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# AN ABSTRACT THEORY OF SINGULAR OPERATORS

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ABSTRACT. We introduce a class of operators on abstract measure spaces that unifies the Calderón–Zygmund operators on spaces of homogeneous type, the maximal functions, the martingale transforms, and Carleson operators. We prove that such operators can be dominated by simple sparse operators with a definite form of the domination constant. Applying these estimates, we improve on several results obtained by different authors in recent years.

## 1. INTRODUCTION

The study of weighted inequalities in harmonic analysis started in the early 1970s. In 1972 Muckenhoupt [27] proved that the maximal function is bounded on $L^p(w)$ for $1 < p < \infty$ if and only if the weight $w$ satisfies the $A_p$ condition

$$[w]_{A_p} = \sup_I \left(\frac{1}{|I|}\int_I w\right)\left(\frac{1}{|I|}\int_I w^{-1/(p-1)}\right)^{p-1} < \infty. \tag{1.1}$$

One year later Hunt, Muckenhoupt, and Wheeden [9] established the same property for Hilbert transform. For the general Calderón–Zygmund operators a weighted $L^p(w)$ bound was first proved by Coifman and Fefferman [4].

In 1993 Buckley [3] discovered that the maximal function $M$ has the sharp estimate

$$\|Mf\|_{L^p(w)\to L^p(w)} \le C\|w\|_{A_p}^{1/(p-1)}, \tag{1.2}$$

resulting in a similar problem for Calderón–Zygmund operators. In the last 15 years there has been activity in the investigation of this problem. For general Calderón–Zygmund operators the conjecture was the bound

$$\|Tf\|_{L^p(w)\to L^p(w)} \le C\|w\|_{A_p}^{\max\{1,1/(p-1)\}}. \tag{1.3}$$

An extrapolation theorem proved in [7] reduced the conjecture to the case $p = 2$, so the inequality (1.3) became known as the $A_2$ conjecture. After first being established for several particular operators [33], [31], [32], [30], [13], in 2010 Hytönen and Kairema [11] proved the $A_2$ conjecture for general Calderón–Zygmund operators.

A series of recent works were motivated by the domination of Calderón–Zygmund operators by very simple sparse operators [22]–[25], [5], [21], [12]–[17]. From such

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results in particular follows (1.3), since the $A_2$ bound is very easy to establish for the sparse operators [24], [14], [28], [29]. A domination of the classical Calderón–Zygmund operators on $\mathbb{R}^n$ by sparse operators was discovered by Lerner [24]. Applying this domination, he gave a simplified proof of the Hytönen $A_2$ theorem. Lacey [21] proved a pointwise domination theorem for more general $\omega$-Calderón–Zygmund operators (see (7.15)–(7.18)), with $\omega$ satisfying the Dini condition

$$\int_0^1 \frac{\omega(t)}{t} dt < \infty, \tag{1.4}$$

deriving weighted bound (1.3) for such operators too. Lacey's result was a stronger version of another pointwise bound independently proved by Conde-Alonso and Rey [5] and Lerner and Nazarov [26]. Moreover, Lacey's inequality assumes only the Dini condition, while prior approaches [5], [26] require $1/t$ in the Dini integral to be replaced by $(\log 2/t)/t$. Hytönen, Roncal, and Tapiola [17] elaborated on the proof of Lacey [21] to get a precise linear dependence of the domination constant on the characteristic numbers of the operator. Lerner [25] gave a simple proof of the Lacey–Hytönen–Roncal–Tapiola theorem.

Anderson and Vagharshakyan [1] proved the $A_2$ theorem for the Calderón–Zygmund operators in general spaces of homogeneous type with modulus of continuity $\omega(t) = t^\alpha$, $\alpha > 0$.

In the late 1970s, several authors also considered martingale analogues of the $A_p$ theory. For instance, Izumisawa and Kazamaki [18] proved a variant of the Muckenhoupt maximal function result [27] in this setting. When it came to martingale transforms, the distinction between the homogeneous and nonhomogeneous cases was already recognized by these authors. Nevertheless, norm inequalities for martingale transforms were proved by Bonami and Lépingle [2]. The $A_2$ theorem for martingale transform was proved recently by Thiele, Treil, and Volberg [34]. Lacey [21] gave a self-contained, short, and elementary proof of this theorem. Moreover, he established a pointwise domination theorem for martingale transforms too.

Grafakos, Martell, and Soria [8] and Di Plinio and Lerner [6] considered weighted estimates for maximal modulations of Calderón–Zygmund operators on $\mathbb{R}^n$, in particular, for Carleson or Walsh–Carleson operators. The paper [8] establishes weighted norm control of the Carleson operators by the maximal function. In [6] the authors proved weighted norm estimates of Carleson and Walsh–Carleson operators with explicit dependence of the constants on $A_p$ characteristics of the weight.

In this paper we introduce so-called BO (bounded oscillation) operators on abstract measure spaces. Those operators unify the Calderón–Zygmund operators and maximal functions in general homogeneous spaces, martingale transforms (nonhomogeneous case), and Carleson operators. The definition of BO operators is motivated by [19], [20], where some exponential estimates for Calderón–Zygmund and other related operators were proved. We shall prove that those operators have pointwise domination by sparse operators and then satisfy the bound (1.3). We derive a variety of other properties of BO operators significant for their further investigation.

To define BO operators, we introduce a concept of ball-basis for an abstract measure space, which is a family of measurable sets holding some common properties of $d$-dimensional balls on $\mathbb{R}^d$ and their analogues in related theories (martingales, dyadic analysis).

**Definition 1.1.** Let $(X, \mathfrak{M}, \mu)$ be a measure space. A family of sets $\mathfrak{B} \subset \mathfrak{M}$ is said to be a ball-basis if it satisfies the following conditions:

(B1) $0 < \mu(B) < \infty$ for any ball $B \in \mathfrak{B}$.

(B2) For any points $x, y \in X$ there exists a ball $B \ni x, y$.

(B3) If $E \in \mathfrak{M}$, then for any $\varepsilon > 0$ there exists a finite or infinite sequence of balls $B_k$, $k = 1, 2, \ldots$ such that $\mu(E \triangle \bigcup_k B_k) < \varepsilon$.

(B4) For any $B \in \mathfrak{B}$ there is a ball $B^* \in \mathfrak{B}$ (called the hull of $B$) satisfying the conditions

$$\bigcup_{A\in\mathfrak{B}:\, \mu(A)\le 2\mu(B),\, A\cap B\neq\varnothing} A \subset B^*, \tag{1.5}$$

$$\mu(B^*) \le \mathcal{K}\mu(B), \tag{1.6}$$

where $\mathcal{K}$ is a positive constant.

One can easily check that the family of Euclidean balls in $\mathbb{R}^n$ forms a ball-basis. Moreover, we will see below (see Theorem 7.1) that if the family of metric balls in spaces of homogeneous type satisfies the density condition (see Definition 3.1), then it is a ball-basis too. The martingale basis considered in Section 8 is an example of ball-bases having the nondoubling property.

**Definition 1.2.** Let $1 \le r < \infty$, let $(X, \mathfrak{M}, \mu)$ be a measure space, and let $L^0(X)$ be the linear space of functions (include nonmeasurable functions) on $X$. $T : L^r(X) \to L^0(X)$ is said to be subadditive if

$$|T(\lambda \cdot f)(x)| = |\lambda| \cdot |Tf(x)|, \quad \lambda \in \mathbb{R}, \quad |T(f+g)(x)| \le |Tf(x)| + |Tg(x)|.$$

*Remark* 1.1. As we will see below, in the definitions of some operators (maximal function, $T^*$) some nonmeasurable functions can appear. To apply the results of the paper to such operators, we allow nonmeasurability of the images in the definition of general subadditive operators. The definitions of $L^p$ (weak-$L^p$) norms and some standard inequalities that we need for nonmeasurable functions will be stated in the next section.

Let $1 \le r < \infty$ be a fixed number in Sections 1–4. For $f \in L^r(X)$ and $B \in \mathfrak{B}$ we set

$$\langle f\rangle_{B,r} = \left(\frac{1}{\mu(B)}\int_B |f|^r\right)^{1/r}, \qquad \langle f\rangle^*_{B,r} = \sup_{A\in\mathfrak{B}:A\supset B} \langle f\rangle_{A,r}.$$

In the case $r = 1$ for those quantities we will use the notations $\langle f\rangle_B$ and $\langle f\rangle^*_B$, respectively (Sections 5–8).

**Definition 1.3.** We say that a subadditive operator $T$ is a bounded oscillation operator with respect to a ball-basis $\mathfrak{B}$ if

(T1) (localization) for every $B \in \mathfrak{B}$ we have

$$\sup_{x,x'\in B,\, f\in L^r(X)} \frac{|T(f\cdot \mathbb{I}_{X\setminus B^*})(x) - T(f\cdot \mathbb{I}_{X\setminus B^*})(x')|}{\langle f\rangle^*_{B,r}} \le \mathcal{L}_1 = \mathcal{L}_1(T) < \infty, \tag{1.7}$$

(T2) (connectivity) for any $A \in \mathfrak{B}$ ($A^* \neq X$) there exists a ball $B \supsetneq A$ (i.e., $B \supset A$, $B \neq A$) such that

$$\sup_{x\in A,\, f\in L^r(X)} \frac{|T(f\cdot \mathbb{I}_{B^*\setminus A^*})(x)|}{\langle f\rangle_{B^*,r}} \le \mathcal{L}_2 = \mathcal{L}_2(T) < \infty, \tag{1.8}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are the least constants satisfying (1.7) and (1.8), respectively. The family of all bounded oscillation operators with respect to a ball-basis $\mathfrak{B}$ will be denoted by $\mathrm{BO}_{\mathfrak{B}}$, or simply BO.

It will be proved below (Theorem 4.4) that if the ball-basis $\mathfrak{B}$ satisfies the doubling condition, then the localization property implies connectivity.

**Definition 1.4.** A collection of balls $\mathcal{S} \subset \mathfrak{B}$ is said to be sparse or $\gamma$-sparse if for any $B \in \mathcal{S}$, there is a set $E_B \subset B$ such that $\mu(E_B) \geq \gamma\mu(B)$ and the sets $\{E_B : B \in \mathcal{S}\}$ are pairwise disjoint, where $0 < \gamma < 1$ is a constant.

Given a family of balls $\mathcal{S}$, we associate the operator

$$\mathcal{A}_{\mathcal{S},r} f(x) = \sum_{A\in\mathcal{S}} \langle f \rangle_{A,r} \cdot \mathbb{I}_A(x).$$

If $\mathcal{S}$ is a sparse collection of balls, then we say that $\mathcal{A}_{\mathcal{S},r}$ is a sparse operator. In the case $r = 1$ we will simply write $\mathcal{A}_{\mathcal{S}}$. Further, positive constants depending on $\mathcal{K}$ (see (1.6)) will be called admissible constants and the relation $a \lesssim b$ will stand for $a \leq c \cdot b$, where $c > 0$ is admissible. We write $a \sim b$ if the relations $a \lesssim b$ and $b \lesssim a$ hold at the same time.

**Theorem 1.1.** *Let an operator $T \in \mathrm{BO}_{\mathfrak{B}}(X)$ satisfy weak-$L^r$ inequality. Then for any function $f \in L^r(X)$ and a ball $B \in \mathfrak{B}$ there exists a family of balls $\mathcal{S}$, which is a union of two $\gamma$-sparse collections and*

$$|Tf(x)| \lesssim (\mathcal{L}_1 + \mathcal{L}_2 + \|T\|_{L^r\to L^{r,\infty}}) \cdot \mathcal{A}_{\mathcal{S},r} f(x) \qquad \text{a.e. } x \in B, \tag{1.9}$$

*where $\mathcal{L}_1$ and $\mathcal{L}_2$ are the constants (1.7) and (1.8), and $0 < \gamma < 1$ is an admissible constant.*

**Definition 1.5.** For $T \in \mathrm{BO}_{\mathfrak{B}}$ define

$$T^* f(x) = \sup_{B\in\mathfrak{B}:\, x\in B} |T(f \cdot \mathbb{I}_{X\setminus B^*})(x)|.$$

We shall prove below (Theorem 4.3) that if $T \in \mathrm{BO}_{\mathfrak{B}}$ satisfies the weak-$L^r$ estimate, then $T^* \in \mathrm{BO}_{\mathfrak{B}}$ and satisfies the weak-$L^r$ bound too. So from Theorem 1.1 we will immediately get the following.

**Theorem 1.2.** *If $T \in \mathrm{BO}_{\mathfrak{B}}(X)$ satisfies weak-$L^r$ inequality, then for any function $f \in L^r(X)$ and a ball $B \in \mathfrak{B}$ there exists a family of balls $\mathcal{S}$, which is a union of two $\gamma$-sparse collections and*

$$|T^* f(x)| \lesssim (\mathcal{L}_1(T) + \mathcal{L}_2(T) + \|T\|_{L^r\to L^{r,\infty}}) \cdot \mathcal{A}_{\mathcal{S},r} f(x) \qquad \text{a.e. } x \in B.$$

Theorems 8.3, 8.1, and 7.5 prove that the $\omega$-Calderón–Zygmund operators on spaces of homogeneous type, the martingale transforms, and the maximal functions are BO operators, so they all satisfy the estimate (1.9). Moreover, combining (1.9) with the weighted bounds for sparse operators, we obtain $A_2$ theorems for all of these operators. Hence, Theorems 1.1 and 1.2 cover the above stated results concerning the weighted bounds and the domination by sparse operators. The Lacey–Hytönen–Roncal–Tapiola theorem [21], [17] is a version of Theorem 1.2 for the $\omega$-Calderón–Zygmund operators on $\mathbb{R}^n$ with the Dini condition. The Lacey domination theorem for martingale transforms [21] is another case of inequality (1.9). In $\mathbb{R}^n$ and in general spaces of homogeneous type, where a doubling condition holds, the proofs of such dominations are based on dyadic decomposition theorems [14], [10]. In the case of martingale transforms [34], instead of dyadic decomposition, the properties of the martingale basis are essentially used. Since a general ball-basis does not always satisfy the doubling condition (the typical example is the martingale basis), there is no dyadic decomposition in general. So our method of proof of Theorem 1.1 is different. It is direct and partially based on [19], [20].

In the last sections we prove several weighted bounds for sparse operators to get $A_2$ theorems for some BO operators. The method of proofs of such theorems are based on a duality argument developed in [25], [5], [14], [21].

Note that Theorem 1.1 also implies the exponential integrability results of our papers [19], [20] proved for the Calderón–Zygmund operators on Euclidean spaces and for the partial sums of Walsh–Fourier series.

Applying Theorem 1.1 to the maximal function corresponding to a general ball-basis, we do not get the full weighted estimate (1.2), which is known to be optimal for the maximal function in Euclidean spaces [3]. In the general case the optimality occurs only when $1 < p \leq 2$. So (1.9) does not cover some estimates that have the maximal function. The reason is that the maximal function has some extra properties that the general BO operators do not have. An example of such a property is the $L^\infty$ bound.

In the last section we prove that the maximal modulation of a BO operator is also a BO operator, deriving pointwise sparse domination for maximal modulated BO operators too.

## 2. Outer measure and $L^p$ norms of nonmeasurable functions

Let $(X, \mathfrak{M}, \mu)$ be a measure space. Define the outer measure of a set $E \subset X$ by $\mu^*(E) = \inf_{F\in\mathfrak{M}:\, F\supset E} \mu(F)$. We say that two sets $A, B \subset X$ (not necessarily measurable) satisfy the relation $A \sim B$ if $\mu^*(A \triangle B) = 0$. Denote by $\bar{\mathfrak{M}}$ the family of sets in $X$, which are equivalent to a measurable set. It is clear that $\bar{\mathfrak{M}}$ is $\sigma$-algebra. For a given set $E \subset X$ we define $\bar{E} = \bigcap_{F\in\mathfrak{M}:\, F\supset E} F$. Observe that $\bar{E} \in \bar{\mathfrak{M}}$, $\mu^*(\bar{E}) = \mu^*(E)$. For any function $f \in L^0(X)$ we denote $G_f(t) = \{x \in X : |f(x)| > t\}$ and $\lambda_f(t) = \mu^*(G_f(t))$. Observe that the function $\bar{f}(x) = \inf\{t \geq 0 : x \in \overline{G_f(t)}\}$ is positive and $\bar{\mathfrak{M}}$-measurable and that $|f(x)| \leq \bar{f}(x)$. Besides, $\bar{f}$ is the smallest $\bar{\mathfrak{M}}$-measurable positive function dominating $|f|$. Namely, if $g(x)$ is $\bar{\mathfrak{M}}$-measurable and satisfies $|f(x)| \leq g(x)$, then $\bar{f}(x) \leq g(x)$. For arbitrary $f \in L^0(X)$ we define

$$\|f\|_{L^p} = \|\bar{f}\|_{L^p} = \left(p\int_0^\infty t^{p-1}\lambda_f(t)dt\right)^{1/p},$$
$$\|f\|_{L^p\to L^{p,\infty}} = \|\bar{f}\|_{L^p\to L^{p,\infty}} = \sup_{t>0} t(\lambda_f(t))^{1/p}.$$

**Definition 2.1.** We say that a subadditive operator $T$ satisfies a weak-$L^p$ or strong-$L^p$ estimate if

$$\|T\|_{L^p\to L^{p,\infty}} = \sup_{t>0,\, f\in L^p(X)} \frac{t\cdot(\lambda_{Tf}(t))^{1/p}}{\|f\|_{L^p}} < \infty,$$
$$\|T\|_{L^p} = \sup_{f\in L^p(X)} \frac{\|Tf\|_{L^p}}{\|f\|_{L^p}} < \infty,$$

respectively.

One can easily check that the standard triangle and Hölder inequalities as well as the Marcinkiewicz interpolation theorem hold in such a setting of $L^p$ norms. We will need the following case of the interpolation theorem.

**Theorem 2.1** (Marcinkiewicz interpolation theorem, [36, ch. 12.4])**.** *If a subadditive operator $T$ satisfies the weak-$L^1$ estimate and the strong-$L^\infty$ estimate, then for $1<p<\infty$ it holds that*

$$\|T\|_{L^p} \le c_p(\|T\|_{L^1\to L^{1,\infty}})^{1/p} \times (\|T\|_{L^\infty})^{1/q}.$$

## 3. Some properties of ball-basis

Let $\mathfrak{B}$ be a ball-basis for the measure space $(X,\mathfrak{M},\mu)$. From the (B4) condition it follows that if balls $A,B$ satisfy $\mu(A)\le 2\mu(B)$, then $A\subset B^*$. This property will be called the two balls relation. Hull levels of a given ball $B\in\mathfrak{B}$ will be denoted by $B^{[0]}=B$, $B^{[n+1]}=\left(B^{[n]}\right)^*$. By property (B4) we have $\mu(B^{[n+1]})\le \mathcal{K}\mu(B^{[n]})$. Applying this inequality consecutively, we get

$$\mu(B^{[n]})\le \mathcal{K}^n\mu(B), \qquad n\ge 0. \tag{3.1}$$

We say a set $E\subset X$ is bounded if $E\subset B$ for some $B\in\mathfrak{M}$.

**Lemma 3.1.** *Let $(X,\mathfrak{M},\mu)$ be a measure space, and let the family of sets $\mathfrak{B}\subset\mathfrak{M}$ satisfy the* (B4) *condition. If $E\subset X$ is bounded and $\mathcal{G}$ is a family of balls with $E\subset\bigcup_{G\in\mathcal{G}}G$, then there exists a finite or infinite sequence of pairwise disjoint balls $G_k\in\mathcal{G}$ such that*

$$E\subset\bigcup_k G_k^{[1]}. \tag{3.2}$$

*Proof.* The boundedness implies that $E\subset B$ for some $B\in\mathfrak{B}$. If there is a ball $G\in\mathcal{G}$ so that $G\cap B\ne\varnothing$ and $\mu(G)>\mu(B)$, then by the two balls relation we have $E\subset B\subset G^{[1]}$. Thus, the desired sequence can be formed by a single element $G$. Hence, we can suppose that any element $G\in\mathcal{G}$ satisfies the conditions $G\cap B\ne\varnothing$ and $\mu(G)\le\mu(B)$. Therefore we get $\bigcup_{G\in\mathcal{G}}G\subset B^{[1]}$. Take $G_1$ to be a ball from $\mathcal{G}$ satisfying $\mu(G_1)>\frac12\sup_{G\in\mathcal{G}}\mu(G)$. Then suppose by induction we have already chosen elements $G_1,\ldots,G_k$ from $\mathcal{G}$. Take $G_{k+1}\in\mathcal{G}$ disjoint with the balls $G_1,\ldots,G_k$ and satisfying $\mu(G_{k+1})>\frac12\sup_{G\in\mathcal{G}\,:\,G\cap G_j=\varnothing,\,j=1,\ldots,k}\mu(G)$. If for some $n$ we will not be able to determine $G_{n+1}$, the process will stop and we will get a finite sequence $G_1,G_2,\ldots,G_n$. Otherwise our sequence will be infinite. We shall consider the infinite case of the sequence (the finite case can be done similarly). Since the balls $G_n$ are pairwise disjoint and $G_n\subset B^{[1]}$, we have $\mu(G_n)\to 0$. Take an arbitrary $G\in\mathcal{G}$ such that $G\ne G_k$, $k=1,2,\ldots$. Let $m$ be the smallest integer such that $\mu(G)>\frac12\mu(G_{m+1})$. Observe that we have $G\cap G_j\ne\varnothing$ for some $1\le j\le m$, since otherwise $G$ had to be chosen instead of $G_{m+1}$. Besides, we have $\mu(G)\le 2\mu(G_j)$, which implies that $G\subset G_j^{[1]}$. Since $G\in\mathcal{G}$ was taken arbitrarily, we get (3.2). □

**Lemma 3.2.** *Let $(X,\mathfrak{M},\mu)$ be a measure space with a ball-basis $\mathfrak{B}$. If balls $B\in\mathfrak{B}$, $G_k\in\mathfrak{B}$, $k=1,2,\ldots$ satisfy the relation*

$$G_k\cap B\ne\varnothing, \qquad \mu(G_k)\to r=\sup_{A\in\mathfrak{B}}\mu(A), \tag{3.3}$$

*then $X\subset\bigcup_k G_k^{[1]}$. Moreover, for any ball $A\in\mathfrak{B}$ we have $A\subset G_k^{[1]}$ for some $k\ge k_0$.*

*Proof.* Since by the (B2) condition every point $x \in X$ is in some ball, it is enough to prove the second part of the theorem. So let $A \in \mathfrak{B}$. Choose points $x \in A$, $y \in B$. According to the (B2) condition, there is a $C \in \mathfrak{B}$ such that $x, y \in C$. Let $G$ be the ball $A, B$, or $C$, which has the biggest measure. Applying the two ball relation twice, one can easily check that $A \cup B \cup C \subset G^{[2]}$. From (3.3) we find an integer $k_0$ such that $\mu(G_k) > \mu(G^{[2]})/2$ for $k > k_0$. Therefore, since $G_k \cap G^{[2]} \neq \varnothing$, we get $A \subset G^{[2]} \subset G_k^{[1]}$, $k > k_0$. □

**Definition 3.1.** For a set $E \in \mathfrak{M}$ a point $x \in E$ is said to be a density point if for any $\varepsilon > 0$ there exists a ball $B$ such that $\mu(B \cap E) > (1-\varepsilon)\mu(B)$. We say that a measure space $(X, \mathfrak{M}, \mu)$ satisfies the density property if for any measurable set $E$, almost all points $x \in E$ are density points.

