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Mathematics

ON THE $\langle \rho_j,\, W_j \rangle$ GENERALIZED COMPLETELY MONOTONE FUNCTIONS

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We consider sequences $\{\rho_j\}_0^\infty$ $(\rho_0 = 1,\ \rho_j \ge 1)$, $\{\alpha_j\}_0^\infty \big(\alpha_0 = 0,$ $\alpha_j = 1 - (1/\rho_j)\big)$, $\{W_j(x)\}_0^\infty \in W$, where

$$W = \left\{ \{W_j(x)\}_0^\infty / W_0(x) \equiv 1,\ W_j(x) > 0,\ W_j'(x) \le 0,\ W_j(x) \in C^\infty[0,a] \right\},$$

$C^\infty[0,a]$ is the class of functions of infinitely differentiable. For such sequences we introduce systems of operators $\left\{A^*_{a,n} f\right\}_0^\infty$, $\left\{\tilde{A}^*_{a,n} f\right\}_0^\infty$ and functions $\{U_{a,n}(x)\}_0^\infty$, $\{\Phi_n(x,t)\}_0^\infty$. For a certain class of functions a generalization of Taylor–Maclaurin type formulae was obtained. We also introduce the concept of $\langle \rho_j,\, W_j \rangle$ generalized completely monotone functions and establish a theorem on their representation.

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Introduction. First of all we note that in author's works [1–5] and jointly with prof. M.M. Dzhrbashyan works [6–8] there were obtained generalized formulas of Taylor–Maclaurin type. In these papers there were introduced the concepts of $\langle \rho \rangle$, $\langle \rho_j \rangle$, $\langle \rho, \lambda_j \rangle$, $\langle \rho_j,\, W_j \rangle$ generalized absolutely monotone functions. The papers study their problems of representation.

The papers [**?**, 3] introduced the following systems of operators $\{A^*_n f\}_0^\infty$, $\left\{\tilde{A}^*_n f\right\}_0^\infty$ and functions $\{U_n(x)\}_0^\infty$, $\{\Phi_n(t,x)\}_0^\infty$:

$$\begin{gathered} A^*_n f(x) \equiv \prod_{j=0}^{n-1} D_j f(x),\ D_j f(x) = D^{1/\rho_j} \left\{ \frac{f(x)}{W_j(x)} \right\},\ A^*_0 f \equiv f,\ j \ge 0,\ n \ge 1, \\ \tilde{A}^*_n f(x) \equiv D^{-\alpha_n} \left\{ \frac{A^*_n f(x)}{W_n(x)} \right\},\ \ n \ge 0, \end{gathered} \tag{1}$$

where $D^{1/\rho} \varphi(x) \equiv \dfrac{d}{dx} D^{-\alpha} \varphi(x)$, $D^{-\alpha} \varphi(x) \equiv \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^x (x-t)^{\alpha-1} \varphi(t)\, dt$ $(\rho \ge 1,\ 1 - \alpha = 1/\rho)$.

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$$U_0(x) \equiv 1, \quad U_1(x) \equiv \frac{1}{\Gamma(\rho_1^{-1})} \int\limits_0^x \xi_1^{1/\rho_1 - 1} W_1(\xi_1) d\xi_1,$$

$$U_n(x) \equiv \frac{1}{\prod\limits_{j=1}^{n} \Gamma(1/\rho_j)} \int\limits_0^x W_1(\xi_1) d\xi_1 \int\limits_0^{\xi_1} (\xi_1 - \xi_2)^{1/\rho_1 - 1} W_2(\xi_2) d\xi_2 \times \cdots \times \\ \times \int\limits_0^{\xi_{n-1}} (\xi_{n-1} - \xi_n)^{1/\rho_{n-1} - 1} \xi_n^{1/\rho_n - 1} W_n(\xi_n) d\xi_n, \ n \geq 2. \tag{2}$$

$$\Phi_0(t,x) \equiv \begin{cases} 1, & 0 \leq t < x, \\ 0, & x \leq t < l, \end{cases}$$

