

Mathematics

S-UNIVERSALITY IN 2-CATEGORIES

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For 2-categories the notion of S -universality is introduced and investigated.

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“The bold step of really formulating the general notion of a universal arrow was taken by Samuel [1] in 1948; the general notion was then lavishly popularized by Bourbaki [2]”. In order to interpret Grothendieck's extension of the fundamental theorem of Galois theory [3] in abstract categories more general notions than the universality is required [4]. Here this notion of S -universality for 2-categories is introduced and investigated. Firstly, we give the definitions of 2-categories, 2-functors and 2-transformations in a form, which is convenient for our goals. For comparison see [5–7].

1. A 2-category \mathcal{C} consists of a date base, satisfying some system of axioms.

The date base for 2-categories is defined as follows:

(i) an arbitrary set of objects $\mathcal{C}_0 = \text{Ob } \mathcal{C}$;

(ii) sets $\mathcal{C}(\mathcal{X}, \mathcal{Y})_0 = \text{Mor}_{\mathcal{C}}(\mathcal{X}, \mathcal{Y})$ of morphisms $\mathcal{X} \xrightarrow{F} \mathcal{Y}$ from \mathcal{X} to \mathcal{Y} for all $\mathcal{X}, \mathcal{Y} \in \text{Ob } \mathcal{C}$;

(iii) sets $\mathcal{C}(F, G) = \text{Trans}(F, G)$ of transformations (or morphisms of rank 2) $F \xrightarrow{\xi} G$ for all $F, G \in \mathcal{C}(\mathcal{X}, \mathcal{Y})_0$ and for all $\mathcal{X}, \mathcal{Y} \in \text{Ob } \mathcal{C}$;

(cm) a composite map for morphisms $F \in \mathcal{C}(\mathcal{X}, \mathcal{Y})_0$ and $K \in \mathcal{C}(\mathcal{Y}, \mathcal{X})_0$ with $F \circ K \in \mathcal{C}(\mathcal{X}, \mathcal{X})_0$, where $\mathcal{X}, \mathcal{Y}, \mathcal{X} \in \text{Ob } \mathcal{C}$;

(ct) a composite map for transformations $\xi \in \mathcal{C}(F, G)$ and $\eta \in \mathcal{C}(G, H)$ with $\xi \circ \eta \in \mathcal{C}(F, H)$, where F, G, H run $\mathcal{C}(\mathcal{X}, \mathcal{Y})_0$ and \mathcal{X}, \mathcal{Y} run $\text{Ob } \mathcal{C}$;

(mt) a multiplication map for transformations $\xi \in \mathcal{C}(F, G)$ and $\rho \in \mathcal{C}(K, L)$ with $\xi \cdot \rho \in \mathcal{C}(F \circ K, G \circ L)$, where F, G run $\mathcal{C}(\mathcal{X}, \mathcal{Y})_0$, K, L run $\mathcal{C}(\mathcal{Y}, \mathcal{X})_0$ and $\mathcal{X}, \mathcal{Y}, \mathcal{X} \in \text{Ob } \mathcal{C}$;

(im) a system of identity morphisms $e(\mathcal{X}) = 1_{\mathcal{X}} \in \mathcal{C}(\mathcal{X}, \mathcal{Y})_0$ for any $\mathcal{X} \in \text{Ob } \mathcal{C}$;

(it) for any $F \in \mathcal{C}(\mathcal{X}, \mathcal{Y})_0$ and any $\mathcal{X}, \mathcal{Y} \in \text{Ob } \mathcal{C}$ a system of identity transformations $1_F \in \mathcal{C}(F, G)$.