**Lemma 3.3.** *Let $(X, \mathfrak{M}, \mu)$ be a measure space. If a family of measurable sets $\mathfrak{B}$ satisfies the density property and the* (B4) *condition, then for any bounded measurable set $E$, $\mu(E) > 0$, and $\varepsilon > 0$ there is a sequence of balls $\{B_k\}$ such that*

$$\mu\left(\bigcup_k B_k \setminus E\right) < \varepsilon, \qquad \mu\left(E \setminus \bigcup_k B_k\right) < \alpha\mu(E) \tag{3.4}$$

*where $0 < \alpha < 1$ is an admissible constant.*

*Proof.* Applying the density property, one can find a family of balls $\mathfrak{B}$ satisfying $E \subset \bigcup_{B \in \mathfrak{B}} B$ a.s. and $\mu(B \cap E) > (1-\delta)\mu(B)$, $B \in \mathfrak{B}$, where $\delta = \min\left\{\frac{\varepsilon}{2\mu(E)}, 1/2\right\}$. Then, applying Lemma 3.1, we get a subfamily of pairwise disjoint balls $B_k$ with (3.2). Thus, we have

$$\mu\left(\bigcup_k B_k \setminus E\right) = \sum_k \mu(B_k \setminus E) < \frac{\delta}{1-\delta}\sum_k \mu(B_k \cap E) \leq 2\delta\mu(E) \leq \varepsilon.$$

On the other hand,

$$\begin{aligned}\mu\left(E \setminus \bigcup_k B_k\right) &= \mu(E) - \sum_k \mu\left(E \cap B_k\right) \leq \mu(E) - (1-\delta)\sum_k \mu\left(B_k\right)\\ &\leq \mu(E) - \frac{1-\delta}{\mathcal{K}}\mu\left(\bigcup_k B_k^{[1]}\right) \leq \mu(E) - \frac{1}{2\mathcal{K}}\mu\left(\bigcup_{B \in \mathfrak{B}} B\right)\\ &\leq \mu(E) - \frac{1}{2\mathcal{K}}\mu(E) = \left(1 - \frac{1}{2\mathcal{K}}\right)\mu(E).\end{aligned}$$

So conditions (3.4) are satisfied with a constant $\alpha = 1 - 1/2\mathcal{K}$. □

Observe that if the (B3) condition holds for the bounded measurable sets, then it holds for all of the measurable sets. Indeed, according to Lemma 3.2, there is a sequence of balls $G_k$ such that $X = \bigcup_k G_k$. This implies that any measurable set $E$ can be written as a countable union of bounded measurable sets $E = \bigcup_k E_k$. Apply the (B3) condition to each set $E_k$ with an approximation number $\varepsilon/2^k$. The union of all obtained approximating balls will give an $\varepsilon$-approximation of $E$.

**Lemma 3.4.** *Let $(X, \mathfrak{M}, \mu)$ be a measurable set. If a family of measurable sets $\mathfrak{B}$ satisfies the* (B4) *condition, then it will satisfy the* (B3) *condition if and only if the density condition holds.*

*Proof.* Let $\mathfrak{B}$ satisfy (B4) and the density conditions, and let $E$ be a measurable set. The remark stated before the lemma allows us to suppose that $E$ is bounded. Applying Lemma 3.3 consecutively, we can find sequences of balls $\mathfrak{B}_k$, $k = 1, 2, \dots$ such that

$$\mu\left(\bigcup_{B\in\mathfrak{B}_k} B \setminus E\right) < \frac{\varepsilon}{2^k}, \qquad k \ge 1, \tag{3.5}$$

$$\mu\left(E \setminus \bigcup_{B\in\bigcup_{j=1}^{k}\mathfrak{B}_j} B\right) < \alpha\mu\left(E \setminus \bigcup_{B\in\bigcup_{j=1}^{k-1}\mathfrak{B}_j} B\right), \qquad k \ge 1, \tag{3.6}$$

where in the case $k = 1$ the right-hand side of (3.6) is assumed to be $E$. Then we denote $\mathfrak{B} = \bigcup_{j=1}^{\infty} \mathfrak{B}_j$. From (3.6) it easily follows that $E \subset \bigcup_{B\in\mathfrak{B}} B$ a.s., while from (3.5) we obtain

$$\mu\left(\left(\bigcup_{B\in\mathfrak{B}} B\right) \setminus E\right) \le \sum_{k=1}^{\infty} \mu\left(\left(\bigcup_{B\in\mathfrak{B}_k} B\right) \setminus E\right) < \varepsilon.$$

To prove the second part of the lemma, let $\mathfrak{B}$ satisfy the (B4) and (B3) conditions. Suppose to the contrary that $\mathfrak{B}$ does not have the density property. That is, there exists a number $\alpha$, $0 < \alpha < 1$, a set $E \in \mathfrak{M}$ together with its subset $F \subset E$, $\mu^*(F) > 0$, such that for any $x \in F$ and $B \in \mathfrak{B}$ with $x \in B$ we have

$$\mu(B \setminus E) > \alpha\mu(B). \tag{3.7}$$

By the (B3) condition for any $\varepsilon > 0$ we can find a sequence of balls $B_k$, $k = 1, 2, \dots$ such that

$$\mu\left(\bar{F} \triangle \bigcup_k B_k\right) < \varepsilon. \tag{3.8}$$

We can suppose that $\mu(B_k \cap \bar{F}) > 0$ for each $B_k$. Observe that it implies that $B_k \cap F \neq \varnothing$. Indeed, suppose to the contrary that $B_{k_0} \cap F = \varnothing$. Then we get $F \subset \bar{F} \setminus B_{k_0}$ and thus a contradiction, $\mu^*(F) \le \mu^*(\bar{F} \setminus B_{k_0}) < \mu^*(\bar{F})$. Thus, by (3.7) we get

$$\mu(B_k \setminus \bar{F}) \ge \mu(B_k \setminus E) > \alpha\mu(B_k), \qquad k = 1, 2, \dots. \tag{3.9}$$

Applying Lemma 3.3, we find a subsequence of pairwise disjoint balls $\tilde{B}_k$, $k = 1, 2, \dots$ such that $\mu\left(\bar{F} \setminus \bigcup_k \tilde{B}_k^{[1]}\right) < \varepsilon$. Thus, applying the (B4) condition, (3.8), and (3.9), we obtain

$$\mu^*(\bar{F}) \le \mu\left(\bigcup_k \tilde{B}_k^{[1]}\right) + \varepsilon \le \mathcal{K}\sum_k \mu(\tilde{B}_k) + \varepsilon \le \frac{\mathcal{K}}{\alpha}\sum_k \mu(\tilde{B}_k \setminus \bar{F}) + \varepsilon$$
$$= \frac{\mathcal{K}}{\alpha}\mu\left(\bar{F} \triangle \bigcup_k B_k\right) + \varepsilon < \varepsilon\left(\frac{\mathcal{K}}{\alpha} + 1\right).$$

Choosing a small enough $\varepsilon$, we get $\mu^*(F) = \mu^*(\bar{F}) = 0$ and thus a contradiction. □

We say a measurable set $E$ is almost surely a subset of a measurable set $F$ if $\mu(E \setminus F) = 0$. We denote this relation by $E \subset F$ a.s.

**Lemma 3.5.** *For any bounded measurable set $E \in \mathfrak{M}$ there exists a sequence of balls $B_k$, $k = 1, 2, \dots$ such that*

$$E \subset \bigcup_k B_k \text{ a.s.}, \qquad \sum_k \mu(B_k) \le 2\mathcal{K}\mu(E). \tag{3.10}$$

*Proof.* Then observe that applying the (B3) condition consecutively, for a given measurable set $E$ and $\varepsilon > 0$ one can find a countable family of balls $\mathfrak{A}$ such that

$$E \subset \bigcup_{A\in\mathfrak{A}} A \text{ a.s.}, \qquad \mu\left(\bigcup_{A\in\mathfrak{A}} A\right) < (1+\varepsilon)\mu(E). \tag{3.11}$$

Applying Lemma 3.1, we find a pairwise disjoint collection $\{A_j\} \subset \mathfrak{A}$ such that $E \subset \bigcup_j {A_j}^{[1]}$ a.s. The (B4) condition and (3.11) yield

$$\sum_j \mu\left({A_j}^{[1]}\right) \le \mathcal{K}\sum_j \mu(A_j) = \mathcal{K}\mu\left(\bigcup_j A_j\right) \le \mathcal{K}\mu\left(\bigcup_{A\in\mathfrak{A}} A\right) < 2\mathcal{K}\mu(E).$$

Defining $B_k = A_k^{[1]}$, one can easily check that (3.10) is satisfied. ☐

We denote by $\#A$ the cardinality of a finite set $A$.

**Lemma 3.6.** *Let $A \in \mathfrak{B}$, and let $\mathcal{G}$ be a family of pairwise disjoint balls such that each $G \in \mathcal{G}$ satisfies the relations*

$$G^{[1]} \cap A \ne \varnothing, \qquad 0 < c_1 \le \mu(G) \le c_2, \tag{3.12}$$

*with some positive constants $c_1, c_2$. Then the number of elements in $\mathcal{G}$ is finite and satisfies the bound*

$$\#\mathcal{G} \le \frac{\min\{\mathcal{K}^3 c_2, \mathcal{K}\mu(A)\}}{c_1}.$$

*Remark* 3.1. One can easily check that this lemma implies a similar lemma with the condition $G^{[2]} \cap A \ne \varnothing$ instead of (3.12).

*Proof.* Suppose that $G_1, G_2, \dots, G_N$ are some elements from $\mathcal{G}$. We can assume that

$$\mu(G_1^{[1]}) \ge \mu(G_i^{[1]}) \tag{3.13}$$

for each $1 \le j \le N$. If $\mu(A) \ge \mu(G_1^{[1]})$, then from (3.12) and the (B4) condition we get $\bigcup_{1\le j\le N} G_k \subset \bigcup_{1\le j\le N} G_j^{[1]} \subset A^{[1]}$. Thus, since $G_k$ are pairwise disjoint, from (3.12) we obtain

$$N \cdot c_1 \le \mu\left(\bigcup_{1\le j\le N} G_k\right) \le \mu(A^{[1]}) \le \mathcal{K}\mu(A),$$

that is,

$$N \le \frac{\mathcal{K}\mu(A)}{c_1}. \tag{3.14}$$

In the case $\mu(A) < \mu(G_1^{[1]})$ we get $A \subset G_1^{[2]}$, and therefore by (3.12), $G_j^{[1]} \cap G_1^{[2]} \ne \varnothing$, $1 \le j \le N$. Thus, applying the two balls relation and (3.13), we obtain

$\bigcup_{1\le j\le N} G_j^{[1]} \subset G_1^{[3]}$. Then, again using (3.12) and (3.1), we get

$$N\cdot c_1 \le \mu\left(\bigcup_k G_k\right) \le \mu(G_1^{[3]}) \le \mathcal{K}^3\mu(G_1) \le \mathcal{K}^3 c_2. \tag{3.15}$$

The combination of (3.14) and (3.15) completes the proof of the lemma. □

## 4. Preliminary properties of bounded oscillation operators

In this section we derive some preliminary properties of BO operators. Let $(X, \mathfrak{M}, \mu)$ be a measure space with a ball-basis $\mathfrak{B}$, and let $1 \le r < \infty$. We will need a weak-$L^r$ inequality of the maximal operator

$$M_r f(x) = \sup_{B\in\mathfrak{B}:\, x\in B} \left(\frac{1}{\mu(B)}\int_B |f(t)|^r d\mu(t)\right)^{1/r} \tag{4.1}$$

associated with a ball-basis $\mathfrak{B}$. The maximal operator corresponding to $r = 1$ will be denoted by $M$.

**Theorem 4.1.** *The maximal operator* (4.1) *satisfies the weak-$L^r$ inequality. Moreover, we have*

$$\|M_r\|_{L^r\to L^{r,\infty}} \le \mathcal{K}^{1/r}. \tag{4.2}$$

*Proof.* Denote $E = \{x \in X : M_r f(x) > \lambda\}$. Note that $E$ can be nonmeasurable. For any $x \in E$ there exists a ball $B(x) \subset X$ such that

$$x \in B(x), \qquad \frac{1}{\mu(B(x))}\int_{B(x)} |f|^r > \lambda^r.$$

We have $E = \bigcup_{x\in E} B(x)$. Given that $B \in \mathfrak{B}$, consider the collection of balls $\{B(x) : x \in E \cap B\}$. Applying Lemma 3.1, we find a sequence of pairwise disjoint balls $\{B_k\}$ taken from this collection such that $E \cap B \subset \bigcup_k B_k^{[1]} = Q(B)$. We have $Q(B)$ being measurable and

$$\mu(Q(B)) \le \sum_k \mu(B_k^{[1]}) \le \mathcal{K}\sum_k \mu(B_k) \le \frac{\mathcal{K}}{\lambda^r}\sum_k \int_{B_k} |f(t)|^r dt \le \frac{\mathcal{K}}{\lambda^r}\int_X |f(t)|^r dt.$$

According to Lemma 3.2, there is a sequence of balls $G_k$ such that $X = \bigcup_{n\ge1}\bigcap_{k\ge n} G_k$, so we get $E \subset \bigcup_{n\ge1}\bigcap_{k\ge n} Q(G_k)$. Hence, we obtain

$$\mu^*(E) = \mu\left(\bigcup_{n\ge1}\bigcap_{k\ge n} Q(G_k)\right) \le \frac{\mathcal{K}}{\lambda^r}\int_X |f(t)|^r dt,$$

and thus (4.2). □

**Theorem 4.2.** *If a subadditive operator $T$ satisfies the* (T1) *condition and the weak-$L^r$ inequality, then $T^*$ satisfies weak-$L^r$ inequality too. Moreover, we have*

$$\|T^*\|_{L^r\to L^{r,\infty}} \lesssim \mathcal{L}_1 + \|T\|_{L^r\to L^{r,\infty}}.$$

*Proof.* Given that $\lambda > 0$, consider the set $E = \{x \in X : T^*f(x) > \lambda\}$, which can be nonmeasurable. For any $x \in E$ there is a ball $B(x) \in \mathfrak{B}$ such that

$$x \in B(x), \qquad |T(f \cdot \mathbb{I}_{X\setminus B^{[1]}(x)})(x)| > \lambda. \tag{4.3}$$

One can check that $E = \bigcup_{x\in E} B(x)$. Given a ball $B$ and applying Lemma 3.1, we find a sequence $x_k \in E$ such that the balls $\{B_k = B(x_k)\}$ are pairwise disjoint and

$$E \cap B \subset \bigcup_k B_k^{[1]} = Q(B). \tag{4.4}$$

Since $T$ satisfies the (T1) condition, we have

$$|T(f \cdot \mathbb{I}_{X\setminus B_k^{[1]}})(x_k) - T(f \cdot \mathbb{I}_{X\setminus B_k^{[1]}})(x)| \le \mathcal{L}_1 \cdot \langle f\rangle^*_{B_k,r}, \qquad x \in B_k.$$

Thus, one can easily conclude from (4.3) that

$$\begin{aligned} |T(f \cdot \mathbb{I}_{X\setminus B_k^{[1]}})(x)| \ge |T(f \cdot \mathbb{I}_{X\setminus B_k^{[1]}})(x_k)| \\ - |T(f \cdot \mathbb{I}_{X\setminus B_k^{[1]}})(x_k) - T(f \cdot \mathbb{I}_{X\setminus B_k^{[1]}})(x)| \\ \ge \lambda - \mathcal{L}_1 \cdot \langle f\rangle^*_{B_k,r}, \qquad x \in B_k. \end{aligned} \tag{4.5}$$

Given that $\beta > 0$, define

$$\tilde{B}_k = \{x \in B_k : |T(f \cdot \mathbb{I}_{B_k^{[1]}})(x)| < \beta \cdot \langle f\rangle^*_{B_k,r}\}. \tag{4.6}$$

Using the weak-$L^r$ inequality of the operator $T$, the measure of the complement of $\tilde{B}_k$ is estimated by

$$\begin{aligned} \mu^*(\tilde{B}_k^c) &\le \frac{\|T\|^r_{L^r\to L^{r,\infty}}}{(\beta \cdot \langle f\rangle^*_{B_k,r})^r} \cdot \int_{B_k^{(1)}} |f|^r \le \left(\frac{\|T\|_{L^r\to L^{r,\infty}}}{\beta}\right)^r \mu(B_k^{(1)}) \\ &\lesssim \left(\frac{\|T\|_{L^r\to L^{r,\infty}}}{\beta}\right)^r \mu(B_k), \end{aligned}$$

so for an appropriate constant $\beta \sim \|T\|_{L^r\to L^{r,\infty}}$ we have

$$\mu^*(\tilde{B}_k) \ge \mu(B_k) - \mu^*(B_k \setminus \tilde{B}_k) \ge \mu(B_k) - \mu^*(\tilde{B}_k^c) \ge \frac{1}{2}\mu(B_k).$$

If $x \in \tilde{B}_k \setminus \{M_r f(x) > \delta\lambda\}$, then, using the subadditivity of $T$ together with relations (4.6), (4.5), we obtain

$$\begin{aligned} |Tf(x)| &\ge |T(f \cdot \mathbb{I}_{X\setminus B_k^{[1]}})(x)| - |T(f \cdot \mathbb{I}_{B_k^{[1]}})(x)| \ge \lambda - \mathcal{L}_1 \cdot \langle f\rangle^*_{B_k,r} - \beta \cdot \langle f\rangle^*_{B_k,r} \\ &\ge \lambda - (\mathcal{L}_1 + \beta) \cdot M_r f(x) \ge \lambda - (\mathcal{L}_1 + \beta)\delta\lambda \ge \lambda/2, \end{aligned}$$

where the last inequality can be satisfied for

$$\delta = 1/2(\mathcal{L}_1 + \beta) \sim (\mathcal{L}_1 + \|T\|_{L^r\to L^{r,\infty}})^{-1}. \tag{4.7}$$

Hence, we conclude that

$$\bigcup_k \tilde{B}_k \subset \{M_r f(x) > \delta\lambda\} \cup \{|Tf(x)| > \lambda/2\}. \tag{4.8}$$

Since the maximal function $M_r$ and the operator $T$ satisfy the weak-$L^r$ bound, from (4.4), (4.7), and (4.8) we get

$$\mu(Q(B)) \le \sum_k \mu(B_k^{[1]}) \le \mathcal{K}\cdot\sum_k \mu(B_k) \le 2\mathcal{K}\cdot\sum_k \mu^*(\tilde{B}_k) \lesssim \mu^*\{M_r f(x) > \delta\lambda\}$$
$$+ \mu^*\{|Tf(x)| > \lambda/2\} \lesssim (\mathcal{L}_1 + \|T\|_{L^r\to L^{r,\infty}})^r \frac{1}{\lambda^r}\int_X |f|^r.$$

The same argument used at the end of the proof of Theorem 4.1 implies that

$$\mu^*(E) \lesssim (\mathcal{L}_1 + \|T\|_{L^r\to L^{r,\infty}})^r \frac{1}{\lambda^r}\int_X |f|^r,$$

so the theorem is proved. □

**Definition 4.1.** Let $T$ be a subadditive operator. Given balls $A$ and $B$ with $A \subset B$, we denote

$$\Delta(A,B) = \Delta_T(A,B) = \sup_{x\in A,\, f\in L^r(X)} \frac{|T(f\cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}})(x)|}{\langle f\rangle_{B^{[1]},r}}.$$

Notice that the (T2) condition for a subadditive operator $T$ means that for any $A \in \mathfrak{B}$ there exists a ball $B$ such that $A \subsetneq B$ and $\Delta(A,B) \le \mathcal{L}_2$.