$$\Phi_1(t,x) = \begin{cases} \dfrac{1}{\Gamma(\rho_1^{-1})} \displaystyle\int\limits_t^x (x - \xi_1)^{\frac{1}{\rho_1} - 1} W_1(\xi_1) d\xi_1, & 0 \leq t < x, \\ 0, & x \leq t < l; \end{cases}$$

$$\Phi_n(t,x) = \begin{cases} \dfrac{1}{\prod\limits_{j=1}^{n} \Gamma(\rho_j^{-1})} \displaystyle\int_t^x W_1(\xi_1) d\xi_1 \int\limits_t^{\xi_1} (\xi_1 - \xi_2)^{\frac{1}{\rho_1} - 1} W_2(\xi_2) d\xi_2 \times \cdots \times \\ \times \displaystyle\int\limits_t^{\xi_{n-1}} (\xi_{n-1} - \xi_n)^{\frac{1}{\rho_{n-1}} - 1} (\xi_n - t)^{\frac{1}{\rho_n} - 1} W_n(\xi_n) d\xi_n, \ 0 \leq t < x, \\ 0, \quad x \leq t < l, \quad n \geq 2. \end{cases} \tag{3}$$

We assume $\rho_j \geq 1$ $(\rho_0 = 1)$, $\alpha_j = 1 - 1/\rho_j$ $(\alpha_0 = 0)$, $W_j(x) \in W$ (see the annotation).

In the works [3, 4] it was obtained a certain class of functions that is a generalization of the Taylor–Maclaurin type formula. The papers also introduce the concept of $\langle \rho_j,\ W_j \rangle$ generallized absolutely monotone functions and study their problems of representations. We note that for $W_j(x) \equiv 1,\ j = 0, 1, \ldots,$ these systems of operators $\{A_n^* f\}_0^\infty$, $\{\tilde{A}_n^* f\}_0^\infty$ and functions $\left\{ \dfrac{x^{\lambda_n}}{\Gamma(1 + \lambda_n)} \right\}_0^\infty$ $\left(\lambda_n = \sum\limits_{j=1}^{n} \dfrac{1}{\rho_j} \right)$ were introduced in [1].

In [1] it was introduced the concept of $\langle \rho_j \rangle$ generalized absolutely monotone functions and studied their representation problems.

For $\rho_j \equiv 1 (j \geq 0)$, $\left\{ W_j(x)_0^\infty \in W \right\}$ these operators were introduced in [9].

For $\rho_j \equiv 1 (j \geq 0)$, $W_j(x) = X^{\gamma_j - \gamma_{j-1} - 1}$ these operators were introduced in [10].

In the present paper we introduce the systems of operators $\left\{ A_{a,n}^* f \right\}_0^\infty$, $\left\{ \tilde{A}_{a,n}^* f \right\}_0^\infty$, and functions $\{U_{a,n}(x)\}_0^\infty$, $\{\Phi_n(x,t)\}_0^\infty$.

In this paper we obtain a generalization of the Taylor–Maclaurin type formula, then we introduce the concept of $\langle \rho_j,\ W_j \rangle$ generalized completely monotone functions and study their representation problems. We note that for $W_j \equiv 1$, $\rho_j \geq 1$,

$j = 0, 1, \dots,$ the concept of $\langle \rho_j \rangle$ generalized completely monotone functions was introduced in [5].

Preliminaries Information. Let $f(x) \in L(0,l)$ $(0 < l < +\infty)$, $\alpha \in (0, +\infty)$. The function

$$ {}_0D^{-\alpha} f(x) \equiv D^{-\alpha} f(x) \equiv \frac{1}{\Gamma(\alpha)} \int\limits_0^x (x-t)^{\alpha-1} f(t) dt $$

is called the Riemann–Liouville integral of order α of function $f(x)$ with a lower integration limit $x = 0$, and the function

$$ D_l^{-\alpha} f(x) \equiv \frac{1}{\Gamma(\alpha)} \int\limits_x^l (t-x)^{\alpha-1} f(t) dt $$

is called the Riemann–Liouville integral of order α of function $f(x)$ with an upper integration limit $x = l$.

