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Algebra

Interassociativity Via Hyperidentities

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Abstract—In this paper the concept of interassociativity via hyperidentities of associativity is extended and describe the set of semigroups which are $\{i, j\}$-interassociative to the free semigroup and the free commutative semigroup, where $i, j = 1, 2, 3$.

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1. INTRODUCTION

It is known (see [1, 2] that if in a q-algebra or in an e-algebra the nontrivial associative hyperidentity is satisfied, then it can be one of the following forms:

$$X(Y(x, y), z) = Y(x, X(y, z)), \qquad (ass)_1$$

$$X(Y(x, y), z) = X(x, Y(y, z)), \qquad (ass)_2$$

$$X(X(x, y), z) = Y(x, Y(y, z)), \qquad (ass)_3$$

where X, Y are functional variables and x, y, z are objective variables.

The concept of interassociativity was first introduced by Zupnik in [3], then this concept has been extended in the papers [4—10]. As a result, the following definition for semigroups was obtained.

Definition 1.1. *A semigroup $Q(\circ)$ is said to be interassociative to a semigroup $Q(\cdot)$ if the following identities are fulfilled:*

$$x \cdot (y \circ z) = (x \cdot y) \circ z, \qquad (1.1)$$
$$x \circ (y \cdot z) = (x \circ y) \cdot z. \qquad (1.2)$$

If, in addition, the following identity is also satisfied:

$$x \circ (y \cdot z) = (x \cdot y) \circ z,$$

then the semigroup $Q(\circ)$ is called strongly interassociative to the semigroup $Q(\cdot)$. Here the following identity is also satisfied:

$$x \cdot (y \circ z) = (x \cdot y) \circ z = x \circ (y \cdot z) = (x \circ y) \cdot z.$$

We now give more general definition.

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Definition 1.2. *Given numbers $i, j = 1, 2, 3$. A semigroup $Q(\circ)$ is said to be $\{i, j\}$-interassociative to a semigroup $Q(\cdot)$ if in the algebra $Q(\circ, \cdot)$ with two binary operations the associative hyperidentities $(ass)_i$ and $(ass)_j$ are satisfied. If $i = j$, then we simply call it $\{i\}$-interassociative. The set of all semigroups that are $\{i, j\}$-interassociative to the semigroup $Q(\cdot)$ is denoted by $Int_{\{i,j\}}Q(\cdot)$. For $i = j$, we use the notation $Int_{\{i\}}Q(\cdot)$.*

Taking $i = j = 1$ and $i = 1$, $j = 2$ in Definition 1.2, we obtain the concepts of interassociativity and strong interassociativity, respectively.

Let X be an arbitrary non-empty set. The free semigroup and the free commutative semigroup over the alphabet X we denote by $\mathcal{F}(X)(\cdot)$ and $\mathcal{FC}(X)(\cdot)$, respectively. The free semigroup with identity we denote by $\mathcal{F}^1(X)(\cdot)$. For the semigroup $Q(\cdot)$ and for its fixed element $x \in Q$ the following binary operation can be defined:

$$a *_x b = a \cdot x \cdot b, \qquad \text{for any } a, b \in Q.$$

As a result, we obtain the semigroup $Q(*_x)$, which is called a variety of the semigroup $Q(\cdot)$ (see [1, 11]).

The concepts of variety and $\{i, j\}$-interassociativity are closely related.

In [12, 13], A. Gorbatkov has proved the following theorems.

Theorem 1.1. *For $|X| \geqslant 4$ the following equalities hold:*

$$Int_{\{1\}}\mathcal{FC}(X)(\cdot) = Int_{\{1,2\}}\mathcal{FC}(X)(\cdot) = \{\mathcal{FC}(X)(*_x) \mid x \in \mathcal{FC}(X)\} \cup \{\mathcal{FC}(X)(\cdot)\}.$$

Theorem 1.2. *The semigroup $\mathcal{F}(X)(\circ)$ is $\{1\}$-interassociative to $\mathcal{F}(X)(\cdot)$ if and only if*

$$u \circ w = u_\ell(u^{(1)} \circ w^{(0)})w_r, \qquad \text{for all } u, w \in \mathcal{F}(X),$$

where $u^{(1)}$ is the last letter of the word u, $w^{(0)}$ is the first letter of the word w, while u_ℓ and w_r are words obtained from the words u and w, by cancelation of the letters $u^{(1)}$ and $w^{(0)}$, respectively.