**Lemma 4.1.** *If $T$ is an arbitrary subadditive operator, then for any balls $A, B$, and $C$ satisfying $A \subset B \subset C$ we have $\Delta(A,B) \le \Delta(A,C)$.*

*Proof.* Given that function $f \in L^r(X)$, denote

$$g(x) = f(x)\cdot \mathbb{I}_{B^{[1]}}(x).$$

Then we get the estimate

$$\sup_{x\in A} |T(f\cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}})(x)| = \sup_{x\in A} |T(g\cdot \mathbb{I}_{C^{[1]}\setminus A^{[1]}})(x)| \le \Delta(A,C)\cdot \langle g\rangle_{C^{[1]},r}$$
$$= \Delta(A,C)\cdot\left(\frac{1}{\mu(C^{[1]})}\int_{B^{[1]}} |f|^r\right)^{1/r} \le \Delta(A,C)\cdot \langle f\rangle_{B^{[1]},r},$$

which implies that $\Delta(A,B) \le \Delta(A,C)$. □

**Lemma 4.2.** *Let a subadditive operator $T$ satisfy the* (T1) *condition and the weak-$L^r$ bound. Then for any balls $A$, $B$ satisfying $A \subset B$ we have*

$$\Delta(A,B) \lesssim (\mathcal{L}_1 + \|T\|_{L^r\to L^{r,\infty}})\left(\frac{\mu(B)}{\mu(A)}\right)^{1/r}. \tag{4.9}$$

*Proof.* Applying the weak-$L^r$ estimate, we get

$$\mu^*\left\{x\in A:\, |T(f\cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}})(x)| > \|T\|_{L^r\to L^{r,\infty}}\left(\frac{2}{\mu(A)}\int_{B^{[1]}}|f|^r\right)^{1/r}\right\} \le \frac{\mu(A)}{2},$$

so we find a point $x_0 \in A$ such that

$$|T(f\cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}})(x_0)| \le \|T\|_{L^r\to L^{r,\infty}}\left(\frac{2}{\mu(A)}\int_{B^{[1]}}|f|^r\right)^{1/r} \tag{4.10}$$
$$\lesssim \|T\|_{L^r\to L^{r,\infty}}\cdot\left(\frac{\mu(B)}{\mu(A)}\right)^{1/r}\cdot \langle f\rangle_{B^{[1]},r}.$$

According to the (T1) condition, for any $x \in A$ we have

$$|T(f \cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}})(x) - T(f \cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}})(x_0)| \le \mathcal{L}_1 \cdot \langle f \cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}}\rangle^*_{A,r}. \tag{4.11}$$

By the definition of $\langle f\rangle^*_{A,r}$ there is a ball $C \supset A$ such that $\langle f \cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}}\rangle^*_{A,r} = \langle f \cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}}\rangle_{C,r}$. If $\mu(C) \le \mu(B^{[1]})$, then we have $C \subset B^{[2]}$ and therefore

(4.12)

$$\begin{aligned}\langle f \cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}}\rangle_{C,r} &\le \left(\frac{\mu(B^{[2]})}{\mu(C)}\right)^{1/r} \langle f \cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}}\rangle_{B^{[2]},r}\\ &= \left(\frac{\mu(B^{[2]})}{\mu(C)}\right)^{1/r} \langle f \cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}}\rangle_{B^{[1]},r} \lesssim \left(\frac{\mu(B)}{\mu(A)}\right)^{1/r} \cdot \langle f\rangle_{B^{[1]},r}.\end{aligned}$$

In the case of $\mu(C) > \mu(B^{[1]})$ we have $B^{[1]} \subset C^{[1]}$. Hence, we get

$$\begin{aligned}\langle f \cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}}\rangle_{C,r} &\le \left(\frac{\mu(C^{[1]})}{\mu(C)}\right)^{1/r} \langle f \cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}}\rangle_{C^{[1]},r}\\ &\lesssim \langle f \cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}}\rangle_{C^{[1]},r} \le \langle f \cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}}\rangle_{B^{[1]},r} \le \langle f\rangle_{B^{[1]},r}.\end{aligned} \tag{4.13}$$

The estimates (4.12) and (4.13) imply the inequality

$$\langle f \cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}}\rangle^*_{A,r} \lesssim \left(\frac{\mu(B)}{\mu(A)}\right)^{1/r} \cdot \langle f\rangle_{B^{[1]},r},$$

which together with (4.10) and (4.11) gives

$$|T(f \cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}})(x)| \lesssim (\mathcal{L}_1 + \|T\|_{L^r\to L^{r,\infty}}) \left(\frac{\mu(B)}{\mu(A)}\right)^{1/r} \langle f\rangle_{B^{[1]},r}, \qquad x \in A.$$

The last inequality completes the proof of the lemma. □

**Lemma 4.3.** *If a subadditive operator $T$ satisfies the* (T1) *condition and the weak-$L^r$ bound, then for any balls $A, B$, and $C$ satisfying $A \subset B \subset C$ we have*

$$\Delta(A, C) \lesssim \left(\frac{\mu(C)}{\mu(B)}\right)^{1/r} \cdot (\mathcal{L}_1 + \|T\|_{L^r\to L^{r,\infty}} + \Delta(A, B)). \tag{4.14}$$

*Proof.* Applying Lemma 4.2 for the sets $B$ and $C$, for $f \in L^r(X)$ and $x \in A$ we obtain

$$\begin{aligned}&|T(f\cdot\mathbb{I}_{C^{[1]}\setminus A^{[1]}})(x)|\\ &\quad\le |T(f \cdot \mathbb{I}_{C^{[1]}\setminus B^{[1]}})(x)| + |T(f \cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}})(x)|\\ &\quad\le \Delta(B, C)\langle f\rangle_{C^{[1]},r} + \Delta(A, B)\langle f\rangle_{B^{[1]},r}\\ &\quad\lesssim (\mathcal{L}_1 + \|T\|_{L^r\to L^{r,\infty}}) \left(\frac{\mu(C)}{\mu(B)}\right)^{1/r} \langle f\rangle_{C^{[1]},r} + \Delta(A, B)\langle f\rangle_{B^{[1]},r}\\ &\quad\le (\mathcal{L}_1 + \|T\|_{L^r\to L^{r,\infty}}) \left(\frac{\mu(C)}{\mu(B)}\right)^{1/r} \langle f\rangle_{C^{[1]},r} + \Delta(A, B) \left(\frac{\mu(C^{[1]})}{\mu(B^{[1]})}\right)^{1/r} \langle f\rangle_{C^{[1]},r}\\ &\quad\lesssim \langle f\rangle_{C^{[1]},r} \left(\frac{\mu(C)}{\mu(B)}\right)^{1/r} \cdot (\mathcal{L}_1 + \|T\|_{L^r\to L^{r,\infty}} + \Delta(A, B)),\end{aligned}$$

so we get (4.14). □

Inequality (3.1) and Lemma 4.3 immediately yield the following.

**Lemma 4.4.** *Let an operator $T \in \mathrm{BO}_{\mathfrak{B}}$, and let it satisfy the weak-$L^r$ bound. Then for any balls $A, B \in \mathfrak{B}$ with $A \subset B$ we have*

$$\Delta(A, B^{[n]}) \lesssim \mathcal{K}^{n/r}(\mathcal{L}_1 + \|T\|_{L^r \to L^{r,\infty}} + \Delta(A, B)).$$

**Theorem 4.3.** *If an operator $T \in \mathrm{BO}_{\mathfrak{B}}$ satisfies the weak-$L^r$ estimate, then the operator $T^* \in \mathrm{BO}_{\mathfrak{B}}$ and it satisfies weak-$L^r$ inequality. Moreover, we have*

$$\mathcal{L}_1(T^*) \lesssim \mathcal{L}_1(T) + \|T\|_{L^r \to L^{r,\infty}}, \qquad \mathcal{L}_2(T^*) = \mathcal{L}_2(T).$$

*Proof.* One can easily check that for any balls $A, B$ satisfying $A \subset B$ we have the equality $\Delta_{T^*}(A, B) = \Delta_T(A, B)$. If balls $A$ and $B$ satisfy the (T2) condition for the operator $T$, then the same conditions will hold also for $T^*$ with $\mathcal{L}_2(T^*) = \mathcal{L}_2(T)$. To prove the (T1) condition, let $B \in \mathfrak{B}$ and $f \in L^r(X)$ satisfy

$$\operatorname{supp} f \in X \setminus B^{[1]}. \tag{4.15}$$

Take arbitrary points $x, x' \in B$, and estimate $|T^* f(x) - T^* f(x')|$. If $T^* f(x) = T^* f(x')$, then the estimation is trivial. So we can suppose that $T^* f(x) > T^* f(x')$. Using the definition of $T^* f(x)$, we find a ball $A \in \mathfrak{B}$ with $x \in A$ such that

$$\frac{T^* f(x) + T^* f(x')}{2} < |T(f \cdot \mathbb{I}_{X \setminus A^{[1]}})(x)|. \tag{4.16}$$

Denote

$$A' = \begin{cases} B & \text{if } \mu(A) < \mu(B), \\ A^{[1]} & \text{if } \mu(A) \ge \mu(B). \end{cases} \tag{4.17}$$

Since $x, x' \in B$, from Lemma 4.2 and (4.15) it follows that

$$\begin{aligned} |T(f \cdot \mathbb{I}_{B^{[2]} \setminus A^{[1]}})(x)| &= |T(f \cdot \mathbb{I}_{B^{[2]} \setminus A^{[1]}} \cdot \mathbb{I}_{B^{[2]} \setminus B^{[1]}})(x)| \\ &\le \Delta(B, B^{[1]}) \langle f \cdot \mathbb{I}_{B^{[2]} \setminus A^{[1]}} \rangle_{B^{[2]}, r} \le \Delta(B, B^{[1]}) \langle f \rangle_{B^{[2]}, r} \\ &\lesssim \left( \frac{\mu(B^{[1]})}{\mu(B)} \right)^{1/r} (\mathcal{L}_1(T) + \|T\|_{L^r \to L^{r,\infty}}) \langle f \rangle^*_{B, r} \\ &\lesssim (\mathcal{L}_1(T) + \|T\|_{L^r \to L^{r,\infty}}) \langle f \rangle^*_{B, r}, \end{aligned} \tag{4.18}$$

and similarly,

$$|T(f \cdot \mathbb{I}_{B^{[2]} \setminus A'^{[1]}})(x')| \lesssim (\mathcal{L}_1(T) + \|T\|_{L^r \to L^{r,\infty}}) \langle f \rangle^*_{B, r}. \tag{4.19}$$

One can easily check that from (4.17) it follows that $B \subset A'$ and thus $x' \in A'$. This implies that $T^* f(x') \ge |T(f \cdot \mathbb{I}_{X \setminus A'^{[1]}})(x')|$, which together with (4.16) yields

$$\begin{aligned} |T^* f(x) - T^* f(x')| &= T^* f(x) - T^* f(x') \\ &\le 2|T(f \cdot \mathbb{I}_{X \setminus A^{[1]}})(x)| - 2T^* f(x') \\ &\le 2\Big( |T(f \cdot \mathbb{I}_{X \setminus A^{[1]}})(x)| - |T(f \cdot \mathbb{I}_{X \setminus A'^{[1]}})(x')| \Big). \end{aligned} \tag{4.20}$$

In the case $\mu(A) < \mu(B)$ we get $A \subset B^{[1]}$ and therefore $A'^{[1]} = B^{[1]} \subset B^{[2]}$, $A^{[1]} \subset B^{[2]}$. Thus, applying the (T1) condition for $T$, from (4.18), (4.19), and

(4.20) we conclude that

$$\begin{aligned}|T^*f(x)-T^*f(x')| &\le 2\Big(|T(f\cdot \mathbb{I}_{X\setminus B^{[2]}})(x)|+|T(f\cdot \mathbb{I}_{B^{[2]}\setminus A^{[1]}})(x)|\\ &\qquad -|T(f\cdot \mathbb{I}_{X\setminus B^{[2]}})(x')|+|T(f\cdot \mathbb{I}_{B^{[2]}\setminus A'^{[1]}})(x')|\Big)\\ &\lesssim |T(f\cdot \mathbb{I}_{X\setminus B^{[2]}})(x)-T(f\cdot \mathbb{I}_{X\setminus B^{[2]}})(x')|\\ &\quad +(\mathcal{L}_1(T)+\|T\|_{L^r\to L^{r,\infty}})\langle f\rangle^*_{B,r}\\ &\lesssim (\mathcal{L}_1(T)+\|T\|_{L^r\to L^{r,\infty}})\langle f\rangle^*_{B,r}.\end{aligned}$$

If $\mu(A)\ge\mu(B)$, then by (4.17) we have $A'=A^{[1]}$ and thus $x,x'\in B\subset A^{[1]}$, $A'^{[1]}=A^{[2]}$. Thus, applying Lemma 4.2 and (4.20), we get

$$\begin{aligned}|T^*f(x)-T^*f(x')| &\le 2\Big(|T(f\cdot \mathbb{I}_{X\setminus A^{[1]}})(x)|-|T(f\cdot \mathbb{I}_{X\setminus A^{[2]}})(x')|\Big)\\ &\le 2\Big(|T(f\cdot \mathbb{I}_{X\setminus A^{[2]}})(x)|-|T(f\cdot \mathbb{I}_{X\setminus A^{[2]}})(x')|\Big)+2|T(f\cdot \mathbb{I}_{A^{[2]}\setminus A^{[1]}})(x)|\\ &\lesssim \mathcal{L}_1(T)\langle f\rangle^*_{A^{[1]},r}+(\mathcal{L}_1(T)+\|T\|_{L^r\to L^{r,\infty}})\langle f\rangle_{A^{[2]},r}\\ &\le (\mathcal{L}_1(T)+\|T\|_{L^r\to L^{r,\infty}})\langle f\rangle^*_{B,r},\end{aligned}$$

and, finally, the (T1) condition. The weak-$L^r$ bound of $T^*$ follows from Theorem 4.2. ☐

We say that a ball-basis $\mathfrak{B}$ in a measure space satisfies the doubling condition if there is a constant $\eta>1$ such that for any ball $A\in\mathfrak{B}$, $A^{[1]}\ne X$, one can find a ball $B$ satisfying

$$A\subsetneq B,\qquad \mu(B)\le\eta\cdot\mu(A). \tag{4.21}$$

**Theorem 4.4.** *Let the ball-basis $\mathfrak{B}$ satisfy the doubling condition. If a subadditive operator $T$ satisfies the* (T1) *condition and the weak-$L^r$ bound, then $T\in \mathrm{BO}_{\mathfrak{B}}$. Moreover, we have $\mathcal{L}_2(T)\lesssim\eta^{1/r}(\mathcal{L}_1+\|T\|_{L^r\to L^{r,\infty}})$, where $\eta$ is the doubling constant from* (4.21).

*Proof.* We need to check the (T2) condition. If balls $A$ and $B$ satisfy conditions (4.21), then applying Lemma 4.2, we get

$$\Delta(A,B)\lesssim(\mathcal{L}_1+\|T\|_{L^r\to L^{r,\infty}})\left(\frac{\mu(B)}{\mu(A)}\right)^{1/r}\le\eta^{1/r}(\mathcal{L}_1+\|T\|_{L^r\to L^{r,\infty}}).$$

Thus, we get $\mathcal{L}_2(T)\lesssim\eta^{1/r}(\mathcal{L}_1+\|T\|_{L^r\to L^{r,\infty}})<\infty$. ☐

## 5. Proof of main theorems

**Lemma 5.1.** *If a subadditive operator $T$ satisfies the* (T1) *condition and the weak-$L^r$ bound, then for any $B\in\mathfrak{B}$ there exists a ball $B'$ such that*

$$B^{[2]}\subset B',\qquad \Delta(B^{[2]},B')\lesssim\mathcal{L}_1+\|T\|_{L^r\to L^{r,\infty}}, \tag{5.1}$$

*and we have either*

$$B'^{[1]}=B'\quad\text{or}\quad \mu(B')\ge 2\mu(B). \tag{5.2}$$

*Proof.* Letting $\mathfrak{A} = \{A \in \mathfrak{B} : A \cap B \neq \varnothing\}$, we denote

$$a = \sup_{A\in\mathfrak{A}:\, \mu(B)\le\mu(A)\le 2\mu(B)} \mu(A) \le 2\mu(B), \tag{5.3}$$

$$b = \inf_{A\in\mathfrak{A}:\, \mu(A)>2\mu(B)} \mu(A) \ge 2\mu(B). \tag{5.4}$$

Observe that there is no ball $A \in \mathfrak{A}$ with $a < \mu(A) < b$ and that there exist balls $G_1, G_2 \in \mathfrak{A}$ such that $a/2 < \mu(G_1) \le a$ and $b \le \mu(G_2) < 2b$. If $b \le \mathcal{K}^2 a$, then we define $B' = G_2^{[3]}$. Since $B \cap G_2 \neq \varnothing$ and $\mu(B) \le a \le b \le \mu(G_2)$, we get $B \subset G_2^{[1]}$ and therefore $B^{[2]} \subset G_2^{[3]} = B'$. Thus, we get (5.1). Taking into account the inequalities

$$\mu(B') = \mu(G_2^{[3]}) \le \mathcal{K}^3\mu(G_2) \le \mathcal{K}^3 \cdot 2b \le 2a\mathcal{K}^5,$$
$$\mu(B^{[2]}) \ge \mu(B) \ge a/2,$$

from Lemma 4.2 it follows that

$$\Delta(B^{[2]}, B') \lesssim \left(\frac{\mu(B')}{\mu(B^{[2]})}\right)^{1/r} (\mathcal{L}_1 + \|T\|_{L^r\to L^{r,\infty}}) \lesssim \mathcal{L}_1 + \|T\|_{L^r\to L^{r,\infty}}.$$

Hence, we obtain (5.1). Then by (5.4) we have

$$\mu(B') \ge \mu(G_2) \ge b \ge 2\mu(B), \tag{5.5}$$

so we get the second relation in (5.2). Now suppose that we have $b > \mathcal{K}^2 a$. Define $B' = G_1^{[1]} \in \mathfrak{A}$. We have $\mu(B'^{[1]}) \le \mathcal{K}^2\mu(G_1) \le a\mathcal{K}^2 < b$. Since there is no ball from $\mathfrak{A}$ with a measure in the interval $(a, b)$, we get $\mu(B'^{[1]}) \le a$. Thus, we get $B'^{[1]} \cap G_1 \neq \varnothing$ and $\mu(B'^{[1]}) \le 2\mu(G_1)$. These relations imply that $B'^{[1]} \subset G_1^{[1]} = B'$, which means that $B'^{[1]} = B'$, and we get the first relation in (5.2). On the other hand, by (5.3) we have

$$\mu(B) \le a \le 2\mu(G_1) \le 2\mu(B') \le 2a \le 4\mu(B).$$

This means that $B \subset B'^{[1]} = B'$, and therefore $B^{[2]} \subset B'^{[2]} = B'$. Hence, we obtain (5.1). Using Lemma 4.2 and (5.5), we get

$$\Delta(B^{[2]}, B') \lesssim \left(\frac{\mu(B')}{\mu(B^{[2]})}\right)^{1/r} (\mathcal{L}_1 + \|T\|_{L^r\to L^{r,\infty}}) \lesssim \mathcal{L}_1 + \|T\|_{L^r\to L^{r,\infty}},$$

which gives (5.1). The lemma is proved. □

**Lemma 5.2.** *Let $T \in \mathrm{BO}_{\mathfrak{B}}$ satisfy the weak-$L^r$ estimate, and for a ball $B \in \mathfrak{B}$ we have $B^{[1]} = B$. Then there exists a ball $B'$ satisfying* (5.1) *and*

$$\Delta(B^{[2]}, B') \le \mathcal{L}_2, \tag{5.6}$$

$$\mu(B') > 2\mu(B). \tag{5.7}$$

*Proof.* Applying the (T2) condition, we find a ball $B' \supsetneq B$ such that $\Delta(B, B') \le \mathcal{L}_2$. On the other hand, from $B^{[1]} = B$ we get $B^{[2]} = B \subset B'$, that is, (5.1). Hence, we get (5.6). Observe that (5.7) holds. Indeed, otherwise by the two balls relation we will have $B' \subset B^{[1]} = B$, which contradicts the condition $B' \supsetneq B$. □

**Lemma 5.3.** *If $T \in \mathrm{BO}_{\mathfrak{B}}$ satisfies the weak-$L^r$ estimate, then for any $B \in \mathfrak{B}$ there exists a sequence of balls $B = B_0, B_1, B_2, \dots$ such that*

$$\bigcup_k B_k = X, \tag{5.8}$$

$$B_{k-1}^{[2]} \subset B_k, \qquad k \geq 1, \tag{5.9}$$

$$\sup_{k\geq 1} \Delta(B_{k-1}^{[2]}, B_k) \lesssim \mathcal{L}_1 + \mathcal{L}_2 + \|T\|_{L^r \to L^{r,\infty}}, \qquad k \geq 1. \tag{5.10}$$