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This data base of the 2-category must satisfy the following system of axioms:

- 1) $F \circ (K \circ R) = (F \circ K) \circ R$; 2) $1_x \circ F = F = F \circ 1_y$; 3) $\xi \circ (\eta \circ \zeta) = (\xi \circ \eta) \circ \zeta$;
 4) $1_F \circ \xi = \xi = \xi \circ 1_G$; 5) $\xi \circ (\rho \circ \alpha) = (\xi \circ \rho) \circ \alpha$; 6) $1_{e(\mathcal{X})} \circ \xi = \xi \circ 1_{e(\mathcal{Y})}$;
 7) $(\xi \circ \eta) \circ (\rho \circ \sigma) = (\xi \circ \rho) \circ (\eta \circ \sigma)$; 8) $1_F \circ 1_K = 1_{F \circ K}$,
 for transformations $\xi \in \mathcal{C}(F, G)$, $\eta \in \mathcal{C}(G, H)$, $\zeta \in \mathcal{C}(H, A)$, $\rho \in \mathcal{C}(K, L)$,
 $\sigma \in \mathcal{C}(L, M)$, $\alpha \in \mathcal{C}(R, S)$, for morphisms $F, G, H, A \in \mathcal{C}(\mathcal{X}, \mathcal{Y})_0$,
 $K, L, M \in \mathcal{C}(\mathcal{Y}, \mathcal{X})_0$, $R, S \in \mathcal{C}(\mathcal{X}, \mathcal{N})_0$ and objects $\mathcal{X}, \mathcal{Y}, \mathcal{X}, \mathcal{N} \in \text{Ob } \mathcal{C}$.

The system of all morphisms of the 2-category \mathcal{C} is denoted by $\text{Mor}_{\mathcal{C}}$ or \mathcal{C}_M , the system of all transformations of \mathcal{C} is denoted by $\text{Trans}_{\mathcal{C}}$ or \mathcal{C}_T . If \mathcal{C} is the 2-category, then for any objects \mathcal{X}, \mathcal{Y} of \mathcal{C} the sets $\mathcal{C}(\mathcal{X}, \mathcal{Y})_0$ and $\mathcal{C}(F, G)$ for all $F, G \in \mathcal{C}(\mathcal{X}, \mathcal{Y})_0$ constitute a usual category (or 1-category), which is denoted as $\mathcal{C}(\mathcal{X}, \mathcal{Y})$. A well-known example of a 2-category is the 2-category **Cat** of categories, covariant functors and natural transformations.

2. Let \mathcal{C} and \mathcal{D} be arbitrary 2-categories. A 2-functor Φ from \mathcal{C} to \mathcal{D} is given by

- (i) a map $\Phi_0: \mathcal{C}_0 \rightarrow \mathcal{D}_0$;
 (ii) maps $\Phi_{\mathcal{X}\mathcal{Y}}: \mathcal{C}(\mathcal{X}, \mathcal{Y})_0 \rightarrow \mathcal{D}(\mathcal{X}\Phi_0, \mathcal{Y}\Phi_0)_0$ for all $\mathcal{X}, \mathcal{Y} \in \text{Ob } \mathcal{C}$;
 (iii) maps $\Phi_{FG}: \mathcal{C}(F, G) \rightarrow \mathcal{D}(F\Phi_{\mathcal{X}\mathcal{Y}}, G\Phi_{\mathcal{X}\mathcal{Y}})$ for all $F, G \in \mathcal{C}(\mathcal{X}, \mathcal{Y})_0$ and $\mathcal{X}, \mathcal{Y} \in \text{Ob } \mathcal{C}$,

which satisfy the following conditions:

- 1) $(F \circ K) \Phi_{\mathcal{X}\mathcal{Y}} = F \Phi_{\mathcal{X}\mathcal{Y}} \circ K \Phi_{\mathcal{Y}\mathcal{X}}$; 2) $e(\mathcal{X}) \Phi_{\mathcal{X}\mathcal{X}} = e(\mathcal{X}\Phi_0)$;
 3) $(\xi \circ \eta) \Phi_{FH} = (\xi \Phi_{FG}) \circ (\eta \Phi_{GH})$; 4) $e(F) \Phi_{FF} = e(F \Phi_{\mathcal{X}\mathcal{Y}})$;
 5) $(\xi \circ \rho) \Phi_{(F \circ K)(G \circ L)} = \xi \Phi_{FG} \circ \rho \Phi_{KL}$.

Remark. The condition of naturality in Gray [8] follows from the conditions 2) and 5). Further the subscripts of the 2-functor are often omitted.