Let $\alpha \in [0,1), 1-\alpha = 1/\rho$ $(\rho \geq 1)$, $f(x) \in L(0,l)$. Then the function $D^{1/\rho} f(x) \equiv \frac{d}{dx} D^{-\alpha} f(x)$ is called the Riemann–Liouville derivative of order $1/\rho$ of $f(x)$ with the initial point $x = 0$, and the function $D_l^{1/\rho} f(x) \equiv \frac{d}{dx} D_l^{-\alpha} f(x)$ is called the Riemann–Liouville derivative of order $1/\rho$ of $f(x)$ with the upper limit $x = l$.

It is known that in all Lebesque points of $f(x)$ $\lim\limits_{\alpha \to +0} D^{-\alpha} f(x) = f(x)$ (and hence almost everywhere) and, therefore, $[D^{-\alpha} f(x)]_{\alpha=0} = f(x)$ and $D^1 f(x) = f'(x)$. The operators $D^0 f(x) = f(x)$, $D^1 f(x) = f'(x)$, $D^{1/\rho} f, \dots, D^{n/\rho} f = D^{1/\rho} D^{n-1/\rho} f$, $n \geq 2$ $\left(D_l^{1/\rho} f, \dots, D_l^{n/\rho} f = D_l^{1/\rho} D_l^{n-1/\rho} f \right)$ are called Riemann–Liouville operators of successive differentiation of order n/ρ of function $f(x)$. For more information on Riemann–Liouville operators see Chapt. IX, [11].

The Mittag–Leffler type function $E_\rho(z,\mu) = \sum\limits_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu + n\rho^{-1})}, \rho > 0$, is an entire function of order ρ with an arbitrary value of parameter μ (see Chapt. VI, §1, [11]).

For any $\mu > 0$, $\alpha > 0$ the following formula holds

$$ \frac{1}{\Gamma(\alpha)} \int\limits_0^z (z-\xi)^{\alpha-1} E_\rho(\lambda \xi^{1/\rho};\, \mu) \xi^{\mu-1} d\xi = z^{\mu+\alpha-1} E_\rho(\lambda z^{1/\rho};\, \mu+\alpha), \qquad (4) $$

where λ is a complex parameter and the integration is taken along the intercept connecting the points 0 and z (see Chapt. III, (1.16), [11]).

Formula of Taylor–Maclaurin Type. Let the sequences $\{\rho_j\}_0^\infty$ $(\rho_0 = 1)$, $\{\alpha_j\}_0^\infty$ $(\alpha_0 = 0)$, $\left\{W_j(x)\right\}_0^\infty$ satisfy the conditions $\rho_j \geq 1, \alpha_j = 1 - 1/\rho_j$, $\left\{W_j(x)\right\}_0^\infty \in W, j = 0, 1, \dots,$ where $W = \left\{ \left\{W_j(x)\right\}_0^\infty / W_0(x) \equiv 1,\ W_j(x) > 0,\ W_j'(x) \leq 0,\ W_j(x) \in C^\infty[0,a] \right\}$.

We introduce the following systems of operators and functions: $\left\{A^*_{a,n}f(x)\right\}_0^\infty$, $\left\{\tilde{A}^*_{a,n}f(x)\right\}_0^\infty$, $\{U_{a,n}(x)\}_0^\infty$, $\{\Phi_n(x,t)\}_0^\infty$:

$$A^*_{a,n}f(x)=\prod_{j=0}^{n-1}D_{a,j}f(x)\ (n\geq 1),\ D_{a,j}f(x)=D_a^{1/\rho_j}\left\{\frac{f(x)}{W_j(x)}\right\}$$
$$(A^*_{a,0}f\equiv f,\ A^*_{a,1}f\equiv f'(x),\ \ j\geq 1), \tag{5}$$
$$\tilde{A}^*_{a,n}f(x)=D_a^{-\alpha_n}\left\{\frac{A^*_{a,n}f(x)}{W_n(x)}\right\},\ \ n\geq 0,\ \ x\in(0,a],$$