*Proof.* We construct a sequence of balls $\{B_k\}$ satisfying (5.9), (5.10), and the relation

$$\mu(B_k) > 2\mu(B_{k-2}), \qquad k \geq 2. \tag{5.11}$$

We do it by induction. Take $B_0 = B$, and suppose that we have already defined $B_k$ for $k = 0, 1, \dots, l$ satisfying the conditions (5.9), (5.10), and (5.11) for $k \leq l$. If $B_l^{[1]} = B_l$, then applying Lemma 5.2, we get a ball $B_{l+1} = B'$ such that

$$\begin{aligned} &B_l^{[2]} \subset B_{l+1}, \\ &\Delta(B_l^{[2]}, B_{l+1}) \leq \mathcal{L}_2, \qquad \mu(B_{l+1}) \geq 2\mu(B_l). \end{aligned} \tag{5.12}$$

In the case of $B_l^{[1]} \neq B_l$ we use Lemma 5.1. At this time the ball $B_{l+1} = B'$ will satisfy (5.12) and the conditions

$$\Delta(B_l^{[2]}, B_{l+1}) \lesssim \mathcal{L}_1 + \|T\|_{L^r \to L^{r,\infty}}, \qquad \mu(B_{l+1}) \geq 2\mu(B_l). \tag{5.13}$$

Applying this process, we get a sequence satisfying the conditions (5.9) and (5.10). Besides, one can easily observe that at least for one of any two consecutive integers $k$ we will have $\mu(B_{k+1}) \geq 2\mu(B_k)$. So the condition (5.11) will also be satisfied. □

**Lemma 5.4.** *Let $T$ be a $\mathrm{BO}_{\mathfrak{B}}$ operator satisfying weak-$L^r$ inequality. If*

$$\lambda \geq 3\mathcal{K}^4, \tag{5.14}$$

*then for every measurable set $F \subset X$ and a ball $A \in \mathfrak{B}$ with*

$$F \cap A \neq \varnothing, \qquad \mu(F) \leq \mu(A)/\lambda, \tag{5.15}$$

*there exists a family of balls $\mathcal{G} \subset \mathfrak{B}$ satisfying the conditions*

$$F \cap A^{[1]} \cap G \neq \varnothing \qquad \text{if } G \in \mathcal{G}, \tag{5.16}$$

$$F \cap A^{[1]} \subset \bigcup_{G\in\mathcal{G}} G \ \text{a.s.}, \tag{5.17}$$

$$\mu\left(\bigcup_{G\in\mathcal{G}} G^{[1]}\right) \lesssim \frac{\mu(A)}{\lambda}. \tag{5.18}$$

*Besides, for each $G \in \mathcal{G}$ there is a ball $\tilde{G}$ (not necessarily in $\mathcal{G}$) such that*

$$\tilde{G} \not\subset F. \tag{5.19}$$

$$G^{[2]} \subset \tilde{G} \subset A^{[1]}, \tag{5.20}$$

$$\Delta(G^{[2]}, \tilde{G}) \lesssim \mathcal{L}_1 + \mathcal{L}_2 + \|T\|_{L^r \to L^{r,\infty}}. \tag{5.21}$$

*Proof.* Applying Lemma 3.5 for the set $E = F \cap A^{[1]}$, we find a family of balls $\mathcal{P} \subset \mathfrak{B}$ such that

$$F \cap A^{[1]} \cap B \neq \varnothing, \qquad B \in \mathcal{P}, \tag{5.22}$$

$$F \cap A^{[1]} \subset \bigcup_{B\in\mathcal{P}} B \text{ a.s.} \tag{5.23}$$

$$\sum_{B\in\mathcal{P}} \mu(B) < 2\mathcal{K}\mu(F \cap A^{[1]}). \tag{5.24}$$

Take an arbitrary element $B \in \mathcal{P}$. Applying Lemma 5.3, we find a sequence of balls $B_k \in \mathfrak{B}$, $k = 0, 1, 2, \ldots$, $B = B_0$, with conditions (5.8)–(5.10). To $B \in \mathcal{P}$ we can attach a ball $G = B_m$, where $m \geq 0$ is the least index satisfying the relation

$$B_{m+1}^{[1]} \not\subset F. \tag{5.25}$$

The collection of all such $G$ defines the family $\mathcal{G}$. If $G \in \mathcal{G}$ is generated by $B \in \mathcal{P}$, then combining the relation

$$B \subset B_0^{[2]} \subset B_m = G \tag{5.26}$$

with (5.22), we obtain (5.16). From (5.26) we also get $\bigcup_{G\in\mathcal{G}} G \supset \bigcup_{B\in\mathcal{P}} B$, which together with (5.23) implies (5.17). Then, according to the definition of the integer $m$ (see (5.25)), we have either $G^{[1]} \subset F$ or $G \in \mathcal{P}$ (the second relation holds only if $m = 0$). This remark together with (5.24) implies that

$$\begin{aligned}\mu\left(\bigcup_{G\in\mathcal{G}} G^{[1]}\right) &\leq \mu(F) + \mu\left(\bigcup_{B\in\mathcal{P}} B^{[1]}\right) \leq \mu(F) + \sum_{B\in\mathcal{P}} \mu\left(B^{[1]}\right) \\ &\leq \mu(F) + \mathcal{K}\sum_{B\in\mathcal{P}} \mu(B) \leq \frac{(2\mathcal{K}^2+1)\mu(A^{[1]})}{\lambda} \leq \frac{3\mathcal{K}^2\mu(A)}{\lambda},\end{aligned} \tag{5.27}$$

and we get (5.18). Now define

$$\tilde{G} = \begin{cases} A^{[1]} & \text{if } \mu(B_{m+1}^{[1]}) > \mu(A), \\ B_{m+1}^{[1]} & \text{if } \mu(B_{m+1}^{[1]}) \leq \mu(A). \end{cases} \tag{5.28}$$

According to (5.15), we have $\mu(A^{[1]}) > \mu(F)$ and thus $A^{[1]} \not\subset F$. This together with (5.25) and (5.28) implies that (5.19). To check condition (5.20), notice that form (5.14) and (5.27) it follows that

$$\mu(G^{[2]}) \leq \mathcal{K}^2 \cdot \mu(G) \leq \mathcal{K}^2 \cdot \frac{3\mathcal{K}^2\mu(A)}{\lambda} \leq \mu(A).$$

Thus, since $G \cap A \neq \varnothing$, we conclude that

$$G^{[2]} \subset A^{[1]}. \tag{5.29}$$

If $\mu(B_{m+1}^{[1]}) > \mu(A)$, then by (5.28) we have $\tilde{G} = A^{[1]}$, and using (5.29) we get (5.20). If $\mu(B_{m+1}^{[1]}) \leq \mu(A)$, then since $B_{m+1}^{[1]} \cap A \neq \varnothing$, we have $B_{m+1}^{[1]} \subset A^{[1]}$. Hence, from (5.9) and (5.28) we obtain $G^{[2]} = B_m^{[2]} \subset B_{m+1}^{[1]} = \tilde{G} \subset A^{[1]}$ and thus (5.20). To prove (5.21), first we suppose that $\mu(B_{m+1}^{[1]}) \leq \mu(A)$ and thus $\tilde{G} = B_{m+1}^{[1]}$.

Applying Lemma 4.4 and (5.10), we get

$$\begin{aligned}\Delta(G^{[2]},\tilde{G}) = \Delta(B_m^{[2]},B_{m+1}^{[1]}) &\lesssim \left(\frac{\mu(B_{m+1}^{[1]})}{\mu(B_{m+1})}\right)^{\frac{1}{r}} (\mathcal{L}_1 + \|T\|_{L^r\to L^{r,\infty}} + \Delta(B_m^{[2]},B_{m+1}))\\ &\lesssim \mathcal{L}_1 + \mathcal{L}_2 + \|T\|_{L^r\to L^{r,\infty}}.\end{aligned}$$

In the case $\mu(B_{m+1}^{[2]}) > \mu(A^{[1]})$ we have $\tilde{G} = A^{[1]} \subset B_{m+1}^{[3]}$ and therefore $\tilde{G}^{[1]} \subset B_{m+1}^{[4]}$. Once again applying Lemma 4.4, we obtain

$$\begin{aligned}\Delta(G^{[2]},\tilde{G}) &\le \Delta(B_m^{[2]},B_{m+1}^{[3]})\\ &\lesssim \left(\frac{\mu(B_{m+1}^{[2]})}{\mu(B_{m+1})}\right)^{1/r} (\mathcal{L}_1 + \mathcal{L}_2 + \|T\|_{L^r\to L^{r,\infty}} + \Delta(B_m^{[2]},B_{m+1}))\\ &\lesssim \mathcal{L}_1 + \mathcal{L}_2 + \|T\|_{L^r\to L^{r,\infty}},\end{aligned}$$

which completes the proof of the lemma. □

**Definition 5.1.** We say a set of balls $\mathfrak{A}$ is a family tree if

(F1) there is an element $A_0 \in \mathfrak{A}$ called the grandparent of $\mathfrak{A}$;
(F2) to each $A \in \mathfrak{A}$ except the grandparent $A_0$ a unique parent $\mathrm{pr}(A) \in \mathfrak{A}$ is attached;
(F3) for each $A \in \mathfrak{A}$, $A \ne A_0$, we have $A_0 = \mathrm{pr}^n(A) = \mathrm{pr}(\mathrm{pr}(\ldots\mathrm{pr}(A)\ldots))$ for some $n \in \mathbb{N}$.

Given ball $A \in \mathfrak{A}$, we denote

$$\mathfrak{Ch}_n(A) = \{B \in \mathfrak{A} : \mathrm{pr}^n(B) = A\}, \quad n = 1,2,\ldots, \quad \mathfrak{Gen}(A) = \bigcup_{n=1}^{\infty} \mathfrak{Ch}_n(A),$$

where $M$ is the maximal operator (4.1). The family $\mathfrak{Ch}(A) = \mathfrak{Ch}_1(A)$ is said to be the children of $A$, and $\mathfrak{Gen}(A)$ is the generation of $A$. The notation $n \ll m$ ($n \gg m$) for two integers $n, m$ denotes $n < m-1$ ($n > m+1$), and $n \asymp m$ stands for the condition $|m-n| \le 1$.

*Proof of Theorem* 1.1. Let an operator $T \in \mathrm{BO}_{\mathfrak{B}}$ satisfy the weak-$L^r$ inequality. Define

$$\Gamma f(x) = \max\{|Tf(x)|, T^*f(x), \mathcal{L}\cdot M_r f(x)\},$$

where $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \|T\|_{L^r\to L^{r,\infty}}$. Applying Theorems 4.1 and 4.2, we conclude that the operator $\Gamma$ satisfies the weak-$L^r$ estimate, and besides that

$$\|\Gamma\|_{L^r\to L^{r,\infty}} \lesssim \mathcal{L}_1 + \mathcal{L}_2 + \|T\|_{L^r\to L^{r,\infty}}. \tag{5.30}$$

Denote

$$T^{**}f(x) = \sup_{B\in\mathfrak{B},\, x\in B} |T(f\cdot \mathbb{I}_{B^{[1]}})(x)|. \tag{5.31}$$

Subadditivity of $T$ implies that

$$T^{**}f(x) \le T^*f(x) + |Tf(x)| \le 2\Gamma f(x), \qquad x \in X. \tag{5.32}$$

Let $f \in L^r(X)$, and let $B$ be the ball from the statement of the theorem. Clearly we can choose a ball $A_0$ such that

$$B^{[1]} \subset A_0, \qquad \int_{A_0} |f| > \frac{\|f\|_1}{2}. \tag{5.33}$$

Let $\lambda > 0$ satisfy (5.14). We shall construct a family tree $\mathfrak{A} \subset \mathfrak{B}$ with the grandparent $A_0$ such that the following apply:

(1) If $G \in \mathfrak{Ch}(A)$, then $A^{[1]} \cap G \neq \varnothing$.

(2) We have

$$\mu\left(\bigcup_{G\in\mathfrak{Ch}(A)} G^{[1]}\right) \lesssim \frac{\mu(A)}{\lambda}. \tag{5.34}$$

(3) If $G \in \mathfrak{Ch}(A)$, then there exist a ball $\tilde{G}$ and a point $\xi \in \tilde{G}$ such that

$$G^{[2]} \subset \tilde{G} \subset A^{[1]}, \tag{5.35}$$

$$\Gamma f_A(\xi) \lesssim \mathcal{L}\lambda \cdot \langle f\rangle_{A^{[3]},r}, \tag{5.36}$$

$$|T(f \cdot \mathbb{I}_{\tilde{G}^{[1]}\setminus G^{[3]}})(x)| \lesssim \mathcal{L}\lambda \cdot \langle f\rangle_{A^{[3]},r}, \qquad x \in G^{[2]}, \tag{5.37}$$

where

$$f_A = \begin{cases} f \cdot \mathbb{I}_{A^{[3]}} & \text{if } A \neq A_0, \\ f & \text{if } A = A_0. \end{cases} \tag{5.38}$$

(4) We have

$$|\Gamma f_A(x)| \lesssim \mathcal{L}\lambda \cdot \langle f\rangle_{A^{[3]},r}$$
$$\text{for almost all } x \in A^{[1]} \setminus \bigcup_{G\in\mathfrak{Ch}(A)} G \qquad \text{if } A \neq A_0. \tag{5.39}$$

The elements of $\mathfrak{A}$ will be determined inductively by an increasing order of generation levels. The first element of $\mathfrak{A}$ is $A_0$. Then we suppose inductively that we have already defined all of the members of $\bigcup_{1\le n\le l}\mathfrak{Ch}_n(A)$ such that any member $A \in \bigcup_{1\le n\le l-1}\mathfrak{Ch}_n(A)$ satisfies conditions (1)–(4). To define the members of $\mathfrak{Ch}_{l+1}(A_0)$, we take an arbitrary $A \in \mathfrak{Ch}_l(A_0)$ and define the children of $A$ as follows. Take a measurable set $F = F_A$ such that

$$F \supset \{x \in X : \Gamma f_A(x) > \beta \cdot \langle f\rangle_{A^{[3]},r}\}, \tag{5.40}$$

$$\mu(F) = \mu^*\{x \in X : \Gamma f_A(x) > \beta \cdot \langle f\rangle_{A^{[3]},r}\}. \tag{5.41}$$

Suppose that $F \neq \varnothing$. Using (5.30), for any $A$ (including the case $A = A_0$) we will have

$$\mu(F) \le \frac{2\mu(A^{[3]})}{\beta^r}\|\Gamma\|^r_{L^r\to L^{r,\infty}} \le \frac{2\mathcal{K}^3\cdot\mu(A)}{\beta^r}\|\Gamma\|^r_{L^r\to L^{r,\infty}} \le \frac{\mu(A)}{\lambda},$$

for a suitable constant $\beta \sim \mathcal{L}\lambda$, since $r \ge 1$ and we have (5.30). Note that in the case $A = A_0$ one needs additionally to use the inequality (5.33). Thus, applying Lemma 5.4 for $A$ and $F = F_A$, we get a family $\mathcal{G}$ satisfying the conditions of the lemma. The family $\mathcal{G}$ will form the children collection of $A$, that is, $\mathfrak{Ch}(A) = \mathcal{G}$. Relations (1) and (2) are immediate consequences of (5.16) and (5.18), respectively, while (5.35) follows from condition (5.20) of Lemma 5.4. From (5.19) follows the existence of $\xi \in \tilde{G} \setminus F$, and by the definition (5.40) we get (5.36) ((5.54) if $A = A_0$). Since $\xi \in \tilde{G}$, using (5.20), (5.21), and (5.36), for $f \in L^r(X)$ we get the inequality

$$\begin{aligned}\sup_{x\in G^{[2]}} |T(f\cdot \mathbb{I}_{\tilde{G}^{[1]}\setminus G^{[3]}})(x)| &= \sup_{x\in G^{[2]}} |T(f\cdot \mathbb{I}_{A^{[3]}}\cdot \mathbb{I}_{\tilde{G}^{[1]}\setminus G^{[3]}})(x)|\\ &\le \Delta(G^{[2]},\tilde{G})\cdot \langle f\cdot \mathbb{I}_{A^{[3]}}\rangle_{\tilde{G}^{[1]},r} \lesssim \mathcal{L}\langle f\cdot \mathbb{I}_{A^{[3]}}\rangle^*_{\tilde{G},r}\\ &\le \mathcal{L}M_r(f\cdot \mathbb{I}_{A^{[3]}})(\xi) \le \Gamma(f\cdot \mathbb{I}_{A^{[3]}})(\xi) = \Gamma f_A(\xi)\\ &\le \beta\cdot \langle f\rangle_{A^{[3]},r} \lesssim \mathcal{L}\lambda\cdot \langle f\rangle_{A^{[3]},r},\end{aligned}$$

which implies (5.37). From (5.17) we get $\mu\left(F\cap\left(A^{[1]}\setminus \bigcup_{G\in\mathfrak{ch}(A)} G\right)\right) = 0$; therefore by the definition of $F$ we have (5.39). Hence, the properties of family $\mathfrak{A}$ are satisfied. Now we construct a sparse subfamily $\mathcal{S} \subset \mathfrak{A}$ consisting of a countable collection of balls, which will satisfy the conditions of the theorem. We will do that by removing some elements of $\mathfrak{A}$. As we will see below, removing an element $A \in \mathfrak{A}$, we also remove all of the elements of its generation $\mathfrak{Gen}(A)$. Thus, one can easily check that properties (1)–(3) will hold during the whole process. To start the description of the process, we let $R = \mathcal{K}^2$, where $\mathcal{K} > 1$ is the constant from (1.6). For $B \in \mathfrak{B}$ denote $\mathrm{r}(B) = [\log_R \mu(B)]$. Observe that the collection of balls

$$\mathfrak{A}_k = \{B\in\mathfrak{A} :\ \mathrm{r}(B) = k\} = \left\{B\in\mathfrak{A} :\ R^k \le \mu(B) < R^{k+1}\right\}, \qquad k\le k_0,$$

gives a partition of $\mathfrak{A}$; i.e., we have $\mathfrak{A} = \bigcup_{k\le k_0}\mathfrak{A}_k$, where $k_0 = \mathrm{r}(A_0)$ and $A_{k_0} = \{A_0\}$. The reduction of the elements of $\mathfrak{A}$ will be realized in different stages. The content of $\mathfrak{A}_{k_0}$ will not be changed. In the $n$th stage only the contents of the families $\mathfrak{A}_k$ with $k \le k_0 - n$ can be changed. Besides, at the end of the $n$th stage $\mathfrak{A}_{k_0-n}$ will be fixed and will remain the same until the end of the process. Suppose by induction that the $l$th stage of reduction has already been finished and thus that the families $\mathfrak{A}_k$, $k = k_0, k_0-1, k_0-2, \dots, k_0-l$ have already been fixed. In the next $(l+1)$th stage we will apply the following two procedures consecutively.

**Procedure 1.** Remove any element $G \in \mathfrak{A}_{k_0-l-1}$ together with all of the elements of its generation $\mathfrak{Gen}(G)$ if there exists a $B \in \mathfrak{A}$ satisfying the conditions

$$G^{[2]}\cap B \ne \varnothing, \tag{5.42}$$

$$\mathrm{r}(\mathrm{pr}^k(G)) \ll \mathrm{r}(B) \ll \mathrm{r}(\mathrm{pr}^{k+1}(G)) \tag{5.43}$$

for some integer $k \ge 0$.

*Remark* 5.1. Observe that if an element $G$ is removed because of a ball $B$ satisfying conditions (5.42) and (5.43) of Procedure 1, then we should have $\mathrm{r}(G) \ll \mathrm{r}(B) \ll \mathrm{r}(\mathrm{pr}(G))$, which means that (5.43) can hold only with $k = 0$. Indeed, the left inequality immediately follows from (5.43). To prove the right one, suppose to the contrary that in (5.43) we have $k \ge 1$. Denote $G' = \mathrm{pr}^k(G) \in \bigcup_{j=0}^{l}\mathfrak{A}_{k_0-j}$. Since $G'^{[2]} \supset G^{[2]}$ (see (5.35)), we have $G'^{[2]}\cap B \ne \varnothing$. On the other hand, (5.43) can be written by $\mathrm{r}(G') \ll \mathrm{r}(B) \ll \mathrm{r}(\mathrm{pr}(G'))$. We thus conclude that $G'$ satisfies the conditions of the Procedure 1, so $G'$ together with its generation $\mathfrak{Gen}(G')$ (including $G$) had to be removed in one of the previous stages of the process, when $B$ was already fixed. This is a contradiction, so $k = 0$.

**Procedure 2.** Apply Lemma 3.1 to the rest of the elements $\mathfrak{A}_{k_0-l-1}$ that we have after Procedure 1. The application of the lemma removes some elements of $\mathfrak{A}_{k_0-l-1}$. If an element $A$ is removed, then the generation $\mathfrak{Gen}(A)$ will also be removed.

*Remark* 5.2. After Procedure 2 the elements of $\mathfrak{A}_{k_0-l-1}$ become pairwise disjoint. Besides, we will have

$$\bigcup_{G\in\mathfrak{A}_{k_0-l-1}(\text{before Procedure 2})} G \subset \bigcup_{G\in\mathfrak{A}_{k_0-l-1}(\text{after Procedure 2})} G^{[1]}. \tag{5.44}$$

After these two procedures the family $\mathfrak{A}_{k_0-l-1}$ will be fixed. Hence, finishing the induction process, we get the final state of $\mathfrak{A}$, which will be denoted by $\mathcal{D}$. Since after Procedure 2 in the $n$th stage $\mathfrak{A}_{k_0-n}$ gets a countable number of balls, family $\mathcal{D}$ will also be countable at the end of the whole process.