3. A natural 2-transformation J of a 2-functor $\mathcal{C} \Phi \mathcal{D}$ into a 2-functor $\mathcal{C} \Psi \mathcal{D}$ is a map $J_0: \text{Ob } \mathcal{C} \rightarrow \text{Mor } \mathcal{D}$ such that

- (a0) $J_0(\mathcal{X})$ is a morphism $(\mathcal{X}\Phi) J_{\mathcal{X}}(\mathcal{X}\Psi)$ for any $\mathcal{X} \in \text{Ob } \mathcal{C}$;
 (a1) $F\Phi \circ J_{\mathcal{Y}} = J_{\mathcal{X}} \circ F\Psi$ for any $\mathcal{X} F \mathcal{Y} \in \text{Mor } \mathcal{C}$;
 (a2) $\xi \Phi \circ e(J_{\mathcal{Y}}) = e(J_{\mathcal{X}}) \circ \xi \Psi$ for any $(F \xi G) \in \text{Trans } \mathcal{C}$, F and $G \in \mathcal{C}(\mathcal{X}, \mathcal{Y})_0$.

Then one can define a map $J_T: \text{Trans } \mathcal{C} \rightarrow \text{Trans } \mathcal{D}$ such that $J_T(F \xi G)$, where $F, G \in \mathcal{C}(\mathcal{X}, \mathcal{Y})_0$, is the transformation $J_{\xi} = \xi \Phi \circ e(J_{\mathcal{Y}}) = e(J_{\mathcal{X}}) \circ \xi \Psi$ of the functor $F\Phi \circ J_{\mathcal{Y}} = J_{\mathcal{X}} \circ F\Psi$ into the functor $G\Phi \circ J_{\mathcal{Y}} = J_{\mathcal{X}} \circ G\Psi$, which both are functors from $\mathcal{X}\Phi$ to $\mathcal{Y}\Psi$.

This map satisfies the following equalities:

- (b1) $J_T(1_F) = e(F\Phi \circ J_{\mathcal{Y}}) = e(J_{\mathcal{X}}) \circ F\Psi$;
 (b2) $J_T((F \xi G) \circ (G \eta H)) = J_T(\xi) \circ J_T(\eta)$;
 (b3) $J_T((F \xi G) \circ (K \rho L)) = J_T(\xi) \circ \rho \Psi = \xi \Phi \circ J_T(\rho)$

for any $F, G, H \in \mathcal{C}(\mathcal{X}, \mathcal{Y})$, for any $K, L \in \mathcal{C}(\mathcal{Y}, \mathcal{X})$ and for any $\mathcal{X}, \mathcal{Y}, \mathcal{X} \in \text{Ob } \mathcal{C}$.

Further, the notion of S -universal morphisms is introduced and investigated.

4. Let S_M be a property of morphisms, S_T be a property of transformations. Usually we suppose that the properties S are

(u) unitary: any identity element (morphism or transformation) has the property S ;

(m) multiplicative: the composite (resp., product) of morphisms or transformations has the property S , if each composant has this property.

Naturally, any property S for an element x can be expressed in the form $x \in \mathcal{M}_S$, where \mathcal{M}_S is the set of all elements satisfying the property S .

5. Definition. Let \mathcal{C} and \mathcal{D} be arbitrary 2-categories, S_M and S_T be arbitrary properties of morphisms and transformations respectively. Let $\mathcal{D}u\mathcal{C}$ be a 2-functor and \mathcal{X} be an object in \mathcal{C} . Then a pair $(\underline{\mathcal{X}}, \mathcal{X} J_{\mathcal{X}}(\underline{\mathcal{X}}u))$, consisting of an object $\underline{\mathcal{X}}$ in \mathcal{D} and a morphism $J_{\mathcal{X}}$ in \mathcal{C} , is called S -universal pair from \mathcal{X} to u , if

(i) for any object \mathcal{P} in \mathcal{D} and for any morphism $\mathcal{X} F(\mathcal{P}u)$ in \mathcal{C} there is a unique morphism $\underline{\mathcal{X}} \underline{F} \mathcal{P}$ with property S_M such that $J_{\mathcal{X}} \circ \underline{F}u = F$;

(ii) for any two morphisms F and G from \mathcal{X} to $\mathcal{P}u$ and for any transformation $F \xi G$ there is the only transformation $\underline{F} \underline{\xi} \underline{G}$ with the property S_T and such that $e(J_{\mathcal{X}} \cdot \underline{\xi}u) = \xi$.