$$U_{a,0}(x)\equiv 1,\quad U_{a,1}(x)\equiv\frac{1}{\Gamma(\rho_1^{-1})}\int_x^a(a-\xi_1)^{1/\rho_1-1}W_1(\xi_1)d\xi_1,$$

$$U_{a,n}(x)\equiv\frac{1}{\prod\limits_{j=1}^{n}\Gamma(\rho_j^{-1})}\int_x^a W_1(\xi_1)d\xi_1\int_{\xi_1}^a(\xi_2-\xi_1)^{1/\rho_1-1}W_2(\xi_2)d\xi_2\times\cdots\times \tag{6}$$
$$\times\int_{\xi_{n-1}}^a(\xi_n-\xi_{n-1})^{1/\rho_{n-1}-1}(a-\xi_n)^{1/\rho_n-1}W_n(\xi_n)d\xi_n,\ x\in(0,a],\ \ n\geq 2.$$

$$\Phi_0(x,t)=\begin{cases}0, & 0\leq t\leq x,\\ 1, & x<t\leq a,\end{cases}\qquad \Phi_1(x,t)=\begin{cases}0, \quad 0\leq t\leq x,\\ \dfrac{1}{\Gamma(\rho_1^{-1})}\displaystyle\int_x^t(t-\xi_1)^{\frac{1}{\rho_1}-1}W_1(\xi_1)d\xi_1,\end{cases}$$

$$\Phi_n(t,x)=\begin{cases}0, \qquad 0\leq t\leq x,\\ \dfrac{1}{\prod\limits_{j=1}^{n}\Gamma(\rho_j^{-1})}\displaystyle\int_x^t W_1(\xi_1)d\xi_1\int_{\xi_1}^t(\xi_2-\xi_1)^{\frac{1}{\rho_1}-1}W_2(\xi_2)d\xi_2\times\cdots\times\\ \times\displaystyle\int_{\xi_{n-1}}^t(\xi_n-\xi_{n-1})^{\frac{1}{\rho_{n-1}}-1}(t-\xi_n)^{\frac{1}{\rho_n}-1}W_n(\xi_n)d\xi_n,\ x<t\leq a,\ n\geq 2.\end{cases} \tag{7}$$

We note that similar operators and functions were introduced in [**?**, 3, 4] for $W_j(x)\equiv 1,\ \rho_j\geq 1,\ j=0,1,\ldots,$ in [1].

Lemma 1. *Let $\varphi(x)\in L(0,a)$. Then the problem of Cauchy type*

$$A^*_{a,n+1}y(x)=\varphi(x),\quad D_a^{-\alpha_j}\left\{\frac{A^*_{a,j}y(x)}{W_j(x)}\right\}\Big|_{x=a}=0,\ j=0,1,\ldots,n, \tag{8}$$

has a unique solution $Y(x)$, which can be expressed in the form

$$Y(x)=(-1)^{n+1}\int_x^a\Phi_n(x,t)\varphi(t)dt. \tag{9}$$

We do not give the proof of Lemma 1 for not loading of work.

Lemma 2. *Let $\rho_j\geq 1$ $(\rho_0=1,\ \ \alpha_j=1-1/\rho_j\ \ (\alpha_0=1),\ \ j\geq 1,$ $\{W_j(x)\}_0^\infty\in W$. Then for any $n\geq 1$ the following relations holds:*

1. $A^*_{a,k}\left\{U_{a,n}(x)\right\}=\tilde{A}^*_{a,k}\left\{U_{a,n}(x)\right\}\equiv 0,\ k\geq n+1,\ x\in(0,a];$ (10)

2. $\tilde{A}^*_{a,n}\{U_{a,n}(x)\} = (-1)^n,$ (11)

3. $\tilde{A}^*_{a,k}\{U_{a,n}(x)\}\Big|_{x=a} = 0,\ 0 \le k \le n-1.$ (12)

Using the definition of operators and functions, we get the proof of Lemma with an easy calculation.