Now we shall prove that for an admissible constant $\lambda > 0$ the family $\mathcal{D}$ is a union of two 1/2-sparse collections of balls. For $A \in \mathcal{D}$ define

$$E(A) = A \setminus \bigcup_{G\in\mathcal{D}\colon \mathrm{r}(G)\ll\mathrm{r}(A)} G = A \setminus \bigcup_{G\in\mathcal{D}\colon G\cap A\neq\varnothing,\, \mathrm{r}(G)\ll\mathrm{r}(A)} G. \tag{5.45}$$

Observe that

$$E(A)\cap E(B) = \varnothing \qquad \text{if } \mathrm{r}(A)\not\asymp \mathrm{r}(B) \text{ or } \mathrm{r}(A) = \mathrm{r}(B). \tag{5.46}$$

Indeed, take arbitrary $A, B \in \mathcal{D}$. If $\mathrm{r}(A) = \mathrm{r}(B)$, then balls $A, B$ are the result of the application of Procedure 2 (Lemma 3.1). This means that we have $A \cap B = \varnothing$, and therefore according to (5.45) it follows that $E(A)\cap E(B) = \varnothing$. If $\mathrm{r}(A) \gg \mathrm{r}(B)$, then $E(A)\cap B = \varnothing$ immediately follows from definition (5.45), so we will get again $E(A)\cap E(B) = \varnothing$. To prove

$$\mu(E(A)) \ge \mu(A)/2, \tag{5.47}$$

take an arbitrary $A \in \mathcal{D}$, and denote $\mathcal{P} = \{P \in \mathcal{D} : \mathrm{r}(P) \asymp \mathrm{r}(A),\ P^{[2]} \cap A \neq \varnothing\}$. We have

$$R^{-3}\cdot\mu(A) \le \mu(P) \le R^3\cdot\mu(A), \qquad P\in\mathcal{P}. \tag{5.48}$$

On the other hand, $\mathcal{P} \subset \mathfrak{A}_{l-1}\cup\mathfrak{A}_l\cup\mathfrak{A}_{l+1}$, where $l = \mathrm{r}(A)$. Hence, $\mathcal{P}$ consists of three families of pairwise disjoint balls. Thus, applying the remark after Lemma 3.6, we get

$$\#\mathcal{P} \lesssim 1. \tag{5.49}$$

Suppose that $G \in \mathcal{D}$ satisfies $\mathrm{r}(G) \ll \mathrm{r}(A)$, $G\cap A \neq \varnothing$, and therefore that $G^{[2]}\cap A \neq \varnothing$. Since $G$ was not removed by Procedure 1, we have $\mathrm{r}(\mathrm{pr}^k(G)) \asymp \mathrm{r}(A)$ for some integer $k \ge 1$. Denote $P = \mathrm{pr}^k(G)$. We have $\mathrm{r}(P) \asymp \mathrm{r}(A)$ and $P^{[2]} \supset G^{[2]}$ by (5.35), so $P^{[2]} \cap A \neq \varnothing$. This implies that $P \in \mathcal{P}$ and $G \in \mathfrak{Gen}(P)$. Hence, from (5.45), (5.34), (5.49), and (5.48), it follows that

$$\begin{aligned}\mu(A\setminus E_A) &\le \mu\left(\bigcup_{G\in\mathcal{D}\colon G\cap A\neq\varnothing,\,\mathrm{r}(G)\ll\mathrm{r}(A)} G\right) \le \mu\left(\bigcup_{P\in\mathcal{P}}\bigcup_{G\in\mathfrak{Gen}(P)} G\right)\\ &\le \sum_{P\in\mathcal{P}}\sum_{k=1}^{\infty}\mu\left(\bigcup_{G\colon \mathrm{pr}^k(G)=P} G\right) \lesssim \sum_{P\in\mathcal{P}}\sum_{k=1}^{\infty}\frac{\mu(P)}{\lambda^k} \lesssim \frac{\mu(A)}{\lambda}.\end{aligned}$$

Thus, for an admissible constant $\lambda$ we get (5.47). From (5.46) and (5.47) we conclude that the two families $\mathcal{D}_1 = \{A\in\mathcal{D} : \mathrm{r}(A) \text{ is odd }\}$ and $\mathcal{D}_2 = \mathcal{D}\setminus\mathcal{D}_1$ are sparse. Further, for an $A \in \mathcal{D}$ we will need the bound

$$|\Gamma f_A(x)| \lesssim \mathcal{L}\lambda\langle f\rangle_{A^{[3]},r} \qquad \text{for a.a. } x \in A^{[1]} \setminus \bigcup_{G\in\mathcal{D}:\, \mathrm{r}(G)<\mathrm{r}(A)} G^{[1]},$$

which is based on the inequality (5.39). It is enough to prove that

$$A^{[1]} \setminus \bigcup_{B\in\mathcal{D}:\, \mathrm{r}(B)<\mathrm{r}(A)} B^{[1]} \subset A^{[1]} \setminus \bigcup_{G\in\mathfrak{A}:\, G\in\mathfrak{Ch}(A)} G$$

or equivalently

$$D = \bigcup_{B\in\mathcal{D}:\, \mathrm{r}(B)<\mathrm{r}(A)} B^{[1]} \supset \bigcup_{G\in\mathfrak{A}:\, G\in\mathfrak{Ch}(A)} G. \tag{5.50}$$

Take $A \in \mathfrak{A}$ and an arbitrary $G \in \mathfrak{Ch}(A)$. We have $\mathrm{r}(G) < \mathrm{r}(A)$. In the case $G \in \mathcal{D}$, that is, $G$ has not been removed during Procedures 1 and 2, $G$ is an element of the left union of (5.50), so $G \subset D$. If $G \notin \mathcal{D}$, then $G$ has been removed during the removal process. If $G$ was removed by an application of Procedure 1, then there exists a ball $B \in \mathcal{D}$ such that $G^{[2]} \cap B \neq \varnothing$ and $\mathrm{r}(G) \ll \mathrm{r}(B) \ll \mathrm{r}(\mathrm{pr}(G)) = \mathrm{r}(A)$ (see Remark 5.1). From the inequality $R \geq \mathcal{K}^3$ it follows that

$$\mu(G^{[2]}) \leq \mathcal{K}^2\mu(G) \leq \mathcal{K}^2 \cdot \frac{\mu(B)}{\mathcal{K}^2} = \mu(B).$$

Thus, we get $G \subset G^{[2]} \subset B^{[1]}$, which means that $G \subset D$. If $G$ was removed by an application of Procedure 2, then according to (5.44), we have $G \subset \bigcup_k B_k^{[1]}$ for a family of balls $B_k$ satisfying $\mathrm{r}(B_k) = \mathrm{r}(G) < \mathrm{r}(A)$, so $B_k^{[1]} \subset D$. This again implies that $G \subset D$, so we get (5.50). To prove the theorem, we need to prove that

$$|Tf(x)| \lesssim \mathcal{L}\lambda \cdot \mathcal{A}_{\mathcal{S},r} f(x) = \mathcal{L}\lambda \cdot \sum_{S\in\mathcal{S}} \langle f\rangle_{S,r} \cdot \mathbb{I}_S(x) \text{ a.e. } x \in X, \tag{5.51}$$

where $\mathcal{S} = \{S^{[3]} : S \in \mathcal{D}\}$ clearly is a union of two sparse collections. Observe that the set $E = \bigcap_{k\leq k_0} \bigcup_{G\in\mathcal{D}:\, \mathrm{r}(G)\leq k} A^{[1]}$ has zero measure, since $\mathcal{D}$ consists of a countable number of balls with bounded sum of their measures. Besides, we can fix a set $F$ of zero measure such that for each $A \in \mathcal{D}$ inequality (5.50) holds for any $x \in \left(A^{[1]} \setminus \bigcup_{G\in\mathcal{D}:\, \mathrm{r}(G)<\mathrm{r}(A)} G^{[1]}\right) \setminus F$. Hence, it is enough to prove the bound (5.51) for arbitrary $x \in A_0 \setminus (E \cup F)$. Observe that for such $x$ there exists a ball $A \in \mathcal{D}$ such that $x \in \left(A^{[1]} \setminus \bigcup_{G\in\mathcal{D}:\, \mathrm{r}(G)<\mathrm{r}(A)} G^{[1]}\right)$, so from (5.50) we conclude that

$$|Tf_A(x)| \leq |\Gamma f_A(x)| \lesssim \mathcal{L}\lambda\langle f\rangle_{A^{[3]},r}. \tag{5.52}$$

According to property (1), one can find a unique sequence of balls $A_0, A_1, A_2, \ldots$, $A_k = A$ in $\mathcal{D}$ such that $A_j = \mathrm{pr}(A_{j+1})$. According to properties (5.35)–(5.37), there exist balls $\tilde{A}_j$ and points $\xi_j$, $j = 0, 1, \ldots, k-1$ such that

$$A_{j+1}^{[3]} \subset \tilde{A}_{j+1}^{[1]} \subset A_j^{[2]}, \tag{5.53}$$

$$\Gamma f_A(\xi_j) \lesssim \mathcal{L}\lambda\langle f\rangle_{A_j^{[3]},r}, \qquad \xi_j \in \tilde{A}_{j+1}, \tag{5.54}$$

$$|T(f \cdot \mathbb{I}_{\tilde{A}_{j+1}^{[1]} \setminus A_{j+1}^{[3]}})(t)| \lesssim \mathcal{L}\lambda\langle f\rangle_{A_j^{[3]},r}, \qquad t \in A_{j+1}^{[2]}. \tag{5.55}$$

Since $x \in A^{[1]} = A_k^{[1]} \subset A_{j+1}^{[2]}$, condition (5.55) holds for $t = x$. Thus, we get

$$T(f \cdot \mathbb{I}_{\tilde{A}_{j+1}^{[1]} \setminus A_{j+1}^{[3]}})(x)| \lesssim \mathcal{L}\lambda \cdot \langle f\rangle_{A_j^{[3]}}. \tag{5.56}$$

We claim that

$$|Tf_{A_j}(x)| \le C\mathcal{L}\lambda \cdot \langle f\rangle_{A_j^{[3]},r} + |Tf_{A_{j+1}}(x)|, \tag{5.57}$$

where $C > 1$ is an admissible constant. Indeed, by the (T1) condition and (5.54), we will have

$$\begin{aligned}\left|T(f_{A_j}\cdot \mathbb{I}_{X\setminus \tilde{A}_{j+1}^{[1]}})(x) - T(f_{A_j}\cdot \mathbb{I}_{X\setminus \tilde{A}_{j+1}^{[1]}})(\xi_j)\right| &\lesssim \mathcal{L}\langle f_{A_j}\rangle^*_{\tilde{A}_{j+1},r}\\ &\le \mathcal{L}M_r f_{A_j}(\xi_j) \le \Gamma f_{A_j}(\xi_j) \lesssim \mathcal{L}\lambda\cdot\langle f\rangle_{A_j^{[3]},r}.\end{aligned} \tag{5.58}$$

Besides, from (5.56) we get

$$\left|T(f_{A_j}\cdot \mathbb{I}_{\tilde{A}_{j+1}^{[1]}\setminus A_{j+1}^{[3]}})(x)\right| = \left|T(f\cdot \mathbb{I}_{\tilde{A}_{j+1}^{[1]}\setminus A_{j+1}^{[3]}})(x)\right| \lesssim \mathcal{L}\lambda\cdot\langle f\rangle_{A_j^{[3]},r}.$$

From the definition of $f_{A_j}$ (see (5.38)) and (5.53) it follows that

$$f_{A_j}\cdot \mathbb{I}_{X\setminus \tilde{A}_{j+1}^{[1]}} = f\cdot \mathbb{I}_{A_j^{[3]}\setminus \tilde{A}_{j+1}^{[1]}} = f\cdot \mathbb{I}_{A_j^{[3]}} - f\cdot \mathbb{I}_{\tilde{A}_{j+1}^{[1]}}. \tag{5.59}$$

Thus, applying (5.32), (5.58), (5.54), (5.59), and (5.59), we conclude that

$$\begin{aligned}\left|Tf_{A_j}(x)\right| &= \left|Tf_{A_j}(x)\right|\\ &\le \left|T(f_{A_j}\cdot \mathbb{I}_{X\setminus \tilde{A}_{j+1}^{[1]}})(x)\right| + \left|T(f_{A_j}\cdot \mathbb{I}_{\tilde{A}_{j+1}^{[1]}})(x)\right|\\ &\le \left|T(f_{A_j}\cdot \mathbb{I}_{X\setminus \tilde{A}_{j+1}^{[1]}})(x) - T(f_{A_j}\cdot \mathbb{I}_{X\setminus \tilde{A}_{j+1}^{[1]}})(\xi_j)\right| + \left|T(f_{A_j}\cdot \mathbb{I}_{X\setminus \tilde{A}_{j+1}^{[1]}})(\xi_j)\right|\\ &\quad + \left|T(f_{A_j}\cdot \mathbb{I}_{\tilde{A}_{j+1}^{[1]}\setminus A_{j+1}^{[3]}})(x)\right| + \left|T(f_{A_j}\cdot \mathbb{I}_{A_{j+1}^{[3]}})(x)\right|\\ &\le C\mathcal{L}\lambda\cdot\langle f\rangle_{A_j^{[3]},r} + \left|Tf_{A_j}(\xi_j)\right| + \left|T(f\cdot \mathbb{I}_{\tilde{A}_{j+1}^{[1]}})(\xi_j)\right| + \left|Tf_{A_{j+1}}(x)\right|\\ &\le C\mathcal{L}\lambda\cdot\langle f\rangle_{A_j^{[3]},r} + 2T^{**}f_{A_j}(\xi_j) + \left|Tf_{A_{j+1}}(x)\right|\\ &\le C\mathcal{L}\lambda\cdot\langle f\rangle_{A_j^{[3]},r} + 4\Gamma f_{A_j}(\xi_j) + \left|Tf_{A_{j+1}}(x)\right|\\ &\le C\mathcal{L}\lambda\cdot\langle f\rangle_{A_j^{[3]},r} + \left|Tf_{A_{j+1}}(x)\right|,\end{aligned}$$

where $C > 0$ is an admissible constant that can vary in the above inequalities. Thus, we get (5.57). Applying (5.57) for each $j = 0, 1, 2, \dots, k-1$, (5.38), and (5.52), we get

$$\begin{aligned}|Tf(x)| = |Tf_{A_0}(x)| &\le C\mathcal{L}\lambda\cdot\sum_{j=0}^{k-1}\langle f\rangle_{A_j^{[3]},r} + |T(f\cdot \mathbb{I}_{A^{[3]}})(x)|\\ &\lesssim \mathcal{L}\cdot\sum_{j=0}^{k}\langle f\rangle_{A_j^{[3]},r} \lesssim \mathcal{L}\cdot\mathcal{A}_{\mathcal{S},r}f(x).\end{aligned}$$

completing the proof of Theorem 1.1. □

*Proof of Theorem* 1.2. Theorem 1.2 immediately follows from Theorems 1.1, 4.2, and 4.3, □

## 6. Weighted estimates in abstract measure spaces

6.1. **The general case.** Let $w$ satisfy the $A_p$ condition. We denote $q = \frac{p}{p-1}$ and let $\sigma = w^{-\frac{1}{p-1}}$ be the dual weight of $w$. Note that if $w \in A_p$, then $\sigma = w^{1/(1-p)} \in A_q$ and

$$[\sigma]_{A_q} = [w]_{A_p}^{1/(p-1)}. \tag{6.1}$$

Besides, we have

$$[w]_{A_p} = \sup_{B\in\mathfrak{B}} \frac{w(B)}{\mu(B)} \left(\frac{\sigma(B)}{\mu(B)}\right)^{p-1}.$$

The notation $dw$ in the integrals will stand for $wd\mu$, where $\mu$ is the basic measure. Any weight $w$ defines a measure on the basic measurable space $(X, \mathfrak{M})$ and the $w$-measure of a set $E$ is defined as $w(E) = \int_E dw$. Everywhere below we denote by $c_p$ different constants depending only on $1 < p < \infty$. In this section we shall consider maximal functions with respect to different measures. So we denote the maximal function associated with a measure $\beta$ by

$$M_\beta f(x) = \sup_{B\in\mathfrak{B}:\, x\in B} \frac{1}{\mu(B)} \int_B |f(t)| d\beta(t).$$

Recall that $\mathcal{A}_{\mathcal{S}}$ denotes the sparse operator corresponding to the case of $r = 1$.

**Lemma 6.1.** *Let $(X, \mathfrak{M}, \mu)$ be a measure space with a ball-basis $\mathfrak{B}$. If $\mathcal{S} \subset \mathfrak{B}$ is a $\gamma$-sparse collection $(0 < \gamma < 1)$ and the weight $w$ satisfies the $A_p$ condition for $1 < p \le 2$, then*

$$\|\mathcal{A}_{\mathcal{S}}\|_{L^p(w)\to L^p(w)} \le \gamma^{-1} [w]_{A_p}^{1/(p-1)} \|M_w\|_{L^p(w)\to L^p(w)}^{1/(p-1)} \cdot \|M_\sigma\|_{L^p(\sigma)\to L^p(\sigma)}. \tag{6.2}$$

*Proof.* Applying the Hölder inequality, for a measurable $E \in \mathfrak{M}$ we have

$$\begin{aligned} \mu(E) &= \int_E w^{1/p} \cdot w^{-1/p} d\mu \\ &\le \left(\int_E w d\mu\right)^{1/p} \cdot \left(\int_E w^{-\frac{1}{p-1}} d\mu\right)^{1/q} = (w(E))^{1/p} (\sigma(E))^{1/q}. \end{aligned} \tag{6.3}$$

Now suppose that $\mathcal{S}$ is a sparse collection of balls and that $f \in L^p(w)$ is positive. Using the inequality $(\sum a_k)^{p-1} \le \sum a_k^{p-1}$ for $1 < p < 2$, we obtain

$$\begin{aligned} \|\mathcal{A}_{\mathcal{S}} f\|_{L^p(w)}^p &= \int_X \left((\mathcal{A}_{\mathcal{S}} f)^{p-1}\right)^{\frac{p}{p-1}} w d\mu = \int_X \left(\left(\sum_{B\in\mathcal{S}} \langle f\rangle_B \cdot \mathbb{I}_B\right)^{p-1}\right)^q w d\mu \\ &\le \int_X \left(\sum_{B\in\mathcal{S}} \langle f\rangle_B^{p-1} \cdot \mathbb{I}_B\right)^q w d\mu = \left\|\sum_{B\in\mathcal{S}} \langle f\rangle_B^{p-1} \cdot \mathbb{I}_B\right\|_{L^q(w)}^q. \end{aligned} \tag{6.4}$$

There is a function $g \in L^p(w)$ with $\|g\|_{L^p(w)} = 1$ such that

$$\begin{aligned} \left\|\sum_{B\in\mathcal{S}} \langle f\rangle_B^{p-1} \cdot \mathbb{I}_B\right\|_{L^q(w)} &= \int_X \sum_{B\in\mathcal{S}} \langle f\rangle_B^{p-1} \mathbb{I}_B g w d\mu = \sum_{B\in\mathcal{S}} \left(\frac{1}{\mu(B)} \int_B f d\mu\right)^{p-1} \int_B g dw \\ &= \sum_{B\in\mathcal{S}} \left(\frac{1}{\sigma(B)} \int_B f d\mu\right)^{p-1} \frac{1}{w(B)} \int_B |g| dw \cdot \frac{w(B)}{\mu(B)} \left(\frac{\sigma(B)}{\mu(B)}\right)^{p-1} \cdot \mu(B). \end{aligned}$$

Thus, applying (6.3), the $A_p$ condition, and Hölder's inequality, we get

(6.5)

$$\left\|\sum_{B\in\mathcal{S}}\langle f\rangle_B^{p-1}\cdot\mathbb{I}_B\right\|_{L^q(w)}\le\gamma^{-1}[w]_{A_p}\sum_{B\in\mathcal{S}}\left(\frac{1}{\sigma(B)}\int_B fd\mu\right)^{p-1}\frac{1}{w(B)}\int_B|g|dw\cdot\mu(E_B)$$

$$\le\gamma^{-1}[w]_{A_p}\sum_{B\in\mathcal{S}}\left(\frac{1}{\sigma(B)}\int_B fd\mu\right)^{p-1}(\sigma(E_B))^{1/q}\frac{1}{w(B)}\int_B|g|dw\cdot(w(E_B))^{1/p}$$

$$\le\gamma^{-1}[w]_{A_p}\left(\sum_{B\in\mathcal{S}}\left(\frac{1}{\sigma(B)}\int_B f\sigma^{-1}d\sigma\right)^p\sigma(E_B)\right)^{1/q}$$

$$\left(\sum_{B\in\mathcal{S}}\left(\frac{1}{w(B)}\int_B|g|dw\right)^p w(E_B)\right)^{1/p}.$$

The last two factors can be estimated with the maximal functions $M_\sigma$ and $M_w$, respectively. Namely, for the second one we have

(6.6)

$$\left(\sum_{B\in\mathcal{S}}\left(\frac{1}{w(B)}\int_B|g|dw\right)^p\cdot w(E_B)\right)^{1/p}\le\|M_w g\|_{L^p(w)}$$

$$\le\|M_w\|_{L^p(w)\to L^p(w)}\cdot\|g\|_{L^p(w)}=\|M_w\|_{L^p(w)\to L^p(w)}.$$

Similarly, the first factor is estimated by

(6.7)

$$\left(\sum_{B\in\mathcal{S}}\left(\frac{1}{\sigma(B)}\int_B f\sigma^{-1}d\sigma\right)^p\sigma(E_B)\right)^{1/q}\le\|M_\sigma\|^{p/q}_{L^p(\sigma)\to L^p(\sigma)}\cdot\|f\sigma^{-1}\|^{p/q}_{L^p(\sigma)}$$

$$=\|M_\sigma\|^{p/q}_{L^p(\sigma)\to L^p(\sigma)}\cdot\left(\int_X f^p\sigma^{-p}\sigma d\mu\right)^{1/q}$$

$$=\|M_\sigma\|^{p/q}_{L^p(\sigma)\to L^p(\sigma)}\cdot\left(\int_X f^p dw\right)^{1/q}=\|M_\sigma\|^{p/q}_{L^p(\sigma)\to L^p(\sigma)}\cdot\|f\|^{p/q}_{L^p(w)}.$$