Dually, the S -couniversality is defined. Note, that if all morphisms and transformations in \mathcal{D} satisfy respectively the properties S_M and S_T , we get the usual definition of (co)universality.

Let $(\underline{\mathcal{X}}, \mathcal{X} J_{\mathcal{X}}(\underline{\mathcal{X}}u))$ and $(\underline{\mathcal{Y}}, \mathcal{Y} J_{\mathcal{Y}}(\underline{\mathcal{Y}}u))$ be two S -universal pairs respectively from \mathcal{X} and \mathcal{Y} to $\mathcal{C}u\mathcal{D}$. A morphism from $(\underline{\mathcal{X}}, \mathcal{X} J_{\mathcal{X}}(\underline{\mathcal{X}}u))$ to $(\underline{\mathcal{Y}}, \mathcal{Y} J_{\mathcal{Y}}(\underline{\mathcal{Y}}u))$ is a pair of morphisms $\mathcal{X} K \mathcal{Y}$ in \mathcal{C} and $\underline{\mathcal{X}} \underline{K} \underline{\mathcal{Y}}$ in \mathcal{D} such that $J_{\mathcal{X}} \circ \underline{K}u = K \circ J_{\mathcal{Y}}$. Emphasize that this condition determines \underline{K} uniquely.

If (K, \underline{K}) and (L, \underline{L}) are two morphisms from $(\underline{\mathcal{X}}, J_{\mathcal{X}})$ to $(\underline{\mathcal{Y}}, J_{\mathcal{Y}})$, then the transformation from (K, \underline{K}) to (L, \underline{L}) is a pair of transformations $K \rho L$ in \mathcal{C} and $\underline{K} \underline{\rho} \underline{L}$ in \mathcal{D} , where the last transformation is uniquely determined by condition $\rho \cdot e(J_{\mathcal{Y}}) = e(J_{\mathcal{X}}) \cdot \underline{\rho}$.

All S -universal pairs from objects in \mathcal{C} to the functor $\mathcal{D}u\mathcal{C}$, their morphisms and the transformations of morphisms together with composition of morphisms, composition and multiplication of transformations, identity morphisms and identity transformations, that are induced by those in \mathcal{C} and \mathcal{D} , from a 2-category (\mathcal{C}, u, S) of S -universal pairs from objects in \mathcal{C} to u .

Dually, the category (v, \mathcal{D}, S^*) of S -couniversal pairs from a 2-functor $\mathcal{C}v\mathcal{D}$ to objects in \mathcal{D} is defined.

6. Proposition. If the properties S_M and S_T are unitary and multiplicative both for the composites and for the products, then for any object \mathcal{X} in \mathcal{C} and any 2-functor $\mathcal{D}u\mathcal{C}$ the S -universal pair $(\underline{\mathcal{X}}, J_{\mathcal{X}})$ from \mathcal{X} to u is determined uniquely up to the isomorphism of the object $\underline{\mathcal{X}}$. More exactly,

(i) if $(\underline{\mathcal{X}}^*, J_{\mathcal{X}}^*)$ is another S -universal pair from \mathcal{X} to u and S_M is multiplicative for composites and unitary, then the only morphism $\underline{\mathcal{X}} \underline{J}_{\mathcal{X}} \underline{\mathcal{X}}^*$, satisfying S_M and $J_{\mathcal{X}} \circ \underline{J}_{\mathcal{X}}^*u = J_{\mathcal{X}}^*$, is the isomorphism;

(ii) if $\underline{\mathcal{X}} \underline{J} \underline{\mathcal{X}}^*$ is an isomorphism such that $\underline{J}, \underline{J}^{-1}, e(\underline{J}), e(\underline{J}^{-1})$ have the properties S_M and S_T respectively, which are multiplicative for composites of morphisms and products of transformations, then $J_{\underline{\mathcal{X}}^*} = J_{\underline{\mathcal{X}}} \circ \underline{J}$ together with $\underline{\mathcal{X}}^*$ constitutes a S -universal pair from \mathcal{X} to u .

