ON DIVERGENCE OF FOURIER–WALSH SERIES
OF CONTINUOUS FUNCTION

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We prove that for any perfect set $P$ of positive measure, for which 0 is a density point, one can construct a function $f(x)$ continuous on $[0,1)$ such that each measurable and bounded function, which coincides with $f(x)$ on the set $P$ has diverging Fourier–Walsh series at 0.

MSC2010: 42C10; 42B08.

Keywords: Fourier–Walsh series, continuous function, divergence.

Introduction. Almost everywhere convergence and divergence problems of Fourier series in different classical orthonormal systems is one of the basic fields in Harmonic analysis. The following theorem was proved by Menshov [1]:

Theorem. For any perfect set $P \subset [-\pi, \pi]$ of positive measure, and for any density point $x_0$ of $P$ one can define a continuous function $f(x)$ on $[-\pi, \pi]$, having the following property: any bounded measurable function $g(x)$, defined on $[-\pi, \pi]$ coinciding with $f(x)$ on $P$, has Fourier series diverging at $x_0$ with respect to the trigonometric system.

In this paper we prove the following theorem.

Theorem 1. For any perfect set $P \subset [0,1)$ of positive measure, for which 0 is a density point, one can define a continuous function $f(x)$ on $[0,1)$ with the following property: any bounded measurable function $g(x)$, defined on $[0,1)$ coinciding with $f(x)$ on $P$, has Fourier series diverging at 0 with respect to the Walsh system.

Definition of Walsh System. The Walsh system $\Phi = \{\phi(x)\}_{n=1}^\infty$ is defined as follows (see [2]):

$$\phi_0(x) = 1, \quad \phi_n(x) = \prod_{s=1}^{k} r_{m_s}(x) \quad \text{for} \quad n = \sum_{s=1}^{k} 2^{m_s}, \quad 0 \leq m_1 < m_2 < \ldots < m_s,$$

where $\{r_k(x)\}_{k=0}^\infty$ is the Rademacher system:

$$r_0(x) = \begin{cases} 1, & x \in [0,1/2), \\ -1, & x \in [1/2,1); \end{cases}$$

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Note that the Walsh system is a basis in $L^p[0,1)$, $1 < p < \infty$.

**Auxiliary Propositions.** Let $D_n(x)$ be the Dirichlet kernel of the Walsh system and $S_n(x, f)$ be the partial sum of Fourier–Walsh series of a function $f(x)$, i.e.

$$D_n(x) = \sum_{k=0}^{n-1} \phi_k(x), \quad S_n(x, f) = \sum_{k=0}^{n-1} c_k \phi_k(x),$$

where $c_k = \int_0^1 f(t) \phi_k(t) dt$, $k = 0, 1, \ldots$

It is known \[2\] that

$$|D_n(x)| < \frac{1}{x}, \quad x \in (0, 1), \quad n = 1, 2, \ldots, \quad (1)$$

$$S_n(x, f) = \int_0^1 f(t \oplus x) D_n(t) dt, \quad (2)$$

where $\oplus$ is the dyadic addition, and

$$\int_{2^{-k}}^{1} |D_{n_k}(t)| dt \geq \frac{k}{4}, \quad k = 1, 2, \ldots, \quad (3)$$

where

$$n_{2s} = \sum_{i=0}^{s-1} 2^{2i+1}, \quad n_{2s-1} = \sum_{i=0}^{s-1} 2^{2i}, \quad s = 1, 2, \ldots \quad (3*)$$

Let $P \subset [0,1)$ be a perfect set of positive measure and $E = [0,1) \setminus P$. Let 0 be a density point of the set $P \subset [0,1)$, i.e.

$$\exists \lim_{h \to 0} \frac{|E \cap (-h, h)|}{2h} = 0.$$  

We will also use the following lemma, which is a direct consequence of Lemma C from [1].

**Lemma.** There exists a positive function $\sigma(\alpha)$, $\alpha \in (0,1)$ with $\lim_{\alpha \to 0} \sigma(\alpha) = 0$ such that

$$0 \leq \int_{E \cap [a_m, \alpha]} \frac{dt}{t} \leq \sigma(\alpha) \int_{0}^{\alpha} \frac{dt}{t}, \quad m = 0, 1, \ldots,$$

where $a_m = \frac{\alpha}{2^m}$

**Proof of Main Result.** We choose a sequence of natural numbers $\{k_m\}_{m=0}^\infty$ such that

$$k_0 = 1, \quad k_m > m^2 k_{m-1}, \quad m = 1, 2, \ldots, \quad (4)$$

$$\sigma\left(\frac{1}{2^{k_m}}\right) < \frac{1}{(m+1)^2}, \quad m = 1, 2, \ldots \quad (5)$$

Denote

$$\Delta_m = \left[\frac{1}{2^{k_m}}, \frac{1}{2^{k_m+1}}\right) (k_1 = 0), \quad \delta_i^m = \left[\frac{i}{2^{k_m}}, \frac{i+1}{2^{k_m}}\right), \quad \gamma_{i}^{0,m} = \left[\frac{i}{2^{k_m}}, \frac{i}{2^{k_m}} + l_m\right),$$

$$\gamma_{i}^{1,m} = \left[\frac{i+1}{2^{k_m}} - l_m, \frac{i+1}{2^{k_m}}\right), \quad i = 0, \ldots, 2^{k_m} - 1, \quad m = 0, 1, \ldots, \quad (6)$$

where $l_m = 1/(2^{2k_m+2})$. 