From (6.4), (6.5), (6.6), and (6.7) we immediately get (6.2). □

**Lemma 6.2.** *Let $(X,\mathfrak{M},\mu)$ be measure space with a ball-basis $\mathfrak{B}$, let $1<p,q<\infty$, and let $p^{-1}+q^{-1}=1$. If $\mathcal{S}$ is a sparse collection and the weight $w$ satisfies the $A_p$ condition, then*

(6.8)
$$\|\mathcal{A}_\mathcal{S}\|_{L^p(w)\to L^p(w)}=\|\mathcal{A}_\mathcal{S}\|_{L^q(\sigma)\to L^q(\sigma)},$$

*where $\sigma$ is the dual weight of $w$.*

*Proof.* We have

$$\|\mathcal{A}_\mathcal{S}\|_{L^p(w)\to L^p(w)}=\sup_{f\in L^p(w),\,g\in L^q(w)}\int_X\mathcal{A}_\mathcal{S}f\cdot gdw.$$

By the duality argument for $f \in L^p(w)$ and $g \in L^q(w)$ we get the estimate

$$\begin{aligned}
\int_X \mathcal{A}_{\mathcal{S}} f \cdot g dw &= \int_X \mathcal{A}_{\mathcal{S}} f \cdot g w d\mu = \sum_{B \in \mathcal{S}} \frac{1}{\mu(B)} \int_B f d\mu \int_B g w d\mu \\
&= \int_X \mathcal{A}_{\mathcal{S}}(gw) \cdot f d\mu = \int_X \frac{\mathcal{A}_{\mathcal{S}}(gw)}{w} \cdot f w d\mu = \int_X \frac{\mathcal{A}_{\mathcal{S}}(gw)}{w} \cdot f dw \\
&\leq \left\| \frac{\mathcal{A}_{\mathcal{S}}(gw)}{w} \right\|_{L^q(w)} \|f\|_{L^p(w)} = \left( \int_X (\mathcal{A}_{\mathcal{S}}(gw))^q w^{-q} dw \right)^{1/q} \cdot \|f\|_{L^p(w)} \\
&= \left( \int_X (\mathcal{A}_{\mathcal{S}}(gw))^q \sigma d\mu \right)^{1/q} \cdot \|f\|_{L^p(w)} \leq \|\mathcal{A}_{\mathcal{S}}\|_{L^q(\sigma)\to L^q(\sigma)} \|gw\|_{L^q(\sigma)} \|f\|_{L^p(w)} \\
&= \|\mathcal{A}_{\mathcal{S}}\|_{L^q(\sigma)\to L^q(\sigma)} \left( \int_X (gw)^q \sigma d\mu \right)^{1/q} \|f\|_{L^p(w)} \\
&= \|\mathcal{A}_{\mathcal{S}}\|_{L^q(\sigma)\to L^q(\sigma)} \|g\|_{L^q(w)} \|f\|_{L^p(w)},
\end{aligned}$$

which implies that $\|\mathcal{A}_{\mathcal{S}}\|_{L^p(w)\to L^p(w)} \leq \|\mathcal{A}_{\mathcal{S}}\|_{L^q(\sigma)\to L^q(\sigma)}$. Similarly, we have the reverse inequality and thus (6.8). □

**Lemma 6.3.** *Let $(X, \mathfrak{M}, \mu)$ be a measure space with a ball-basis $\mathfrak{B}$, and let $w$ be a weight satisfying the $A_p$ condition, $1 < p < \infty$, with respect to the measure $\mu$. Then for any balls $A, B$ with $A \subset B$ we have*

$$\frac{w(B)}{w(A)} \leq 2^p \cdot [w]_{A_p} \cdot \left( \frac{\mu(B)}{\mu(A)} \right)^p. \tag{6.9}$$

*Proof.* Denote

$$a = \frac{1}{\mu(A)} \int_A w d\mu = \frac{w(A)}{\mu(A)}. \tag{6.10}$$

By Chebyshev's inequality we find that $\mu\{t \in A : w \leq 2a\} > \frac{\mu(A)}{2}$. Thus, we get

$$\left( \int_B w^{1/(1-p)} \right)^{p-1} \geq \left( \frac{\mu(A)}{2} (2a)^{1/(1-p)} \right)^{p-1} = \frac{(\mu(A))^p}{2^p w(A)}, \tag{6.11}$$

and then by (1.1) and (6.11) we obtain

$$\begin{aligned}
[w]_{A_p} &\geq \frac{1}{\mu(B)} \int_B w \cdot \left( \frac{1}{\mu(B)} \int_B w^{1/(1-p)} \right)^{p-1} \\
&\geq \frac{w(B)}{(\mu(B))^p} \cdot \left( \int_B w^{1/(1-p)} \right)^{p-1} \geq 2^{-p} \cdot \frac{w(B)}{w(A)} \cdot \frac{(w(A))^p}{(\mu(B))^p},
\end{aligned}$$

so (6.10). □

**Lemma 6.4.** *Let $(X, \mathfrak{M}, \mu)$ be a measure space with a ball-basis $\mathfrak{B}$, and let $w$ be a weight, satisfying the $A_p$ condition, $1 < p < \infty$. Then the maximal function associated with the measure $w$ satisfies the inequalities*

$$\|M_w\|_{L^1(w)\to L^{1,\infty}(w)} \leq (2\mathcal{K})^p \cdot [w]_{A_p}, \tag{6.12}$$

$$\|M_w\|_{L^{p'}(w)\to L^{p'}(w)} \leq c(p,p') \cdot \mathcal{K}^{p/p'} [w]_{A_p}^{1/p'}, \qquad 1 < p' < \infty, \tag{6.13}$$

*where the constant $c(p, p')$ depends on $p$ and $p'$.*

*Proof.* Denote $E = \{x \in B : M_w f(x) > \lambda\}$. For any $x \in E$ there exists a ball $B(x) \subset X$ such that

$$x \in B(x), \quad \frac{1}{w(B(x))} \int_{B(x)} |f| dw > \lambda.$$

Given that ball $B \in \mathfrak{B}$, consider the collection of balls $\{B(x) : x \in E \cap B\}$. Applying Lemma 3.1, we find a sequence of pairwise disjoint balls $\{B_k\}$ taken from this collection satisfying $E \cap B \subset \bigcup_k B_k^{[1]}$. Note that $B_k^{[1]}$ as usual is defined with respect to the measure $\mu$. From Lemma 6.3 we easily get $w(B_k^{[1]}) \le (2\mathcal{K})^p \cdot [w]_{A_p} w(B_k)$, and hence

$$\begin{aligned} w\left(\bigcup_k B_k^{[1]}\right) &\le \sum_k w(B_k^{[1]}) \le (2\mathcal{K})^p \cdot [w]_{A_p} \sum_k w(B_k) \\ &\le (2\mathcal{K})^p \cdot [w]_{A_p} \frac{1}{\lambda} \sum_k \int_{B_k} |f| dw \le \frac{(2\mathcal{K})^p \cdot [w]_{A_p}}{\lambda} \int_X |f| dw. \end{aligned}$$

Similarly, as in the proof of Theorem 4.1, thus we conclude that

$$w^*(E) \le \frac{(2\mathcal{K})^p \cdot [w]_{A_p}}{\lambda} \int_X |f| dw,$$

and thus (6.12). Applying Theorem 2.1 (Marcinkiewicz interpolation theorem), we then get (6.13). □

**Theorem 6.1.** *If $\mathcal{S}$ is a $\gamma$-sparse collection and the weight $w$ satisfies the $A_p$ condition for some $1 < p < \infty$, then the corresponding sparse operator satisfies the bound*

$$\|\mathcal{A}_{\mathcal{S}} f\|_{L^p(w) \to L^p(w)} \le c(p, \mathcal{K}) \cdot \gamma^{-1} [w]_{A_p}^{\max\left\{\frac{p+2}{p(p-1)}, \frac{3p-2}{p}\right\}}. \tag{6.14}$$

*Proof.* First we suppose that $1 < p \le 2$. Applying Lemmas 6.1 and 6.4 and equation (6.1), we obtain

$$\|\mathcal{A}_{\mathcal{S}}\|_{L^p(w) \to L^p(w)} \le c(p, \mathcal{K}) \gamma^{-1} [w]_{A_p}^{1/(p-1)} \cdot [w]_{A_p}^{1/p(p-1)} \cdot [\sigma]_{A_q}^{1/p} = c(p, \mathcal{K}) \gamma^{-1} [w]_{A_p}^{\frac{p+2}{p(p-1)}},$$

and thus (6.14). If $2 < p < \infty$, then by Lemma 6.2 and (6.1) we obtain

$$\begin{aligned} \|\mathcal{A}_{\mathcal{S}} f\|_{L^p(w) \to L^p(w)} &= \|\mathcal{A}_{\mathcal{S}} f\|_{L^q(\sigma) \to L^q(\sigma)} \\ &\le c(q, \mathcal{K}) \gamma^{-1} [\sigma]_{A_q}^{\frac{q+2}{q(q-1)}} = c(p, \mathcal{K}) \gamma^{-1} [w]_{A_p}^{\frac{3p-2}{p}}. \end{aligned}$$

The theorem is proved. □

Combining Theorems 1.1 and 6.1, we obtain the following.

**Theorem 6.2.** *If $(X, \mathfrak{M}, \mu)$ is a measure space with a ball-basis $\mathfrak{B}$ and the operator $T \in \mathrm{BO}_{\mathfrak{B}}(X)$ satisfies weak-$L^1$ inequality, then*

$$\|Tf\|_{L^p(w) \to L^p(w)} \le c(p, \mathcal{K})(\mathcal{L}_1 + \mathcal{L}_2 + \|T\|_{L^1 \to L^{1,\infty}}) [w]_{A_p}^{\max\left\{\frac{p+2}{p(p-1)}, \frac{3p-2}{p}\right\}}.$$

6.2. **The case of the Besicovitch condition.**

**Definition 6.1.** Let $\mathfrak{B}$ be a family of sets of an arbitrary set $X$. We say that $\mathfrak{B}$ satisfies the Besicovitch $D$-condition with a constant $D \in \mathbb{N}$ if for any collection $\mathcal{A} \subset \mathfrak{B}$ one can find a subscollection $\mathcal{A}' \subset \mathcal{A}$ such that

$$\bigcup_{A\in\mathcal{A}} A = \bigcup_{A\in\mathcal{A}'} A, \qquad \sum_{A\in\mathcal{A}'} \mathbb{I}_A(x) \le D.$$

We say that $\mathfrak{B}$ is a martingale system if $D = 1$.

**Theorem 6.3.** *Let $(X, \mathfrak{M}, \mu)$ be a measure space, and let the collection of measurable sets $\mathfrak{B} \subset \mathfrak{M}$ satisfy the Besicovitch $D$-condition. Then the maximal operator $M_\mu$ satisfies the bounds*

$$\begin{aligned} &\|M_\mu\|_{L^1(\mu)\to L^{1,\infty}(\mu)} \le D, \\ &\|M_\mu\|_{L^p(\mu)\to L^p(\mu)} \le c_p \cdot D^{1/p}, \qquad 1 < p < \infty. \end{aligned} \tag{6.15}$$

*Proof.* Define $E = \{x \in X : M_\mu f(x) > \lambda\}$. For any $x \in E$ there exists a set $B(x) \subset \mathfrak{B}$ such that

$$\frac{1}{w(B(x))}\int_{B(x)} |f|dw > \lambda, \qquad x \in B(x). \tag{6.16}$$

According to the Besicovitch condition, there is a subcollection $\mathfrak{A} \subset \{B(x) : x \in E\}$ such that

$$\bigcup_{A\in\mathfrak{A}} A = \bigcup_{x\in E} B(x), \qquad \sum_{A\in\mathfrak{A}} \mathbb{I}_A(x) \le D.$$

Thus, we get

$$\mu^*(E) \le \sum_{A\in\mathfrak{A}} \mu(A) \le \frac{1}{\lambda}\sum_{A\in\mathfrak{A}} \int_A |f| \le \frac{D}{\lambda}\int_X |f|d\mu.$$

The second inequality immediately follows from (6.15), according to the Marcinkiewicz interpolation theorem (Theorem 2.1). □

The following theorem gives a sharp weighted estimate in general measure spaces with a ball-basis satisfying the Besicovitch condition.

**Theorem 6.4.** *Let $\mathfrak{B}$ be a family of sets in a measure space $(X, \mathfrak{M}, \mu)$ satisfying the Besicovitch $D$-condition. If $\mathcal{S}$ is a $\gamma$-sparse collection and the weight $w$ satisfies the $A_p$ condition for $1 < p < \infty$, then*

$$\|\mathcal{A}_{\mathcal{S}} f\|_{L^p(w)\to L^p(w)} \lesssim c_p \gamma^{-1} D^{\max\{1/(p-1),p-1\}} \cdot [w]_{A_p}^{\max\{1,1/(p-1)\}}. \tag{6.17}$$

*Proof.* First suppose that $1 < p \le 2$. Applying Lemma 6.1 and Theorem 6.3, we obtain

$$\|\mathcal{A}_{\mathcal{S}}\|_{L^p(w)\to L^p(w)} \le c_p\gamma^{-1}[w]_{A_p}^{1/(p-1)} \cdot D^{1/p(p-1)} \cdot D^{1/p} = c_p\gamma^{-1}[w]_{A_p}^{1/(p-1)} \cdot D^{1/(p-1)},$$

and thus (6.17). In the case $2 < p < \infty$ we use the same argument as in the proof of Theorem 6.1. □

Applying Theorems 1.1 and 6.4, we immediately get the following.

**Theorem 6.5.** *Let a family of measurable sets* $\mathfrak{B}$ *in a measure space* $(X,\mathfrak{M},\mu)$ *satisfy the Besicovitch D-condition, and let* $w$ *be a* $A_p$ *weight with* $1<p<\infty$. *Then if an operator* $T\in \mathrm{BO}_{\mathfrak{B}}(X)$ *satisfies the weak-*$L^1$ *inequality, then*

$$\|T\|_{L^p(w)\to L^p(w)}\lesssim C(\mathcal{L}_1+\mathcal{L}_2+\|T\|_{L^1\to L^{1,\infty}})\cdot [w]_{A_p}^{\max\{1,1/(p-1)\}},$$

*where* $C$ *is a constant depending on* $p$ *and the Besicovitch constant.*

## 7. Bounded oscillation operators on spaces of homogeneous type

**Definition 7.1.** A quasi metric on a set $X$ is a function $\rho : X\times X\to[0,\infty)$ satisfying the conditions

(1) $\rho(x,y)\ge 0$ for every $(x,y)\in X$ and $\rho(x,y)=0$ if and only if $x=y$;

(2) $\rho(x,y)=\rho(y,x)$ for every $x,y\in X$;

(3) $\rho(x,y)\le \mathcal{D}(\rho(x,z)+\rho(z,y))$ for every $x,y,z\in X$, where $\mathcal{D}>1$ is a fixed constant.

Define the ball of a center $x$ and a radius $r$ by

$$B(x,r)=\{y\in X:\ \rho(x,y)<r\},\quad x\in X,\quad 0<r<\infty,$$

and denote by $\mathfrak{U}(\rho)$ the family of all such balls, calling them $\rho$-balls. A "quasi metric" defines a topology for which the $\rho$-balls form a base. In general, the balls need not be open sets in this topology. For $B\in\mathfrak{U}(\rho)$ denote by $c(B)$ and $r(B)$, respectively, the center and the radius of $B$. For any $t>0$ we set $tB=B(c(B),tr(B))$. We define also an enlarged family of balls $\mathfrak{U}'(\rho)$ as follows: if $\mu(X)=\infty$, then $\mathfrak{U}'(\rho)$ coincides with $\mathfrak{U}(\rho)$. In the case $\mu(X)<\infty$ we include in $\mathfrak{U}'(\rho)$ additionally the set $X$.

**Definition 7.2.** Let $\rho$ be a quasi metric on $X$, and let $\mu$ be a positive measure defined on a $\sigma$-algebra $\mathfrak{M}$ of subsets of $X$ containing the $\rho$-open sets and the $\rho$-balls. The collection $(X,\rho,\mathfrak{M},\mu)$ is said to be a space of homogeneous type if

$$\mu(2B)\le\mathcal{H}\cdot\mu(B) \tag{7.1}$$

for any ball $B\in\mathfrak{U}(\rho)$.

Note that (7.1) implies a more general inequality. Namely,

$$\mu(a\cdot B)\le\mathcal{H}(a)\cdot\mu(B),\qquad a>0, \tag{7.2}$$

where $\mathcal{H}(a)$ is a constant depending on $a$ and $\mathcal{H}$. From property (3) of the quasi metric it easily follows that for any $B\in\mathfrak{U}(\rho)$ the inequality

$$\operatorname{diam}B=\sup_{x,y\in B}\rho(x,y)\le 2\mathcal{D}\cdot r(B)$$

holds. In this section the notation $a\lesssim b$ will stand for $a\le c\cdot b$, where $c>0$ is a constant depending on the constants $\mathcal{H}$ and $\mathcal{D}$ of a space of homogeneous type .

**Theorem 7.1.** *Let* $(X,\rho,\mathfrak{M},\mu)$ *be a space of homogeneous type such that* $\mathfrak{U}(\rho)$ *satisfies the density condition. Then the enlarged family of balls* $\mathfrak{U}'(\rho)$ *forms a ball-basis for the measure space* $(X,\mathfrak{M},\mu)$ *and satisfies the doubling condition. Besides, the hull ball of any* $B=B(x_0,r)\in\mathfrak{U}(\rho)$ *has the form* $B^*=B(x_0,R)$, $2r\le R\le\infty$.

The proof of the theorem is based on the following lemmas.

**Lemma 7.1.** *If $(X,\rho,\mathfrak{M},\mu)$ is a space of homogeneous type, then for any point $x_0\in X$ and ball $G\in\mathfrak{U}(\rho)$ we have*

$$G\subset B(x_0,2\mathcal{D}^2(\operatorname{dist}(x_0,G)+r(G))). \tag{7.3}$$

*Proof.* Fix a point $y\in G$ with $\rho(x_0,y)<2\operatorname{dist}(x_0,G)$. For arbitrary $x\in G$ we have

$$\begin{aligned}\rho(x,x_0)&\le\mathcal{D}(\rho(x,y)+\rho(y,x_0))\\&\le\mathcal{D}(2\mathcal{D}r(G)+2\operatorname{dist}(x_0,G))\le 2\mathcal{D}^2(\operatorname{dist}(x_0,G)+r(G)),\end{aligned}$$

which means that $x$ belongs to the right side of (7.3). □

**Lemma 7.2.** *If $(X,\rho,\mathfrak{M},\mu)$ is a space of homogeneous type and the balls $B\in\mathfrak{U}(\rho)$, $G_k\in\mathfrak{U}(\rho)$, $k=1,2,\dots$, satisfy the relations*

$$B\cap G_k\neq\varnothing,\qquad r(G_k)\to\infty \text{ as } k\to\infty,$$

*then $\mu(X)\lesssim\limsup_{k\to\infty}\mu(G_k)$.*

*Proof.* Without loss of generality we can suppose that $r(G_k)>r(B)$. From (7.4) it follows that $\operatorname{dist}(c(G_k),B)<r(G_k)$. Thus, applying Lemma 7.1, we get

$$B(c(B),r(G_k))\subset B(c(G_k),2\mathcal{D}^2(\operatorname{dist}(c(G_k),B)+r(G_k)))\subset B(c(G_k),4\mathcal{D}^2r(G_k)),$$

and therefore by (7.2) we obtain $\mu(B(c(B),r(G_k)))\le\mathcal{H}(4\mathcal{D}^2)\cdot\mu(G_k)$. On the other hand, since $r(G_k)\to\infty$, we have $X=\bigcup_k B(c(B),r(G_k))$. Therefore we get

$$\mu(X)=\lim_{k\to\infty}\mu(B(c(B),r(G_k)))\lesssim\limsup_{k\to\infty}\mu(G_k). \qquad \square$$

**Lemma 7.3.** *Let $(X,\rho,\mathfrak{M},\mu)$ be a space of homogeneous type. Then for any $B=B(x_0,r)\in\mathfrak{U}(\rho)$ there exists a ball $B^*=B(x_0,R)$ with $2r\le R\le\infty$ such that*

$$\mu(B^*)\lesssim\mu(B), \tag{7.4}$$

$$\bigcup_{A\in\mathfrak{U}(\rho):\,\mu(A)\le 2\mu(B),\,A\cap B\neq\varnothing}A\subset B^*. \tag{7.5}$$

*Proof.* For a given $B\in\mathfrak{U}(\rho)$ let $\mathfrak{A}$ be the family of balls $A\in\mathfrak{U}(\rho)$ satisfying $A\cap B\neq\varnothing$ and $\mu(A)\le 2\mu(B)$. First suppose that $\gamma=\sup_{A\in\mathfrak{A}}r(A)<\infty$. Applying Lemma 7.1, for an arbitrary $A\in\mathfrak{A}$, we get

$$A=B(c(A),r(A))\subset B(c(B),2\mathcal{D}^2(r(B)+r(A)))\subset B(c(B),4\mathcal{D}^2\gamma).$$