In [14] was considered $\{3\}$-interassociativity, and was obtained the following result.

Theorem 1.3. *If $Q(\circ) \in Int_{\{3\}}Q(\cdot)$ and the following quasiidentity is satisfied:*

$$x \cdot x = y \cdot y \Rightarrow x = y,$$

then the semigroups $Q(\circ)$ and $Q(\cdot)$ coincide.

2. THE MAIN RESULTS

In this section, for an arbitrary set X we obtain descriptions of the sets $Int_{\{2\}}\mathcal{F}(X)(\cdot)$, $Int_{\{1,2\}}\mathcal{F}(X)(\cdot)$, $Int_{\{3\}}\mathcal{F}(X)(\cdot)$ and $Int_{\{3\}}\mathcal{FC}(X)(\cdot)$, and the description of the set $Int_{\{2\}}\mathcal{FC}(X)(\cdot)$ for $|X| \geqslant 4$.

We first prove two lemmas.

Lemma 2.1. *Let $Q(\circ) \in Int_{\{2\}}Q(\cdot)$. Assume that for some $a \in Q$ the mapping $\chi_a : Q \to Q$, $\chi_a(x) = ax$ is injective. Then $Q(\circ) \in Int_{\{1\}}Q(\cdot)$.*

Proof. We have the following identities:

$$(x \circ y)z = x(y \circ z), \tag{2.1}$$

$$(xy) \circ z = x \circ (yz). \tag{2.2}$$

Hence, from the following chain of equalities:

$$x((yz) \circ t) \overset{(2.1)}{=} (x \circ (yz))t \overset{(2.2)}{=} ((xy) \circ z)t \overset{(2.1)}{=} xy(z \circ t) = x(y(z \circ t))$$

and conditions of the lemma, we obtain

$$(yz) \circ t = y(z \circ t),$$

that is, the identity (1.1). Next, combining the identities (1.1), (2.1) and (2.2) we get the identity (1.2). Finally, the identities (1.1) and (1.2), together with semigroup associative identities, yield the hyperidentity $(ass)_1$. Lemma 2.1 is proved.

If in the definition of the mapping χ_a, each element we first multiply by a and then impose the corresponding conditions, then the corresponding lemma can be proved similarly.

Lemma 2.2. *Assume that $Q(\circ) \in Int_{\{2\}}Q(\cdot)$, then in the algebra $Q(\circ, \cdot)$ the following identity is fulfilled:*

$$(a \circ b)cd = ab(c \circ d).$$

Proof. Indeed, we have

$$(a \circ b)cd \overset{(2.1)}{=} a(b \circ c)d \overset{(2.1)}{=} ab(c \circ d).$$

Lemma 2.2 is proved.

Theorem 2.1. *For $|X| \geqslant 3$ the following equalities hold:*

$$Int_{\{2\}}\mathcal{F}(X)(\cdot) = Int_{\{1,2\}}\mathcal{F}(X)(\cdot) = \{\mathcal{F}(X)(\cdot)\}.$$

Proof. The first equality immediately follows from Lemma 2.1. Next, we take $a, b \in X$, $a \neq b$, and use Lemma 2.2 and the identity

$$(a \circ b)ab = ab(a \circ b) \tag{2.3}$$

to obtain

$$a \circ b = ab\,\varphi(a, b), \qquad \varphi : X \times X \to \mathcal{F}^1(X),$$
$$a \circ b = \psi(a, b)\,ab, \qquad \psi : X \times X \to \mathcal{F}^1(X).$$

Substituting the last equality into (2.3), we obtain

$$\varphi(a, b) = \psi(a, b), \qquad a \neq b,\ a, b \in X.$$

Now we take $c, d \in X$, $c \neq d$, and apply Lemma 2.2 to get

$$(a \circ b)cd = ab(c \circ d),$$
$$ab\,\varphi(a, b)\,cd = ab\,\varphi(c, d)\,cd \ \Rightarrow\ \varphi(a, b) = \varphi(c, d),$$

where φ is a constant mapping if its arguments do not coincide. Next, let

$$\varphi(a, b) = x \in \mathcal{F}^1(X) \qquad a, b \in X,\ a \neq b,$$
$$a \circ b = abx = xab,$$

and let $a, b, c \in X$, $a \neq b$, $b \neq c$. Then using (2.1) we get

$$(a \circ b)c = a(b \circ c) \ \Rightarrow\ abxc = abcx \ \Rightarrow\ xc = cx.$$

Using the induction on the length of the word x, we easily deduce that $x = c^n$ for $n \in \mathbb{N}$. If $c = a$, then $x = a^m$, $m \in \mathbb{N}$, and if $c = d$, where $d \neq a$ and $d \neq b$, then we obtain $x = d^k$, $k \in \mathbb{N}$. Therefore, $x = \emptyset$. Also, we have $x = \emptyset \Rightarrow a \circ b = ab$, where $a \neq b$.