Constant on each $\delta_m$, $m = 1, 2, \ldots$.

From (3) obviously follows, that for all $m \geq 0$ the function $D_{n_m}(x)$ is constant on each $\delta_m \subset \Delta_m$.

Then

$$[0, 1) = \delta^0 \cup \Delta_m \cup \left[2^{-k_m}, 1\right), \quad \Delta_m = \bigcup_{i=1}^{2^{k_m}-k_m-1} \delta^i, \quad m = 1, 2, \ldots$$

(7)

We define functions $f_0(x)$ and $f(x)$ as follows,

$$f_0(x) = \begin{cases} \frac{1}{m} \text{sign}\, D_{n_m}(x), & \text{if } x \in \Delta_m, m = 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

(8)

$$f(x) = \begin{cases} f_0 \left(\frac{i}{2^m}\right), & \text{if } x \in \delta^i \setminus (\delta^0 \cup \delta^i) \subset \Delta_m, \\ \frac{1}{m} (x - \frac{i}{2^m}) f_0 \left(\frac{i}{2^m}\right), & \text{if } x \in \delta^i, \subset \Delta_m, \\ \frac{1}{m} (x - \frac{i + 1}{2^m}) f_0 \left(\frac{i}{2^m}\right), & \text{if } x \in \delta^i, \subset \Delta_m, \\ 0, & \text{if } x = 0, \end{cases}$$

(9)

where $m = 0, 1, \ldots$

From (6)–(9) it is easy to notice that $f(x)$ is continuous on $[0, 1)$.

Let $g(x)$ be an arbitrary measurable and bounded function defined on $[0, 1)$ and coinciding with $f(x)$ on $P$. Then let $m$ be a natural number. We put

$$I_m^{(1)} = \int_{\delta^0} g(t) D_{n_m}(t) dt, \quad I_m^{(2)} = \int_{\Delta_m} g(t) D_{n_m}(t) dt, \quad I_m^{(3)} = \int_{[2^{-k_m}, 1)} g(t) D_{n_m}(t) dt.$$  

(10)

From (2), (7) and (10) we get

$$S_{n_m}(0, g) = \int_0^1 g(t) D_{n_m}(t) dt = I_m^{(1)} + I_m^{(2)} + I_m^{(3)}.$$  

(11)

It follows from (6) and (10) that

$$|I_m^{(1)}| \leq C \int_{\delta^0} |D_{n_m}(t)| dt = C \frac{n_m}{2^{k_m}} \leq C, \quad C = \sup_{x \in [0, 1]} g(x).$$  

(12)

From (11) and (10) we obtain

$$|I_m^{(3)}| \leq C \int_{[2^{-k_m}, 1)} |D_{n_m}(t)| dt \leq C \int_{[2^{-k_m}, 1)} \frac{1}{t} dt = C k_m - \ln 2.$$  

(13)

Then

$$I_m^{(2)} = \int_{P \cap \Delta_m} g(t) D_{n_m}(t) dt + \int_{E \cap \Delta_m} g(t) D_{n_m}(t) dt.$$

Obviously

$$I_m^{(2)} = B_m^{(1)} + B_m^{(2)},$$  

(14)

where

$$B_m^{(1)} = \int_{\Delta_m} f(t) D_{n_m}(t) dt, \quad B_m^{(2)} = \int_{E \cap \Delta_m} |g(t) - f(t)| D_{n_m}(t) dt.$$  

(15)
From (1), (5), (15) and Lemma we conclude
\[ |B_m^{(2)}| \leq 2C \sigma (2^{-k_m-1}) \int_{\Delta_m} t \, dt \leq \frac{2C}{m^2} (k_m - k_{m-1}) \ln 2. \] (16)

From (8) and (15) we have
\[ B_m^{(1)} = \frac{1}{m} \int_{\Delta_m} |D_{n_m}(t)| \, dt - \int_{\Delta_m} [f_0(t) - f(t)] D_{n_m}(t) \, dt. \] (17)

From (1), (3) and (6)–(9) we obtain
\[ \int_{\Delta_m} |f_0(t) - f(t)| D_{n_m}(t) \, dt < 1, \quad \int_{\Delta_m} |D_{n_m}(t)| \, dt \geq \frac{k_m}{4} - k_{m-1} \ln 2. \] (18)

From (11)–(14) and (16)–(18) we get
\[ S_{n_m}(0,g) > \frac{1}{m} \left( \frac{k_m}{4} - k_{m-1} \ln 2 \right) - \frac{2C}{m^2} (k_m - k_{m-1}) \ln 2 - Ck_{m-1} \ln 2 - C - 1. \] (19)

From (4) and (19) it follows that
\[ S_{n_m}(0,g) \to \infty \quad \text{when} \quad m \to \infty, \]
which completes the proof of the Theorem.

Received 13.04.2015

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