7. Proposition. Consider an arbitrary object $\underline{\mathcal{D}}$ in \mathcal{D} , an object \mathcal{X} and a morphism $\mathcal{X}J\underline{\mathcal{D}}$ in \mathcal{C} . Then for any object \mathcal{P} in \mathcal{D} the maps

$$\varphi_M: \mathcal{D}(\underline{\mathcal{D}}, \mathcal{P})_0 \rightarrow \mathcal{C}(\mathcal{X}, \mathcal{N})_0, \quad \underline{F} \mapsto J \circ Fu,$$

$$\varphi_T: \mathcal{D}(\underline{F}, \underline{G}) \rightarrow \mathcal{C}(\underline{F} \varphi_M, \underline{G} \varphi_M), \quad \underline{\xi} \mapsto 1_J \cdot \underline{\xi} \text{ for all } \underline{F}, \underline{G} \in \mathcal{D}(\underline{\mathcal{D}}, \mathcal{P})$$

constitute a 1-functor of 1-categories $\varphi: \mathcal{D}(\underline{\mathcal{D}}, \mathcal{P}) \rightarrow \mathcal{C}(\mathcal{X}, \mathcal{P}u)$.

Dually, a functor $\varphi^*: \mathcal{C}(\mathcal{X}, \mathcal{P}) \rightarrow \mathcal{D}(\mathcal{X}v, \mathcal{P})$ is determined.

8. Proposition.

(a) The pair $(\underline{\mathcal{D}}, \mathcal{X}J\underline{\mathcal{D}})$ is a S -universal pair from \mathcal{X} to u , if and only if the components of the functor $\varphi = (\varphi_M, \varphi_T)$ associated with J have sections, i.e. left inverse maps

$$\psi_M: \mathcal{C}(\mathcal{X}, \mathcal{P}u)_0 \rightarrow \mathcal{D}(\underline{\mathcal{D}}, \mathcal{P})_0,$$

$$\psi_T: \mathcal{C}(\underline{F} \varphi_M, \underline{G} \varphi_M) \rightarrow \mathcal{D}(\underline{F}, \underline{G}), \quad \underline{F}, \underline{G} \in \mathcal{D}, \quad \underline{\mathcal{D}} \in \mathcal{P}$$

for all $\mathcal{P} \in \mathcal{D}_0$.

Moreover, one can take

(i) the property S_M for a functor \underline{F} is $\underline{F} \in \text{Im } \psi_M$;

(ii) the property S_T for a transformation $\underline{F} \underline{\xi} \underline{G}$ with $\underline{F}, \underline{G} \in \mathcal{D}(\underline{\mathcal{D}}, \mathcal{P})$ is $\underline{\xi} \in \text{Im } \psi_T$.

(b) The pair $\psi = (\psi_M, \psi_T)$ gives a functor from $\mathcal{C}(\mathcal{X}, \mathcal{P}u)$ to $\mathcal{D}(\underline{\mathcal{D}}, \mathcal{P})$, if and only if S_T is a unitary and multiplicative for composites property.

A dual necessary and sufficient condition is true for a S -couniversal pair $(\underline{\mathcal{P}}, \mathcal{P}v)$ from v to \mathcal{P} and the pair of sections

$$\psi_M^*: \mathcal{D}(\mathcal{X}v, \mathcal{P})_0 \rightarrow \mathcal{C}(\mathcal{X}, \underline{\mathcal{D}})_0,$$

$$\psi_T^*: \mathcal{D}(\underline{U} \varphi_M^*, \underline{V} \varphi_M^*)_0 \rightarrow \mathcal{C}(U, V)_0, \quad U, V \in \mathcal{C}(\mathcal{X}, \underline{\mathcal{D}})_0$$

gives a functor from $\mathcal{D}(\mathcal{X}v, \mathcal{P})$ to $\mathcal{C}(\mathcal{X}, \underline{\mathcal{D}})$, if and only if S_T^* is a multiplicative for composite and unitary property.

9. Let $\mathcal{D}u\mathcal{C}$ be an arbitrary 2-functor, $\underline{\mathcal{D}}$ and \mathcal{X} be arbitrary objects in \mathcal{D} and \mathcal{C} respectively.

