curve is upward-sloping, par yield curve is also upward-sloping. In order to prove this two par bonds with n and $n-1$ years to maturity and c_n and c_{n-1} coupons respectively are considered. By definition, the difference in prices of these two bonds is zero since both of them are priced at par:

$$0 = \left(\frac{c_n}{(1+s_1)^1} + \frac{c_n}{(1+s_2)^2} + \cdots + \frac{c_n+100}{(1+s_n)^n}\right) - \left(\frac{c_{n-1}}{(1+s_1)^1} + \frac{c_{n-1}}{(1+s_2)^2} + \cdots + \frac{c_{n-1}+100}{(1+s_{n-1})^{n-1}}\right) \quad (13)$$

The definition of a par bond also implies that $\frac{c_n}{100} = y_n$ and $\frac{c_{n-1}}{100} = y_{n-1}$, so it follows from equation 13 that:

$$(y_n - y_{n-1})\left(\frac{1}{(1+s_1)^1} + \frac{1}{(1+s_2)^2} + \cdots + \frac{1}{(1+s_{n-1})^{n-1}}\right)(1+s_n)^n = \frac{(1+s_n)^n}{(1+s_{n-1})^{n-1}} - (y_n + 1) \quad (14)$$

Equation 14 allows to conclude that if spot curve is upward sloping, then $\frac{(1+s_n)^n}{(1+s_{n-1})^{n-1}} > \frac{(1+s_n)^n}{(1+s_n)^{n-1}} = 1 + s_n$, which in turn implies that $\frac{(1+s_n)^n}{(1+s_{n-1})^{n-1}} - (y_n + 1) > 1 + s_n - y_n - 1 > 0$. Consequently, $y_n > y_{n-1}$, because $\left(\frac{1}{(1+s_1)^1} + \frac{1}{(1+s_2)^2} + \cdots + \frac{1}{(1+s_{n-1})^{n-1}}\right)(1+s_n)^n > 0$. In summary it follows from the analysis that:

8.1) if spot curve is upward-sloping, par curve is also upward-sloping.

9.1) if spot curve is downward-sloping, par curve is also downward-sloping.

4. Conclusion

The academic literature focuses on the classic theories of the term structure of interest rates. The pattern and ordering of the par, spot and forward curves are also well documented and is usually explained in terms of averages. We think that the pattern and ordering of the par, spot and forward curves requires more attention. The article mathematically presents the joint behavior of par, spot, and forward curves in discrete time setting and provides an alternative explanation of the above-mentioned patterns of yield curves.

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Notes

Note 1. The expectation, segmented market, and liquidity preference theories.

Note 2.CFA Institute Investment Series mathematically represent only the ordering of the forward and spot curves.

Note 3.Spot rate is the geometrical average of forward rates. In addition, the yield-to-maturity is some weighted average of spot rates (CFA Institute Investment Series, 2015).