It is clear that $B^*=B(c(B),4\mathcal{D}^2\gamma)$ satisfies (7.5). Take a ball $G\in\mathfrak{A}$ such that $r(G)>\gamma/2$. Again applying Lemma 7.1, we get

$$\begin{aligned}B^*=B(c(B),4\mathcal{D}^2\gamma)&\subset B(c(G),2\mathcal{D}(r(G)+4\mathcal{D}^2\gamma))\\&\subset B(c(G),10\mathcal{D}^3\gamma)\subset B(c(G),20\mathcal{D}^3r(G))=20\mathcal{D}^3\cdot G.\end{aligned}$$

Thus, we conclude that

$$\mu(B^*)\le\mu(20\mathcal{D}^3\cdot G)\le\mathcal{H}(20\mathcal{D}^3)\mu(G)\lesssim\mu(B),$$

which is just (7.4). Now consider the case $\gamma=\infty$. There is a sequence of balls $G_k\in\mathfrak{A}$ such that $r(G_k)\to\infty$. Applying Lemma 7.2, we get $\mu(X)\lesssim\limsup_{k\to\infty}\mu(G_k)\le 2\mu(B)$. Obviously $B^*=B(x_0,\infty)=X$ satisfies (7.4) and (7.5). □

*Proof of Theorem* 7.1. We need to check conditions (B1)–(B4) of the definition of a ball-basis. Conditions (B1) and (B2) immediately follow from the axioms of quasi-metric space and (B4) follows from Lemma 7.3, and moreover for $B = B(x_0) \in \mathfrak{U}(\rho)$ the hull ball $B^*$ has the form $B(x_0, R)$. The (B3) condition follows from the density property since by Lemma 3.4 those are equivalent. In order to prove the doubling condition, take a ball $A = B(x_0, r)$ such that $A^* = B(x, R) \neq X$. Denote $R' = \sup_{r' \geq R:\, B(x_0') = B(x_0, R)} r'$. Since $B(x_0, R) \neq X$, one can check that $R' < \infty$ and $A^* = B(x_0, R) = B(x_0, R') \subsetneq B(x_0, 2R')$. Thus, defining $B = B(x_0, 2R')$, we get $A \subsetneq B$ and

$$\mu(B) = \mu(B(x_0, 2R')) \lesssim \mu(B(x_0, R')) = \mu(A^*) \lesssim \mu(A),$$

which proves the doubling condition. □

**Theorem 7.2.** *Let $(X, \rho, \mathfrak{M}, \mu)$ be a space of homogeneous type satisfying the density condition. If $\mathcal{S} \subset \mathfrak{U}(\rho)$ is a sparse collection of balls and the weight $w$ satisfies the $A_p$ condition for $1 < p < \infty$ (with respect to the family $\mathfrak{U}(\rho)$), then the corresponding sparse operator satisfies the bound*

$$\|\mathcal{A}_{\mathcal{S}} f\|_{L^p(w) \to L^p(w)} \lesssim c_p [w]_{A_p}^{\max\{1, 1/(p-1)\}}. \tag{7.6}$$

The proof of this theorem is based on the Hytönen-Kairema [10] dyadic decomposition theorem, which reduces Theorem 7.2 to its martingale version (the case of $D = 1$ in Theorem 6.4).

**Definition 7.3.** Let $(X, \mathfrak{M}, \mu)$ be a measure space. For two families of measurable sets $\mathfrak{B}$ and $\mathfrak{B}'$ we write $\mathfrak{B} \prec \mathfrak{B}'$ if for any $B \in \mathfrak{B}$ there exists a $B' \in \mathfrak{B}'$ such that $B \subset B'$, $\mu(B') \leq \gamma \mu(B)$, where $\gamma > 0$ is a constant. The minimum value of such constants $\gamma$ will be denoted by $\gamma(\mathfrak{B} \prec \mathfrak{B}')$. If the relations $\mathfrak{B} \prec \mathfrak{B}'$ and $\mathfrak{B}' \prec \mathfrak{B}$ hold simultaneously, then we write $\mathfrak{B} \sim \mathfrak{B}'$ and denote

$$\gamma(\mathfrak{B} \sim \mathfrak{B}') = \max\{\gamma(\mathfrak{B} \prec \mathfrak{B}'), \gamma(\mathfrak{B}' \prec \mathfrak{B})\}.$$

*Remark* 7.1. One can verify that if for two families of measurable sets in $(X, \mathfrak{M}, \mu)$ we have $\mathfrak{B} \sim \mathfrak{B}'$, then the $A_p$ characteristics with respect to these families are equivalent. That is,

$$0 < c_1 < \frac{\sup_{B \in \mathfrak{B}} \left(\frac{1}{|B|} \int_B w\right) \left(\frac{1}{|B|} \int_B w^{-1/(p-1)}\right)^{p-1}}{\sup_{B \in \mathfrak{B}'} \left(\frac{1}{|B|} \int_B w\right) \left(\frac{1}{|B|} \int_B w^{-1/(p-1)}\right)^{p-1}} < c_2$$

for some constants $c_1$ and $c_2$ depending on $\gamma(\mathfrak{B} \sim \mathfrak{B}')$.

**Theorem 7.3** (Hytönen and Kairema [10])**.** *If $(X, \rho, \mathfrak{M}, \mu)$ is a space of homogeneous type, then there exist martingale systems $\mathfrak{B}_k \subset \mathfrak{M}$, $k = 1, 2, \ldots, l$, such that $\mathfrak{U}(\rho) \sim \mathfrak{B} = \bigcup_{j=1}^{l} \mathfrak{B}_j$, where $l$ and $\gamma(\mathfrak{U}(\rho) \sim \mathfrak{B})$ are constants depending on $\mathcal{H}$ and $\mathcal{D}$.*

*Proof of Theorem* 7.2. Apply Theorem 7.3. For every $B \in \mathfrak{U}(\rho)$ there exists a set $Q(B) \in \mathfrak{B}$ such that

$$B \subset Q(B), \qquad \mu(Q(B)) \lesssim \mu(B). \tag{7.7}$$

We shall consider the sparse operators

$$\mathcal{A}_k f(x) = \sum_{B \in \mathcal{S}:\, Q(B) \in \mathfrak{B}_k} \langle f \rangle_{Q(B)} \mathbb{I}_{Q(B)}(x), \qquad k = 1, 2, \ldots, l.$$

From (7.7) it follows that

$$
\begin{aligned}
\mathcal{A}_{\mathcal{S}} f(x) &= \sum_{B\in\mathcal{S}} \langle f\rangle_B \mathbb{I}_B(x) \lesssim \sum_{B\in\mathcal{S}} \langle f\rangle_{Q(B)} \mathbb{I}_{Q(B)}(x) \\
&\le \sum_{k=1}^{l} \sum_{B\in\mathcal{S}:\, Q(B)\in\mathfrak{B}_k} \langle f\rangle_{Q(B)} \mathbb{I}_{Q(B)}(x) = \sum_{k=1}^{l} \mathcal{A}_k f(x).
\end{aligned}
\tag{7.8}
$$

Since each $\mathfrak{B}_k$ is a martingale system, by Theorem 6.4 (for $D = 1$) we conclude that $\|\mathcal{A}_k f\|_{L^p(\omega)} \lesssim c_p[\omega]_{A_p}^{\max\{1,1/(p-1)\}}$. Combining this and (7.8), we get (7.6). □

Let $(X, \rho, \mathfrak{M}, \mu)$ be a space of homogeneous type, and let $K(x, y) : X \times X \to \mathbb{R}$ be a measurable function. Given that ball $\mathfrak{B} \in \mathfrak{U}(\rho)$, define the function

$$
\phi_B(t) = \sup_{x,x'\in B,\, y\in X\setminus B(c(B),t)} |K(x, y) - K(x', y)| \qquad \text{if } t \ge 2r(B), \tag{7.9}
$$

$$
\phi_B(t) = \phi_B(2r(B)) \qquad \text{if } 0 \le t < 2r(B), \tag{7.10}
$$

which is clearly decreasing on $[0,\infty)$. Denote

$$
R = \sup_{B\in\mathfrak{B}} \int_X \phi_B\left(\rho(y, c(B))\right) d\mu(y), \qquad d_B = \sup_{x\in B,\, y\in X\setminus 2B} |K(x, y)|. \tag{7.11}
$$

**Definition 7.4.** An operator $T : L^1(X) \to L^0(X)$ is said to be of Calderón–Zygmund type if for any $B \in \mathfrak{U}(\rho)$, it admits the representation

$$
Tf(x) = \int_X K(x, y)f(y)d\mu(y) \qquad \text{whenever } x \in B,\ \operatorname{supp} f \subset X \setminus 2B, \tag{7.12}
$$

where the kernel $K(x, y)$ satisfies the conditions

$$
R < \infty, \quad d_B < \infty \quad \text{for any } B \in \mathfrak{U}(\rho). \tag{7.13}
$$

**Theorem 7.4.** *If $(X, \rho, \mathfrak{M}, \mu)$ is a space of homogeneous type such that $\mathfrak{U}(\rho)$ satisfies the density condition, then any Calderón–Zygmund type operator* (7.12) *is a* BO *operator with respect to the ball-basis $\mathfrak{U}'(\rho)$. Moreover, we have $\mathcal{L}_1(T) \le R$, where $R$ is* (7.11).

*Proof.* First note that Theorem 7.1 implies that $\mathfrak{U}'(\rho)$ is a ball-basis having the doubling property, and for $B = B(x_0, r) \in \mathfrak{U}(\rho)$ the hull ball $B^*$ has the form $B(x_0, R)$, $2r \le R < \infty$. Since $\mathfrak{U}'(\rho)$ satisfies the doubling condition, according to Theorem 4.4, we need to verify only the (T1) condition. Since $\phi_B(t)$ is decreasing, we can prove that

$$
\int_X \phi_B(\rho(y, c(B)))|f(y)|d\mu(y) \le R \cdot \langle f\rangle_B^*. \tag{7.14}
$$

Indeed, one can easily find a step function $\psi(t)$ on $[0,\infty)$ such that

$$
\phi_B(t) \le \psi(t) = \sum_{k=1}^{\infty} a_k \mathbb{I}_{[0,r_k]}(t), \qquad a_k > 0,\ 2r(B) = r_1 < r_2 < \cdots,
$$

$$
\int_X \psi\left(\rho(y, c(B))\right) d\mu(y) < \int_X \phi_B\left(\rho(y, c(B))\right) d\mu(y) + \delta \le R + \delta,
$$

where $\delta > 0$ can be small enough. We have

$$\begin{aligned}\int_X \phi_B(\rho(y, c(B)))|f(y)|d\mu(y) &\le \int_X \psi(\rho(y, c(B)))|f(y)|d\mu(y)\\ &= \sum_{k=1}^\infty a_k \int_{B(c(B),r_k)} |f(y)|d\mu(y) \le \langle f\rangle_B^* \sum_{k=1}^\infty a_k\mu(B(c(B), r_k))\\ &= \langle f\rangle_B^* \int_X \psi\left(\rho(y, c(B))\right) d\mu(y) \le \langle f\rangle_B^*(R + \delta).\end{aligned}$$

Since $\delta > 0$ is small enough, we get (7.14). Now take $B \in \mathfrak{U}(\rho)$, $f \in L^1(X)$, and suppose that $x, x' \in B$. From (7.9) it follows that

$$|K(x, y) - K(x', y)| \le \phi_B(\rho(y, c(B)))| \text{ whenever } y \in X \setminus 2B.$$

Thus, using (7.14) and the relation $B^* \supset 2B$, we get the bound

$$\begin{aligned}|T(f\cdot \mathbb{I}_{X\setminus B^*})(x) - T(f\cdot \mathbb{I}_{X\setminus B^*})(x')| &= \left|\int_{X\setminus B^*} (K(x, y) - K(x', y))f(y)d\mu(y)\right|\\ &\le \int_{X\setminus 2B} |K(x, y) - K(x', y)||f(y)|d\mu(y)\\ &\le \int_X \phi_B(\rho(y, c(B)))|f(y)|d\mu(y) \le R\cdot \langle f\rangle_B^*,\end{aligned}$$

which gives the (T1) condition. □

Let $(X, \rho, \mathfrak{M}, \mu)$ be a space of homogeneous type, and let $\omega : [0,\infty) \to [0,\infty)$ be an increasing function satisfying $\omega(t + s) \le \omega(t) + \omega(s)$, $\omega(0) = 0$, and the Dini condition

$$C_1 = \int_0^1 \frac{\omega(t)}{t} dt < \infty. \tag{7.15}$$

An operator $T : L^1(X) \to L^0(X)$ is said to be an $\omega$-Calderón–Zygmund operator if it has the representation (7.12) and for any ball $B \in \mathfrak{U}(\rho)$ we have

$$\sup_{x\in B,\, y\in X\setminus B(c(B),t)} |K(x, y)| \le \frac{C_2}{\mu(B(c(B), t))}, \tag{7.16}$$

$$\sup_{x,x'\in B,\, y\in X\setminus B(c(B),t)} |K(x, y) - K(x', y)| \le \frac{\omega\left(\frac{r(B)}{t}\right)}{\mu(B(c(B), t))}, \tag{7.17}$$

$$\sup_{y,y'\in B,\, x\in X\setminus B(c(B),t)} |K(x, y) - K(x, y')| \le \frac{\omega\left(\frac{r(B)}{t}\right)}{\mu(B(c(B), t))}, \tag{7.18}$$

where all of these inequalities hold for any $t > 2r(B)$.

**Theorem 7.5.** *Let $(X, \rho, \mathfrak{M}, \mu)$ be a space of homogeneous type such that $\mathfrak{U}(\rho)$ satisfies the density condition. If $T$ is an $\omega$-Calderón–Zygmund operator, then it is a* BO *operator with respect to the ball-basis $\mathfrak{U}'(\rho)$. Moreover, we have the estimates*

$$\mathcal{L}_1(T) \lesssim C_1, \qquad \mathcal{L}_2(T) \lesssim C_2. \tag{7.19}$$

*Proof.* Taking into account (7.17) and the definition of function $\phi_B$ in (7.9), (7.10), for every $B = B(x_0, r) \in \mathfrak{U}(\rho)$ we have

$$\phi_B(t) \le \omega\left(\frac{r(B)}{t}\right)\frac{1}{\mu(B(c(B),t))} \qquad \text{if } t \ge 2r(B),$$
$$\phi_B(t) \le \omega\left(\frac{1}{2}\right)\frac{1}{\mu(2B)} \qquad \text{if } 0 \le t < 2r(B).$$

Thus, applying the doubling property (7.1), (7.15), and the subadditivity of $\omega$, we get

$$\int_X \phi_B(\rho(y, c(B)))\, d\mu(y) = \int_{2B} \phi_B(\rho(y, c(B)))\, d\mu(y)$$
$$+ \sum_{k=1}^{\infty} \int_{(2^{k+1}B)\setminus(2^k B)} \phi_B(\rho(y, c(B)))\, d\mu(y)$$
$$\le \omega\left(\frac{1}{2}\right) + \sum_{k=1}^{\infty} \omega\left(2^{-k}\right) \frac{\mu(2^{k+1}B) - \mu(2^k B)}{\mu(2^k B)} \lesssim \sum_{k=1}^{\infty} \omega(2^{-k}) \lesssim C_1.$$

Since this inequality holds for any ball $B$ applying Theorem 7.4, we get the first estimate in (7.19). To estimate $\mathcal{L}_2(T)$, take a ball $A = B(x_0, r) \in \mathfrak{U}(\rho)$ with $A^* \ne X$. According to Theorem 7.1, $A^* = B(x_0, R)$, $R \ge 2r$. Denote

$$L = \sup_{r' \ge R:\, B(x_0, r') = B(x_0, R)} r',$$

and let $B = B(x_0, 2L)$. Since $B(x_0, R) \ne X$, we have $L < \infty$ and

$$A^* = B(x_0, R) = B(x_0, L) \subsetneq B(x_0, 2L) = B.$$

Thus, we get $A \subsetneq B$ and

$$\mu(B^*) \lesssim \mu(B(x_0, 2L)) \le \mathcal{H}\mu(B(x_0, L)) = \mathcal{H}\mu(A^*). \tag{7.20}$$

Since $r(A^*) \ge 2r(A)$, from (7.16) we obtain

$$\sup_{x \in A,\, y \in X \setminus A^*} |K(x, y)| \le \frac{C_2}{\mu(A^*)}.$$

Thus, also using (7.20), for any point $x \in A$ we get

$$|T(f \cdot \mathbb{I}_{B^* \setminus A^*})(x)| \le \int_{B^* \setminus A^*} |K(x, y)||f(y)|dy \le \frac{C_2}{\mu(A^*)} \int_{B^*} |f(y)|dy \lesssim C_2 \langle f \rangle_{B^*}.$$

Hence, we obtain $\mathcal{L}_2(T) \lesssim C_2$, completing the proof of the theorem. □

Combining Theorems 1.1, 7.2 and 7.5, we immediately get the following.

**Theorem 7.6.** *Let $(X, \rho, \mathfrak{M}, \mu)$ be a space of homogeneous type such that $\mathfrak{U}(\rho)$ satisfies the density condition. If $T$ is an $\omega$-Calderón–Zygmund operator and the weight $w$ satisfies the $A_p$ condition with respect to the ball-basis $\mathfrak{U}(\rho)$, $1 < p < \infty$, then we have*

$$\|T\|_{L^p(w) \to L^p(w)} \le c_p(C_1 + C_2 + \|T\|_{L^1 \to L^{1,\infty}})[w]_{A_p}^{\max\{1, 1/(p-1)\}}. \tag{7.21}$$

It is well known that any $\omega$-Calderón–Zygmund operator, which is bounded on $L^2(X)$ satisfies the bound $\|T\|_{L^1 \to L^{1,\infty}} \lesssim \|T\|_{L^2 \to L^2}$. So in (7.21) $\|T\|_{L^1 \to L^{1,\infty}}$ can be replaced by $\|T\|_{L^2 \to L^2}$. Note that the Hytönen–Roncal–Tapiola [17] inequality is the case of (7.21) for the $\omega$-Calderón–Zygmund operators on $\mathbb{R}^n$. Besides, (7.21) is a stronger version of the Anderson–Vagharshakyan [1] inequality, where the case of $\omega(t) = t^\delta$ was considered.

## 8. Other examples of BO operators

**Theorem 8.1.** *If $(X, \mathfrak{M}, \mu)$ is a measure space with a ball-basis $\mathfrak{B}$, then the maximal operator $M$ corresponding to $r = 1$ in (4.1) is the* BO *operator with respect to* $\mathfrak{B}$.

*Proof.* In order to establish the (T1) condition, we let $B$ be an arbitrary ball. Take two points $x, x' \in B$ and a nonzero function $f \in L^1(X)$ with

$$\operatorname{supp} f \in X \setminus B^{[1]}. \tag{8.1}$$

Suppose that

$$Mf(x) \ge Mf(x'). \tag{8.2}$$

We have $\langle f \rangle_B^* > 0$. Thus, by the definition of a maximal operator we get

$$Mf(x) \le \frac{1}{\mu(A)} \int_A |f| + \langle f \rangle_B^* \tag{8.3}$$

for some ball $A \ni x$. If $\mu(A) \le \mu(B)$, then by the two balls relation we have $A \subset B^{[1]}$, so by (8.1) we get

$$\frac{1}{\mu(A)} \int_A |f| = 0.$$

Therefore according to (8.2) and (8.3), we have

$$|Mf(x) - Mf(x')| = Mf(x) - Mf(x') \le \langle f \rangle_B^*.$$

If $\mu(A) > \mu(B)$, then we get $B \subset A^{[1]}$, so

$$\begin{aligned} Mf(x) - Mf(x') &\le \frac{1}{\mu(A)} \int_A |f| + \langle f \rangle_B^* \\ &\lesssim \frac{1}{\mu(A^{[1]})} \int_{A^{[1]}} |f| + \langle f \rangle_B^* \lesssim \langle f \rangle_B^*. \end{aligned}$$

This gives the (T1) condition. To prove the (T2) condition, fix a ball $B$ and set

$$\mathcal{A} = \{A \in \mathfrak{B} : A \cap B \ne \varnothing, \mu(A) > \mu(B)\}, \qquad \gamma = \inf_{A \in \mathcal{A}} \mu(A).$$

There exists a ball $A \in \mathcal{A}$ such that $\gamma \le \mu(A) < 2\gamma$. Define $B' = A^{[1]}$. One can check that

$$B \subsetneq A^{[1]} = B'. \tag{8.4}$$

On the other hand, for any function $f \in L^1(X)$ and any point $x \in B$ we have

$$M(f \cdot \mathbb{I}_{B'^{[1]} \setminus B^{[1]}})(x) = \frac{1}{\mu(C)} \int_C |f| \cdot \mathbb{I}_{B'^{[1]} \setminus B^{[1]}} + \delta \tag{8.5}$$

for some ball $C \ni x$ and a number $\delta > 0$ that can be taken as arbitrarily small. We can suppose that $\mu(C) > \mu(B)$, since otherwise we should have $C \subset B^{[1]}$, which will imply that

$$\frac{1}{\mu(C)} \int_C |f| \cdot \mathbb{I}_{B'^{[1]} \setminus B^{[1]}} = 0.$$

Hence, since we also have $C \cap B \neq \varnothing$, we get $C \in \mathcal{A}$. Thus, we obtain $\mu(C) \geq \gamma$, and therefore

$$\mu(C) > \frac{\mu(A)}{2} \geq \frac{\mu(A^{[2]})}{2\mathcal{K}^2} = \frac{\mu(B'^{[1]})}{2\mathcal{K}^2}.$$

Hence, we have

$$\frac{1}{\mu(C)} \int_C |f| \cdot \mathbb{I}_{B'^{[1]} \setminus B^{[1]}} \lesssim \frac{1}{\mu(B'^{[1]})} \int_{B'^{[1]}} |f| = \langle f \rangle_{B'^{[1]}}. \tag{8.6}$$

Combining (8.5) and (8.6), we get $M(f \cdot \mathbb{I}_{B'^{[1]} \setminus B^{[1]}})(x) \lesssim \langle f \rangle_{B^{[1]}}$, so the (T2) condition is proved. □

Thus, applying Theorem 1.1, we get the following.