Proposition. By assigning to each

(i) object $\mathcal{P} \in \mathcal{D}_0$ the objects $\mathcal{D}(\underline{\mathcal{D}}, \mathcal{P})$ and $\mathcal{C}(\mathcal{X}, \mathcal{N})$;

(ii) morphism $\mathcal{P}U\underline{\mathcal{D}}$ in \mathcal{D} the morphisms.

(a) $\mathcal{D}(\underline{\mathcal{D}}, U): \mathcal{D}(\underline{\mathcal{D}}, \mathcal{P}) \rightarrow \mathcal{D}(\underline{\mathcal{D}}, \underline{\mathcal{D}})$ such that $\underline{F} \mapsto \underline{F} \circ U$, $\underline{F} \underline{\xi} \underline{G} \mapsto (\underline{F} \circ U) (\underline{\xi} \cdot 1_U) (\underline{G} \circ U)$;

(b) $\mathcal{C}(\mathcal{X}, Uu): \mathcal{C}(\mathcal{X}, \mathcal{N}) \rightarrow \mathcal{C}(\mathcal{X}, \underline{\mathcal{D}}u)$ such that $F \mapsto F \circ Uu$,

$$F \underline{\xi} G \mapsto (F \circ Uu) (\underline{\xi} \cdot 1_{Uu}) (G \circ Uu).$$

(iii) transformation $U \kappa V$ with $U, V \in \mathcal{D}(\underline{\mathcal{D}}, \underline{\mathcal{D}})$ the transformations

(a) $\mathcal{D}(\underline{\mathcal{D}}, \kappa): \mathcal{D}(\underline{\mathcal{D}}, U) \rightarrow \mathcal{D}(\underline{\mathcal{D}}, V)$ such that for any $\underline{F} \in \mathcal{D}(\underline{\mathcal{D}}, \mathcal{P})$ $\mathcal{D}(\underline{\mathcal{D}}, \kappa)_F = 1_F \cdot \kappa: \underline{F} \circ U \mapsto \underline{F} \circ V$;

(b) $\mathcal{C}(\mathcal{X}, \kappa u): \mathcal{C}(\mathcal{X}, Uu) \rightarrow \mathcal{C}(\mathcal{X}, Vu)$ such that for any $F \in \mathcal{C}(\mathcal{X}, \mathcal{N})$ $\mathcal{C}(\mathcal{X}, \kappa u)_F = 1_F \cdot \kappa u: F \circ Uu \rightarrow F \circ Vu$, we define 2-functors $\mathcal{D}(\underline{\mathcal{D}}, \cdot)$ and $\mathcal{C}(\mathcal{X}, \cdot u)$ from \mathcal{D} to **Cat**.

Dually, the 2-functors $\mathcal{D}(\cdot v, \mathcal{P})$ and $\mathcal{C}(\cdot, \underline{\mathcal{D}})$ from \mathcal{C}^{op} to **Cat** are defined. So, if $\mathcal{C}v\mathcal{D}$ and $\mathcal{D}u\mathcal{C}$ are arbitrary 2-functors, then we have two 2-bifunctors $\mathcal{D}(\cdot v, \cdot)$ and $\mathcal{C}(\cdot, \cdot u)$ from $\mathcal{C}^{op} \times \mathcal{D}$ to **Cat**.

10. Proposition.

(a) Let $\mathcal{D} u \mathcal{C}$ be an arbitrary 2-functor, \mathcal{X} and $\underline{\mathcal{X}}$ be arbitrary objects in \mathcal{C} and \mathcal{D} respectively, $\mathcal{X} J (\underline{\mathcal{X}} u)$ be an arbitrary morphism. Then the system of functors φ for all $\mathcal{P} \in \mathcal{D}_0$ constitutes a natural 2-transformation $\varphi: \mathcal{D}(\underline{\mathcal{X}}, \cdot) \rightarrow \mathcal{C}(\mathcal{X}, \cdot u)$ of 2-functors from \mathcal{D} to **Cat**.