L e m m a 3. *For any $n \ge 0$ in a sum of the form*

$$P_n(x) = \sum_{k=0}^{n} C_k U_{a,k}(x), \tag{13}$$

the coefficients $\{C_k\}_0^n$ may be determined for the formulas

$$C_k = (-1)^k \widetilde{A}^*_{a,k}\{P_n(x)\}\big|_{x=a},\ \ 0 \le k \le n. \tag{14}$$

P r o o f . Assuming $0 \le j \le n$, we apply the operator $\tilde{A}^*_{a,j}$ to the function $P_n(x)$. Then using (10)–(12), we obtain

$$\tilde{A}^*_{a,j}\{P_n(x)\} = \sum_{k=0}^{n} C_k \tilde{A}^*_{a,j}\{U_{a,k}(x)\} = C_j \tilde{A}^*_{a,j}\{U_{a,j}(x)\} + \sum_{k=j+1}^{n} C_k \tilde{A}^*_{a,j}\{U_{a,k}(x)\}. \tag{15}$$

From (15) we get

$$\tilde{A}^*_{a,j}\{P_n(x)\}\big|_{x=a} = (-1)^j C_j, \ \text{ i.e. } \ C_j = (-1)^j \tilde{A}^*_{a,j}\{P_n(x)\}\big|_{x=a}.$$

We denote by $C_{n+1}\{(0,a), \langle \rho_j,\, W_j \rangle\}$ the set of functions $f(x)$ satisfying the following conditions:

1) the functions $\tilde{A}^*_{a,k} f(x),\ \ k = 0,1,\dots,n$, are continuous on $[0,\ a]$;

2) the functions $A^*_{a,k} f(x),\ \ k = 0,1,\dots,n,n+1$, are continuous on $(0,\ a)$ and belongs to $L(0,\ a)$.

It is easy to see that each function $U_{a,n}(x),\ n = 0,1,\dots,$ and each polinom $P_n(x) = \sum\limits_{k=0}^{n} C_k U_{a,k}(x)$ belongs to the class $C_{n+1}\{(0,a), \langle \rho_j,\, W_j \rangle\}$.

T h e o r e m 1. *If $f(x) \in C_{n+1}\{(0,a), \langle \rho_j,\, W_j \rangle\}$, then for any $n \ge 1$*

$$f(x) = \sum_{k=0}^{n} (-1)^k \tilde{A}^*_{a,k} f(a) U_{a,k}(x) + R_n(x), \tag{16}$$

where

$$R_n(x) = (-1)^{n+1} \int_x^a \Phi_n(x,t) A^*_{a,n+1} f(t)\, dt. \tag{17}$$

P r o o f . We put

$$P_n(x,f) = \sum_{k=0}^{n} (-1)^k \tilde{A}^*_{a,k} f(a) U_{a,k}(x) \text{ and } f(x) = P_n(x,f) + R_n(x).$$

It is easy to see that

$$\tilde{A}^*_{a,k}\{R_n(x)\}\big|_{x=a} = 0,\ \ k = 0,1,\dots,n, \text{ and } \ A^*_{a,n+1}\{R_n(x)\} = A^*_{a,n+1} f(x).$$

We notice that the function $R_n(x)$ satisfies the conditions of Lemma 1, consequently

$$R_n(x) = (-1)^{n+1}\int_x^a \Phi_n(x,t)A^*_{a,n+1}f(t)dt, \quad \text{i.e.}$$

$$f(x) = \sum_{k=0}^{n}(-1)^k\tilde{A}^*_{a,k}f(a)U_{a,k}(x) + (-1)^{n+1}\int_x^a \Phi_n(x,t)A^*_{a,n+1}f(t)dt. \qquad \square$$

$\langle\rho_j,\ W_j\rangle$ **Generalized Completely Monotone Functions.** We denote by $C_\infty\{(0,a),\langle\rho_j,\ W_j\rangle\}$ the set of functions $f(x)\in C_{n+1}\{(0,a),\langle\rho_j,\ W_j\rangle\}$ for any $n\geq 0$. We say that $f(x)$ is $\langle\rho_j,\ W_j\rangle$ generalized completely monotone, if

$$\begin{aligned}&1.\ f(x)\in\{C_\infty(0,a),\langle\rho_j,\ W_j\rangle\};\\ &2.\ (-1)^nA^*_{a,n}f(x)\geq 0,\ \ n\geq 0,\ \ x\in(0,a].\end{aligned} \tag{18}$$