**Theorem 8.2.** *Let $(X, \mathfrak{M}, \mu)$ be a measure space with a ball-basis $\mathfrak{B}$, and let $w$ be a $A_p$ weight with $1 < p < \infty$. Then the maximal function* (4.1) *satisfies the bound*

$$\|M\|_{L^p(w) \to L^p(w)} \lesssim [w]_{A_p}^{\max\left\{\frac{p+2}{p(p-1)}, \frac{3p-2}{p}\right\}}.$$

*If in addition $\mathfrak{B}$ satisfies the Besicovitch condition, then*

$$\|M\|_{L^p(w) \to L^p(w)} \lesssim [w]_{A_p}^{\max\{1, 1/(p-1)\}}.$$

*Remark* 8.1. Theorem 8.2 does not give the full weighted estimate like (1.2), which is known to be optimal for the maximal function in Euclidean spaces [3]. In the general case the optimality occurs only when $1 < p \leq 2$. The Buckley [3] argument can be applied to get the full bound (1.2) in the case of the Besicovitch condition.

Another example of a BO operator is the martingale transform. Let $(X, \mathfrak{M}, \mu)$ be a measure space, and let $\{\mathfrak{B}_n : n \in \mathbb{Z}\}$ be a collection of measurable sets such that the following apply:

(1) Each $\mathfrak{B}_n$ is a finite or countable partition of $X$.
(2) For each $n$ and $A \in \mathfrak{B}_n$ the set $A$ is the union of sets from $\mathfrak{B}_{n+1}$.
(3) The collection $\mathfrak{B} = \bigcup_{n \in \mathbb{Z}} \mathfrak{B}_n$ generates the $\sigma$-algebra $\mathfrak{M}$.
(4) For any points $x, y \in X$ there is a set $A \in X$ such that $x, y \in A$.

For a given $A \in \mathfrak{B}$ let $\mathrm{pr}(A)$ (parent of $A$) be the minimal element of $\mathfrak{B}$ satisfying $A \subsetneq \mathrm{pr}(A)$. One can easily check that $\mathfrak{B}$ satisfies the ball-basis conditions (B1)–(B4). Moreover, for $A \in \mathfrak{B}$ we can define

$$A^* = \begin{cases} A & \text{if } \mu(\mathrm{pr}(A)) > 2\mu(A), \\ \mathrm{pr}^n(A) & \text{if } \mu(\mathrm{pr}^n(A)) \leq 2\mu(A) < \mu(\mathrm{pr}^{n+1}(A)) \end{cases} \tag{8.7}$$

and take $\mathcal{K} = 2$. Consider a function $f \in L^1(X)$. The martingale difference associated with $A \in \mathfrak{B}$ is

$$\Delta_A f(x) = \sum_{B:\, \mathrm{pr}(B) = A} \left( \frac{1}{\mu(B)} \int_B f - \frac{1}{\mu(A)} \int_A f \right) \mathbb{I}_B(x).$$

The martingale transform operator is defined by

$$Tf(x) = \sum_{A \in \mathfrak{B}} \varepsilon_A \Delta_A f(x),$$

where $\varepsilon_A = \pm 1$ are fixed.

**Lemma 8.1.** *Any martingale transform $T$ satisfies the* (T1) *condition and moreover, $\mathcal{L}_1(T) = 0$.*

*Proof.* Take a function $f \in L^1(X)$ with $\operatorname{supp} f \in X \setminus A^*$ and two points $x, x' \in A$. Observe that

$$\Delta_B f(x) = \Delta_B f(x') \qquad \text{if } B \supset A^*,$$
$$\Delta_B f(x) = \Delta_B f(x') = 0 \qquad \text{if } B \subseteq A^* \text{ or } B \cap A^* = \varnothing.$$

Thus, we have $\Delta_B f(x) = \Delta_B f(x')$ for any ball $B \in \mathfrak{B}$, so $Tf(x) = Tf(x')$. This implies that $\mathcal{L}_1(T) = 0$. $\square$

**Lemma 8.2.** *If $T$ is a martingale transform, then for any ball $A \in \mathfrak{B}$ we have*

$$\sup_{x \in A,\, f \in L^1(X)} \frac{|T(f \cdot \mathbb{I}_{\mathrm{pr}(A)\setminus A})(x)|}{\langle f \rangle_{\mathrm{pr}(A)}} \le 2. \tag{8.8}$$

*Proof.* Take a function $f \in L^1(X)$ with $\operatorname{supp} f \subset \mathrm{pr}(A) \setminus A$. Consider the sequence $A_k$, $k = 1, 2, \dots$, defined by $A_0 = A$, $A_{k+1} = \mathrm{pr}(A_k)$. For every point $x \in A$ we have

$$\Delta_{A_k} f(x) = \left(\frac{1}{\mu(A_{k-1})} - \frac{1}{\mu(A_k)}\right) \int_{\mathrm{pr}(A)} f \qquad \text{if } k > 1,$$
$$\Delta_B f(x) = 0 \qquad \text{if } B \subseteq A,$$
$$\Delta_{A_1} f(x) = -\frac{1}{\mu(A_1)} \int_{\mathrm{pr}(A)} f.$$

Hence, we get

$$|Tf(x)| \le \sum_{k \ge 1} |\Delta_k f(x)| \le \frac{1}{\mu(A_1)} \int_{\mathrm{pr}(A)} |f| + \sum_{k>1} \left(\frac{1}{\mu(A_{k-1})} - \frac{1}{\mu(A_k)}\right) \int_{\mathrm{pr}(A)} |f|$$
$$\le \frac{2}{\mu(\mathrm{pr}(A))} \int_{\mathrm{pr}(A)} |f| = 2\langle f \rangle_{\mathrm{pr}(A)}.$$

The lemma is proved. $\square$

**Lemma 8.3.** *If $T$ is a martingale transform, then $T$ satisfies the* (T2) *condition and $\mathcal{L}_2(T)$ is bounded by an absolute constant.*

*Proof.* We need to prove the inequality

$$\sup_{x \in A,\, f \in L^1(X)} \frac{|T(f \cdot \mathbb{I}_{\mathrm{pr}(A)^* \setminus A^*})(x)|}{\langle f \rangle_{\mathrm{pr}(A)^*}} \le c, \tag{8.9}$$

where $c > 0$ is an absolute constant (see the definition of a (T2) condition). If $\mu(\mathrm{pr}(A)) \le 2\mu(A)$, then applying Lemmas 4.2 and 8.1, the left-hand side of (8.9) can be estimated by $c \cdot \|T\|_{L^1 \to L^{1,\infty}}$. Since $\mathcal{K} = 2$, we can say that here $c > 0$ is an absolute constant. It is well known that $\|T\|_{L^1 \to L^{1,\infty}}$ is estimated by an absolute constant too. This implies (8.9). In the case $\mu(\mathrm{pr}(A)) > 2\mu(A)$, applying $\mathcal{K} = 2$, we obtain

$$\sup_{x \in A,\, f \in L^1(X)} \frac{|T(f \cdot \mathbb{I}_{\mathrm{pr}(A)^* \setminus A^*})(x)|}{\langle f \rangle_{\mathrm{pr}(A)^*}}$$
$$\le \sup_{x \in A,\, f \in L^1(X)} \frac{|T(f \cdot \mathbb{I}_{\mathrm{pr}(A) \setminus A^*})(x)|}{\langle f \rangle_{\mathrm{pr}(A)^*}} + \sup_{x \in A,\, f \in L^1(X)} \frac{|T(f \cdot \mathbb{I}_{\mathrm{pr}(A)^* \setminus \mathrm{pr}(A)})(x)|}{\langle f \rangle_{\mathrm{pr}(A)^*}}$$
$$\le 2 \sup_{x \in A,\, f \in L^1(X)} \frac{|T(f \cdot \mathbb{I}_{\mathrm{pr}(A) \setminus A})(x)|}{\langle f \rangle_{\mathrm{pr}(A)}} + \sup_{x \in A,\, f \in L^1(X)} \frac{|T(f \cdot \mathbb{I}_{\mathrm{pr}(A)^* \setminus \mathrm{pr}(A)})(x)|}{\langle f \rangle_{\mathrm{pr}(A)^*}}.$$

The first term in the last sum is estimated by (8.8). Now let us estimate the second term. Applying the weak-$L^1$ inequality, for a $\lambda > 0$ we can write

$$\mu\{x \in \mathrm{pr}(A) : |T(f \cdot \mathbb{I}_{\mathrm{pr}(A)^* \setminus \mathrm{pr}(A)})(x)| > \lambda \langle f \rangle_{\mathrm{pr}(A)^*}\} \le \frac{\|T\|_{L^1 \to L^{1,\infty}}}{\lambda \cdot \langle f \rangle_{\mathrm{pr}(A)^*}} \int_{\mathrm{pr}(A)^*} |f|$$
$$= \frac{\|T\|_{L^1 \to L^{1,\infty}} \cdot \mu(\mathrm{pr}(A)^*)}{\lambda} \le \frac{\mu(\mathrm{pr}(A))}{2},$$

where the last inequality is obtained with a suitable absolute constant $\lambda > 0$. This implies that the inequality

$$|T(f \cdot \mathbb{I}_{\mathrm{pr}(A)^* \setminus \mathrm{pr}(A)})(x)| \le \lambda \langle f \rangle_{\mathrm{pr}(A)^*} \tag{8.10}$$

holds for some point $x \in \mathrm{pr}(A)$. Observe that the function $T(f \cdot \mathbb{I}_{\mathrm{pr}(A)^* \setminus \mathrm{pr}(A)})(x)$ is constant on $\mathrm{pr}(A)$, and it can be shown likewise for the proof of Lemma 8.1. Hence, we will have (8.10) for any $x \in \mathrm{pr}(A)$. Thus, we will give a bound on the second term by an absolute constant. □

Lemmas 8.1 and 8.3 immediately imply the following.

**Theorem 8.3.** *The martingale transform is a* BO *operator with respect to the ball-basis* $\mathfrak{B}$. *Moreover,* $\mathcal{L}_1(T) = 0$, *and* $\mathcal{L}_2(T)$ *is bounded by an absolute constant.*

Applying Theorems 1.1, 6.4 and 8.3, we deduce the following results.

**Theorem 8.4** (Lacey [21])**.** *Let* $T$ *be a martingale transform. If* $B \in \mathfrak{B}$ *and* $f \in L^1(X)$, *then there is a sparse operator* $\mathcal{A}$ *such that*

$$|Tf(x)| \le C \cdot \mathcal{A}f(x), \qquad x \in B,$$

*where* $C$ *is an absolute constant.*

**Theorem 8.5** (Thiele, Treil, and Volberg [34])**.** *If* $T$ *is a martingale transform and the weight* $w$ *satisfies the* $A_p$ *condition with respect to the ball-basis* $\mathfrak{B}$, $1 < p < \infty$, *then we have*

$$\|T\|_{L^p(w) \to L^p(w)} \le c_p [w]_{A_p}^{\max\{1, 1/(p-1)\}}.$$

## 9. Bounded oscillation of Carleson operators

Let $\{T_\alpha\}$ be a family of BO operators. In this section we prove that if the characteristic constants of operators $T_\alpha$ are uniformly bounded, then the domination operator

$$Tf(x) = \sup_\alpha |T_\alpha f(x)| \tag{9.1}$$

is also a BO operator. More precisely, we have the following.

**Theorem 9.1.** *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space with a ball-basis* $\mathfrak{B}$. *If a* BO *family of operators* $\{T_\alpha\}$ *satisfies weak-*$L^r$ *inequality, then the operator* (9.1) *satisfies the bounds*

$$\mathcal{L}_1(T) \lesssim \sup_\alpha \mathcal{L}_1(T_\alpha), \tag{9.2}$$

$$\mathcal{L}_2(T) \lesssim \sup_\alpha \mathcal{L}_1(T_\alpha) + \sup_\alpha \mathcal{L}_2(T_\alpha) + \sup_\alpha \|T_\alpha\|_{L^r \to L^{r,\infty}}. \tag{9.3}$$

*Proof.* Let $A \in \mathfrak{B}$ be an arbitrary ball. Take two points $x, x' \in A$ and a nonzero function $f \in L^r(X)$ with

$$\operatorname{supp} f \subset X \setminus A^{[1]}.$$

Suppose that

$$Tf(x) \geq Tf(x'). \tag{9.4}$$

According to the definition of $T$, for any $\delta > 0$ there exists an index $\alpha$ such that

$$Tf(x) \leq |T_\alpha f(x)| + \delta. \tag{9.5}$$

On the other hand, for the same $\alpha$ we have $Tf(x') \geq |T_\alpha f(x')|$. Thus, applying (9.4), (9.5), and the localization property of $T_\alpha$, we obtain

$$\begin{aligned} |Tf(x) - Tf(x')| &= Tf(x) - Tf(x') \\ &\leq |T_\alpha f(x)| + \delta - |T_\alpha f(x')| \\ &\leq |T_\alpha f(x) - T_\alpha f(x')| + \delta \\ &\leq \mathcal{L}_1(T_\alpha)\langle f\rangle^*_{A,r} + \delta. \end{aligned}$$

Since $\delta > 0$ can be taken as small enough, we get (9.2).

To prove (9.3), fix a ball $A$ ($A^* \neq X$) and consider the number

$$\gamma = \inf_{B \in \mathfrak{B}:\, B \supsetneq A} \mu(B).$$

Since $A \neq X$, from Lemma 3.2 it easily follows that the set of balls $B$ satisfying $B \supsetneq A$ is nonempty. So there exists a ball $B \supsetneq A$ such that

$$\gamma \leq \mu(B) < 2\gamma.$$

Take $f \in L^r(X)$ and a point $x \in A$. We have

$$T(f \cdot \mathbb{I}_{B^{[1]} \setminus A^{[1]}})(x) = |T_\alpha(f \cdot \mathbb{I}_{B^{[1]} \setminus A^{[1]}})(x)| + \delta \tag{9.6}$$

for an index $\alpha$, where $\delta > 0$ can be arbitrarily small. Since $T_\alpha$ satisfies the (T2) condition, there exists a ball $C \supsetneq A$ such that

$$T_\alpha(g \cdot \mathbb{I}_{C^{[1]} \setminus A^{[1]}})(x) \lesssim \mathcal{L}_2(T_\alpha)\langle g\rangle_{C^{[1]},r} \tag{9.7}$$

holds for any $g \in L^r(X)$. Consider the function $g = f \cdot \mathbb{I}_{B^{[1]} \setminus A^{[1]}}$. If $\mu(C) > \mu(B)$, then we have $B \subset C^{[1]}$, and therefore $B^{[1]} \subset C^{[2]}$. Thus, applying (9.7) and Lemma 4.2, we obtain

(9.8)

$$\begin{aligned} |T_\alpha g(x)| &= |T_\alpha(g \cdot \mathbb{I}_{C^{[2]} \setminus A^{[1]}})(x)| \\ &\leq |T_\alpha(g \cdot \mathbb{I}_{C^{[1]} \setminus A^{[1]}})(x)| + |T_\alpha(g \cdot \mathbb{I}_{C^{[2]} \setminus C^{[1]}})(x)| \\ &\leq \mathcal{L}_2(T_\alpha)\langle g\rangle_{C^{[1]},r} + \sup_{x \in C,\, u \in L^r(X)} \frac{|T_\alpha(u \cdot \mathbb{I}_{C^{[2]} \setminus C^{[1]}})(x)|}{\langle u\rangle_{C^{[2]},r}} \cdot \langle g\rangle_{C^{[2]},r} \\ &\lesssim \mathcal{L}_2(T_\alpha)\left(\frac{1}{\mu(C^{[1]})}\int_{B^{[1]}} |f|^r\right)^{1/r} \\ &\quad + (\mathcal{L}_1(T_\alpha) + \|T_\alpha\|_{L^r \to L^{r,\infty}})\left(\frac{\mu(C^{[1]})}{\mu(C)}\right)^{1/r}\left(\frac{1}{\mu(C^{[2]})}\int_{B^{[1]}} |f|^r\right)^{1/r} \\ &\lesssim \sup_\alpha(\mathcal{L}_2(T_\alpha) + \mathcal{L}_1(T_\alpha) + \|T_\alpha\|_{L^r \to L^{r,\infty}}) \cdot \langle f\rangle_{B^{[1]},r}. \end{aligned}$$

In the case $\mu(C) \le \mu(B)$ we have $C \subset B^{[1]}$, and since $C \supsetneq A$, we obtain

$$\mu(B) \gtrsim \mu(B^{[1]}) \ge \mu(C) \ge \gamma \ge \frac{\mu(B)}{2}.$$

Thus, again applying Lemma 4.2 and (9.7), we conclude that

$$\begin{aligned} |T_\alpha(g)(x)| &\le |T_\alpha(g \cdot \mathbb{I}_{B^{[2]}\setminus C^{[1]}})(x)| + |T_\alpha(g \cdot \mathbb{I}_{C^{[1]}\setminus A^{[1]}})(x)| \\ &\le \sup_{x\in C,\, u\in L^r(X)} \frac{|T_\alpha(u \cdot \mathbb{I}_{B^{[2]}\setminus C^{[1]}})(x)|}{\langle u\rangle_{B^{[2]},r}} \cdot \langle g\rangle_{B^{[2]},r} + \mathcal{L}_2(T_\alpha)\langle g\rangle_{C^{[1]},r} \\ &\lesssim (\mathcal{L}_1(T_\alpha)) + \|T_\alpha\|_{L^r\to L^{r,\infty}}\langle g\rangle_{B^{[2]},r} + \mathcal{L}_2(T_\alpha)\langle g\rangle_{C^{[1]},r} \\ &\lesssim \sup_\alpha(\mathcal{L}_1(T_\alpha)) + \|T_\alpha\|_{L^r\to L^{r,\infty}} + \mathcal{L}_2(T_\alpha)\cdot\langle f\rangle_{B^{[1]},r}. \end{aligned} \tag{9.9}$$

Observe that the admissible constants used in (9.8) and (9.9) are independent of $f$, point $x$, and the number $\delta$ from (9.6). Hence, since $\delta$ can be taken as arbitrarily small, from (9.6), (9.8), and (9.9) we get the inequality

$$\begin{aligned} \sup_{x\in A,\, f\in L^r(X)} & \frac{T(f\cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}})(x)}{\langle f\rangle_{B^{[1]},r}} \\ & \lesssim \mathcal{L}_1(T_\alpha) + \mathcal{L}_2(T_\alpha) + \|T_\alpha\|_{L^r\to L^{r,\infty}}, \end{aligned}$$

which implies (9.3). □

Let $T$ be a BO operator, and let $\mathcal{G} = \{g_\alpha\} \subset L^\infty(X)$ be a family of functions such that

$$\beta = \sup_\alpha \|g_\alpha\|_\infty < \infty, \qquad \|T\|_{L^r\to L^{r,\infty}} < \infty. \tag{9.10}$$

One can easily check that the operators

$$T_\alpha f(x) = T(g_\alpha \cdot f)(x) \tag{9.11}$$

are BO operators. Moreover, we have

$$\mathcal{L}_1(T_\alpha) \le \beta\mathcal{L}_1(T), \quad \mathcal{L}_2(T_\alpha) \le \beta\mathcal{L}_2(T), \quad \|T_\alpha\|_{L^r\to L^{r,\infty}} \le \beta\|T\|_{L^r\to L^{r,\infty}}. \tag{9.12}$$

Define the maximal modulation of the operator $T$ by

$$T^{\mathcal{G}} f(x) = \sup_\alpha |T_\alpha f(x)|. \tag{9.13}$$

According to Theorem 9.1 and relations (9.12), we conclude that $T^{\mathcal{G}}$ is also a BO operator. Hence, applying Theorem 1.1, we obtain the following.

**Theorem 9.2.** *If $T \in \mathrm{BO}_{\mathfrak{B}}(X)$ satisfies (9.10), then for any function $f \in L^r(X)$ and a ball $B \in \mathfrak{B}$ there exists a family of balls $\mathcal{S}$, which is a union of two sparse collections and*

$$|T^{\mathcal{G}} f(x)| \lesssim \sup_\alpha \|g_\alpha\|_\infty(\mathcal{L}_1(T) + \mathcal{L}_2(T) + \|T^{\mathcal{G}}\|_{L^r\to L^{r,\infty}})\cdot \mathcal{A}_{\mathcal{S},r} f(x),$$

*for a.e. $x \in B$.*

Weighted estimates of the maximal modulations of Calderón–Zygmund operators on $\mathbb{R}^n$ (in particular Carleson or Walsh–Carleson operators) were considered in [6,8]. Theorem 9.2 implies a pointwise sparse domination of such operators, which is the strongest version of the weighted norm domination of Carleson operators by sparse operators proved in [6].

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