(b) If $(\underline{\mathcal{X}}, \mathcal{X} J (\underline{\mathcal{X}} u))$ is a S -universal pair from \mathcal{X} to u , then the system of pairs $\psi = (\psi_M, \psi_T)$ for all $\mathcal{P} \in \mathcal{D}_0$ is natural in the second variable. It gives a natural 2-transformation $\psi: \mathcal{C}(\mathcal{X}, \cdot u) \rightarrow \mathcal{D}(\underline{\mathcal{X}}, \cdot)$ of 2-functors from \mathcal{D} to **Cat**, if the property S_T is unitary and multiplicative for composites.

11. Theorem. Given a 2-functor $\mathcal{D} u \mathcal{C}$, objects \mathcal{X} in \mathcal{C} and $\underline{\mathcal{X}}$ in \mathcal{D} , then there exists a canonical bijection between all

(i) S -universal pairs $(\underline{\mathcal{X}}, \mathcal{X} J_{\mathcal{X}} (\underline{\mathcal{X}} u))$ with a unitary and multiplicative for composites property S_T ;

(ii) pairs (φ, ψ) , consisting of natural 2-transformations

$$\varphi: \mathcal{D}(\underline{\mathcal{X}}, \cdot) \rightarrow \mathcal{C}(\mathcal{X}, \cdot u), \quad \psi: \mathcal{C}(\mathcal{X}, \cdot u) \rightarrow \mathcal{D}(\underline{\mathcal{X}}, \cdot u),$$

of 2-functors from \mathcal{D} to **Cat**, satisfying the condition $\psi \circ \varphi = 1$.

Dually, given a 2-functor $\mathcal{C} v \mathcal{D}$, objects \mathcal{P} in \mathcal{D} and $\underline{\mathcal{P}}$ in \mathcal{C} there exists a canonical bijection between all

(i) S^* -couniversal pairs $(\underline{\mathcal{P}}, \mathcal{P} I_{\mathcal{P}} \underline{\mathcal{P}})$ with a unitary and multiplicative for property S_T^* ;

(ii) pairs of natural 2-transformations

$$\varphi^*: \mathcal{C}(\cdot, \underline{\mathcal{P}}) \rightarrow \mathcal{D}(\cdot v, \underline{\mathcal{P}}), \quad \psi^*: \mathcal{D}(\cdot v, \underline{\mathcal{P}}) \rightarrow \mathcal{C}(\cdot, \underline{\mathcal{P}})$$

of 2-functors from \mathcal{C} to **Cat** such that $\psi^* \circ \varphi^* = 1$.

12. We denote by \mathcal{C}_u the full 2-subcategory of \mathcal{C} , for which the objects are all objects \mathcal{X} in \mathcal{C} such that there is a S -universal pair $(\underline{\mathcal{X}}, \mathcal{X} J (\underline{\mathcal{X}} u))$ from \mathcal{X} to u and $\gamma: \mathcal{C}_u \rightarrow \mathcal{C}$ is the canonical embedding.

Fix a S -universal pair $(\underline{\mathcal{X}}, \mathcal{X} J_{\mathcal{X}} (\underline{\mathcal{X}} u))$ from \mathcal{X} to u for any object \mathcal{X} in \mathcal{C}_u . Then assigning to each

(i) object \mathcal{X} the object $\underline{\mathcal{X}}$;

(ii) morphism $\mathcal{X} K \mathcal{Y}$ the morphism $\underline{\mathcal{X}} \underline{K} \underline{\mathcal{Y}}$, which is uniquely determined by condition $J_{\mathcal{X}} \circ \underline{K} u = K \circ J_{\mathcal{Y}}$ and the property S_M ;

(iii) transformation $(\mathcal{X} K \mathcal{Y}) \rho (\mathcal{X} L \mathcal{Y})$ the transformation $(\underline{\mathcal{X}} \underline{K} \underline{\mathcal{Y}}) \underline{\rho} (\underline{\mathcal{X}} \underline{L} \underline{\mathcal{Y}})$, which is uniquely determined by properties $\rho \circ e(J_{\mathcal{Y}}) = e(J_{\mathcal{X}}) \cdot \underline{\rho} u$ and S_T ;

(iv) object \mathcal{X} the morphism $\mathcal{X} J_{\mathcal{X}} \mathcal{A}(v \circ u)$ we get a 2-functor $\mathcal{C}_u v \mathcal{D}$ and a natural 2-transformation from γ to $v \circ u$.