We denote by $\{C^*_\infty(0,a),\langle\rho_j,\ W_j\rangle\}$ the class of $\langle\rho_j,\ W_j\rangle$ generalized completely monotone functions. We note that in [5] in the case of $W_j(x)\equiv 1$, $\rho_j\geq 1$ $(\rho_j=1)$, $j\geq 1$, it was introduced the concept $\langle\rho_j\rangle$ generalized completely monotone functions and studied their problems of representation. Note that in the case of $\rho_j=1, W_j(x)=x^{\gamma_j-\gamma_{j-1}-1}$ $\gamma_0=0<\gamma_1\leq\gamma_2<\ldots,\ j=1,2,\ldots,$ in [12] it was introduced the concept of regular monotone functions and studied their problems of representation.

Theorem 2. *Let* $f(x)\in C^*_\infty\{(0,a),\langle\rho_j,\ W_j\rangle\}$ *and*

$$\lim_{n\to\infty}\prod_{j=1}^{n}\frac{W_j(\vartheta x_0)}{W_j(a)}\left(\frac{a-\vartheta x_0}{a-x_0}\right)^{\lambda_n}=0, \tag{19}$$

where $\forall x_0\in(0,a),\ \ x_0<\vartheta x_0<a\ \ \left(1<\vartheta<\dfrac{a}{x_0}\right),\ \ \lambda_n=\sum_{j=1}^{n}\dfrac{1}{\rho_j}$. *Then*

$$f(x)=\sum_{k=0}^{\infty}(-1)^k\tilde{A}^*_{a,k}f(a)U_{a,k}(x))dx. \tag{20}$$

Proof. Notice that from $(16),(17)$ we have

$$f(x)=\sum_{k=0}^{n}(-1)^k\tilde{A}^*_{a,k}f(a)U_a,k(x)+R_n(x),$$

where

$$R_n(x)=(-1)^{n+1}\int_x^a\Phi_n(x,t)A^*_{a,n+1}f(t)dt.$$

First of all we note that $(-1)^k\tilde{A}^*_{a,k}f(x)\geq 0,\ \ k\geq 0$, since

$$(-1)^k\tilde{A}^*_{a,k}f(x)=(-1)^kD_a^{-\alpha_k}\left\{\frac{A_{a,k}f(x)}{W_k(x)}\right\}\geq 0,\ \ k\geq 0.$$

Notice that

$$\begin{aligned}
R_n(\vartheta x_0) &= \int_{\vartheta x_0}^{a} \Phi_n(\vartheta x_0,t)\left\{(-1)^{n+1}A^*_{a,n+1}f(t)\right\}dt = \\
&= \int_{\vartheta x_0}^{a} \frac{\Phi_n(\vartheta x_0,t)}{\Phi_n(x_0,t)}\Phi_n(x_0,t)\left\{(-1)^{n+1}A^*_{a,n+1}f(t)\right\}dt \le \\
&\le \max_{\vartheta x_0\le t\le a}\left\{\frac{\Phi_n(\vartheta x_0,t)}{\Phi_n(x_0,t)}\right\}\int_{\vartheta x_0}^{a}\Phi_n(x_0,t)\left\{(-1)^{n+1}A^*_{a,n+1}f(t)\right\}dt \le \\
&\le \max_{\vartheta x_0\le t\le a}\left\{\frac{\Phi_n(\vartheta x_0,t)}{\Phi_n(x_0,t)}\right\}\int_{x_0}^{a}\Phi_n(x_0,t)\left\{(-1)^{n+1}A^*_{a,n+1}f(t)\right\}dt = \\
&= \max_{\vartheta x_0\le t\le a}\left\{\frac{\Phi_n(\vartheta x_0,t)}{\Phi_n(x_0,t)}\right\}R_n(x_0) \le \max_{\vartheta x_0\le t\le a}\left\{\frac{\Phi_n(\vartheta x_0,t)}{\Phi_n(x_0,t)}\right\}f(x_0).
\end{aligned} \tag{21}$$