If $a \neq c$, then from (2.1) we get

$$(a \circ a)c = a(a \circ c) = aac \ \Rightarrow\ a \circ a = aa,$$

and hence the operations $\circ$ and $\cdot$ coincide on the set $X \times X$. Finally, applying Theorem 1.2, we conclude that these operations coincide also on the set $\mathcal{F}(X) \times \mathcal{F}(X)$. Theorem 2.1 is proved.

It is not difficult to see that $\mathcal{FC}(X)(\cdot)$ also satisfies the conditions of Lemma 2.1, and hence, in view of Theorem 1.1, we obtain the following result.

Theorem 2.2. *For $|X| \geqslant 4$ the following equalities hold:*

$$Int_{\{2\}}\mathcal{FC}(X)(\cdot) = \{\mathcal{FC}(X)(*_x) \,|\, x \in \mathcal{FC}(X)\} \cup \{\mathcal{FC}(X)(\cdot)\}.$$

In [13], B. Gorbatkov has described the set $Int_{\{1\}}\mathcal{F}(X)$ in the case where $|X| = 2$. Using the method applied in [13], by direct verification we conclude that the result of Theorem 2.1 remains true also in the case where $|X| = 2$.

Theorem 2.3. *For $|X| = 2$ the following equalities hold:*

$$Int_{\{2\}}\mathcal{F}(X)(\cdot) = Int_{\{1,2\}}\mathcal{F}(X)(\cdot) = \{\mathcal{F}(X)(\cdot)\}.$$

Theorem 2.4. *Let $|X| = 1$ and $X = \{a\}$. Then the following equalities hold:*

$$Int_{\{1\}}\mathcal{F}(X)(\cdot) = Int_{\{2\}}\mathcal{F}(X)(\cdot) = Int_{\{1,2\}}\mathcal{F}(X)(\cdot)$$

$$= \{\mathcal{F}(X)(*_x) \,|\, x \in \mathcal{FC}(X)\} \cup \{\mathcal{F}(X)(\cdot)\} \cup \{\mathcal{F}(X)(\Delta)\},$$

where $a^m \Delta a^n = a^{m+n-1}$, $m, n \in \mathbb{N}$.

Proof. The first and second equalities follow from Lemma 2.1 and the commutativity of the operation. Next, assume that $\mathcal{F}(X)(\circ) \in Int_{\{1\}}\mathcal{F}(X)(\cdot)$. Then for $m, n > 1$ we have

$$a^m \circ a^n = (a^{m-1}a) \circ a^n \overset{(1.1)}{=} a^{m-1}(a \circ a^n) = a^{m-1}(a \circ (aa^{n-1})) \overset{(1.2)}{=} a^{m-1}(a \circ a)a^{n-1}.$$

Indeed, if either $m = 1$ or $n = 1$, then this equality is obvious. We consider the following cases:

(i) $a \circ a = a \Rightarrow a^m \circ a^n = a^{m+n-1} = a^m \Delta a^n$, that is, we obtain the semigroup $\mathcal{F}(X)(\Delta)$;

(ii) $a \circ a = aa$. In this case the operations $\circ$ and $\cdot$ coincide;

(iii) $a \circ a = a^k$, $k > 2$, then $a^m \circ a^n = a^m a^{k-2} a^n$.

Denoting $a^{k-2} = x$, we conclude that the semigroups $\mathcal{F}(X)(\circ)$ and $\mathcal{F}(X)(*_x)$ coincide. Theorem 2.4 is proved.

In conclusion, observe that the semigroups $\mathcal{F}(X)(\cdot)$ and $\mathcal{FC}(X)(\cdot)$ satisfy the requirement of Theorem 1.3, and hence, the following equalities hold:

$$Int_{\{3\}}\mathcal{F}(X)(\cdot) = \{\mathcal{F}(X)(\cdot)\}, \quad Int_{\{3\}}\mathcal{FC}(X)(\cdot) = \{\mathcal{FC}(X)(\cdot)\}.$$

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