It is easy to see that

$$\begin{aligned}
\Phi_n(\vartheta x_0,t) &= \frac{1}{\prod_{j=1}^{n}\Gamma(\rho_j^{-1})}\int_{\vartheta x_0}^{t} W_1(\xi_1)d\xi_1\int_{\xi_1}^{t}(\xi_2-\xi_1)^{1/\rho_1-1}W_2(\xi_2)d\xi_2\times\cdots\times \\
&\times\int_{\xi_{n-1}}^{t}(\xi_n-\xi_{n-1})^{1/\rho_{n-1}-1}(t-\xi_n)^{1/\rho_n-1}W_n(\xi_n)d\xi_n < \\
&< \frac{1}{\prod_{j=1}^{n}\Gamma(\rho_j^{-1})}\prod_{j=1}^{n}W_j(\vartheta x_0)\int_{\vartheta x_0}^{t}d\xi_1\int_{\xi_1}^{t}(\xi_2-\xi_1)^{1/\rho_1-1}d\xi_2\times\cdots\times \\
&\times\int_{\xi_{n-1}}^{t}(\xi_n-\xi_{n-1})^{1/\rho_{n-1}-1}(t-\xi_n)^{1/\rho_n-1}d\xi_n.
\end{aligned} \tag{22}$$

It is obvious that

$$\begin{aligned}
&\frac{1}{\prod_{j=1}^{n}\Gamma(\rho_j^{-1})}\int_{\vartheta x_0}^{t}d\xi_1\int_{\xi_1}^{t}(\xi_2-\xi_1)^{1/\rho_1-1}d\xi_2\times\cdots\times \\
&\times\int_{\xi_{n-1}}^{t}(\xi_n-\xi_{n-1})^{1/\rho_{n-1}-1}(t-\xi_n)^{1/\rho_n-1}d\xi_n = \frac{(t-\vartheta x_0)^{\lambda_n}}{\Gamma(1+\lambda_n)},\ \ n\ge 2.
\end{aligned} \tag{23}$$

From (22) and (23) we get

$$\Phi_n(\vartheta x_0,\,t) \le \prod_{j=1}^{n}W_j(\vartheta x_0)\frac{(t-\vartheta x_0)^{\lambda_n}}{\Gamma(1+\lambda_n)},\ \ n\ge 1,\ \ \vartheta x_0\le t\le a. \tag{24}$$

Further

$$\Phi_n(x_0,t) \geq \frac{1}{\prod\limits_{j=1}^{n}\Gamma(\rho_j^{-1})}\prod_{j=1}^{n}W_j(t)\int\limits_{x_0}^{t}d\xi_1\int\limits_{\xi_1}^{t}(\xi_2-\xi_1)^{1/\rho_1-1}d\xi_2\times\cdots\times$$

$$\times\int\limits_{\xi_{n-1}}^{t}(\xi_n-\xi_{n-1})^{1/\rho_{n-1}-1}(t-\xi_n)^{1/\rho_n-1}d\xi_n = \prod_{j=1}^{n}W_j(t)\frac{(t-x_0)^{\lambda_n}}{\Gamma(1+\lambda_n)}, \tag{25}$$

$$\vartheta x_0 \leq t \leq a, \ \ n \geq 1.$$

From (24) and (25) we get

$$\max_{\vartheta x_0\leq t\leq a}\left\{\frac{\Phi_n\left(\vartheta x_0,t\right)}{\Phi_n\left(x_0,t\right)}\right\} \leq \max_{\vartheta x_0\leq t\leq a}\prod_{j=1}^{n}\frac{W_j(\vartheta x_0)}{W_j(t)}\left(\frac{t-\vartheta x_0}{t-x_0}\right)^{\lambda_n}. \tag{26}$$

From (21) and (26) we obtain

$$R_n(\vartheta x_0) \leq \prod_{j=1}^{n}\frac{W_j(\vartheta x_0)}{W_j(a)}\left(\frac{a-\vartheta x_0}{a-x_0}\right)^{\lambda_n} f(x_0) \longrightarrow 0, \text{ as } n\longrightarrow\infty, \tag{27}$$

consequently $\lim\limits_{n\longrightarrow\infty} R_n(\vartheta x_0) = 0$. Since $x > \vartheta x_0, \ \ R_n(x) < R_n(\vartheta x_0)$, we have $\lim\limits_{n\longrightarrow\infty} R_n(x) = 0, \quad \forall x \in [\vartheta x_0, \ a]$.

So $\ f(x) = \sum\limits_{k=0}^{\infty}(-1)^k\tilde{A}^*_{a,k}f(a)U_{a,k}(x), \ \ x \in (0, \ a]$. □

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Բ. Հ. ՍԱՀԱԿՅԱՆ

$\langle \rho_j, W_j \rangle$ ԸՆԴՀԱՆՐԱՑՎԱԾ ԼԻՈՎԻՆ ՄՈՆՈՏՈՆ ՖՈՒՆԿՑԻԱՆԵՐԻ ՄԱՍԻՆ

Ներկա աշխատանքում տրված $\{\rho_j\}_0^\infty$ $(\rho_0 = 1,\ \rho_j \geq 1)$, $\{\alpha_j\}_0^\infty$ $\left(\alpha_0 = 0,\ \alpha_j = 1 - (1/\rho_j)\right)$, $\{W_j(x)\}_0^\infty \in W$ հաջորդականությունների հետ, որտեղ

$$W = \left\{\left\{W_j(x)\right\}_0^\infty / W_0(x) \equiv 1,\ W_j(x) > 0,\ W_j'(x) \leq 0,\ W_j(x) \in C^\infty[0,a]\right\},$$

ասոցացվում է $\left\{A_{a,n}^* f\right\}_0^\infty$, $\left\{\tilde{A}_{a,n}^* f\right\}_0^\infty$ օպերատորների և $\{U_{a,n}(x)\}_0^\infty$, $\{\Phi_n(x,t)\}_0^\infty$ ֆունկցիաների համակարգեր: Աշխատանքում որոշակի դասի ֆունկցիաների համար ստացվել է Թեյլոր–Մակլորենի տիպի ընդհանրացված բանաձև, մտցվել է $\langle \rho_j, W_j \rangle$ ընդհանրացված լիովին մոնոտոն ֆունկցիայի գաղափարը և ուսումնասիրվել է նրանց ներկայացման հարցերը:

Б. А. СААКЯН

ОБ ОБОБЩЕННОЙ ВПОЛНЕ МОНОТОННОЙ ФУНКЦИИ $\langle \rho_j, W_j \rangle$

В настоящей работе с последовательностями $\{\rho_j\}_0^\infty$ $(\rho_0 = 1,\ \rho_j \geq 1)$, $\{\alpha_j\}_0^\infty$ $\left(\alpha_0 = 0,\ \alpha_j = 1 - (1/\rho_j)\right)$, $\{W_j(x)\}_0^\infty \in W$, где

$$W = \left\{\left\{W_j(x)\right\}_0^\infty / W_0(x) \equiv 1,\ W_j(x) > 0,\ W_j'(x) \leq 0,\ W_j(x) \in C^\infty[0,a]\right\},$$

будут ассоциироваться системы операторов $\left\{A_{a,n}^* f\right\}_0^\infty$, $\left\{\tilde{A}_{a,n}^* f\right\}_0^\infty$ с системами функций $\{U_{a,n}(x)\}_0^\infty$, $\{\Phi_n(x,t)\}_0^\infty$. В работе для функций определенного класса получена обобщенная формула типа Тейлора–Маклорена, введено понятие обобщенной вполне монотонной функции $\langle \rho_j, W_j \rangle$ и исследуются вопросы их представления.