In the dual case we denote by \mathcal{D}^v the full 2-subcategory of \mathcal{D} defined by all objects \mathcal{P} in \mathcal{D} such that there is a S^* -couniversal pair $(\underline{\mathcal{P}}, \underline{\mathcal{P}} v) I \mathcal{P}$ from $\mathcal{C} v \mathcal{D}$ to \mathcal{P} and by $\delta: \mathcal{D}^v \rightarrow \mathcal{D}$ the canonical embedding. Then the 2-functor $\mathcal{D}^v u \mathcal{C}$ and the natural 2-transformation from $u \circ v$ to δ are similarly determined.

13. Let $\mathcal{C}_u v \mathcal{D}$ be the 2-functor associated with a system of fixed S -universal pairs from all $\mathcal{X} \in (\mathcal{C}_u)_0$ to u . Suppose that S_T is unitary and multiplicative for composite property. Then for any object \mathcal{P} in \mathcal{D} the systems of functors

$$\varphi: \mathcal{D}(\mathcal{X} v, \mathcal{P}) \rightarrow \mathcal{C}(\mathcal{X}, \mathcal{P} u), \quad \psi: \mathcal{C}(\mathcal{X}, \mathcal{P} u) \rightarrow \mathcal{D}(\mathcal{X} v, \mathcal{P}) \text{ for all } \mathcal{X} \in \mathcal{C}_0$$

constitute the natural 2-transformations

$$\varphi: \mathcal{D}(\cdot v, \mathcal{P}) \rightarrow \mathcal{C}(\cdot, \mathcal{P}u), \quad \psi: \mathcal{C}(\cdot, \mathcal{P}u) \rightarrow \mathcal{D}(\cdot v, \mathcal{P})$$

of 2-functors from \mathcal{C}_u^{op} to **Cat**.

Dual assertions are true for the 2-functor $\mathcal{D}^V u \mathcal{C}$ associated with a system of fixed S^* -couniversal pairs from $\mathcal{C} v \mathcal{D}$ to all $\mathcal{P} \in (\mathcal{D}^V)_0$.

14. Theorem. Given a 2-functor $\mathcal{D} u \mathcal{C}$ and a full 2-subcategory \mathcal{E} of \mathcal{C} with an embedding functor $\mathcal{E} \gamma \mathcal{C}$, there exists a canonical bijection between all

(i) pairs (v, J) , where $\mathcal{E} v \mathcal{D}$ is a 2-functor and $J: \gamma \rightarrow v \circ u$ is a natural 2-transformation such that for any $\mathcal{X} \in \mathcal{E}_0$, $(\mathcal{X} v, \mathcal{X} J_{\mathcal{X}} \mathcal{D}(v \circ u))$ is a S -universal pair from \mathcal{X} to u with unitary and multiplicative for property S_T ;

(ii) triples (v, φ, ψ) , where $\mathcal{E} v \mathcal{D}$ is a 2-functor,

$$\varphi: \mathcal{D}(\cdot v, \cdot) \rightarrow \mathcal{C}(\cdot, \cdot u), \quad \psi: \mathcal{C}(\cdot, \cdot u) \rightarrow \mathcal{D}(\cdot v, \cdot)$$

are binatural 2-transformations of 2-functors from $\mathcal{E}^{op} \times \mathcal{D}$ to **Cat** such that $\psi \circ \varphi = 1$.

A similar canonical bijection exists in the dual case for a 2-functor $\mathcal{C} v \mathcal{D}$ and a full 2-subcategory \mathcal{F} of \mathcal{D} with an embedding functor $\mathcal{F} \delta \mathcal{D}$.

15. Remarks.

(a) In the case, when all morphisms and transformations of the category \mathcal{D} have the property S , we have both equalities $\psi \circ \varphi = 1$ and $\varphi \circ \psi = 1$. Consequently, in this case φ is a bijection, ψ is the inverse bijection, therefore, ψ is functorial and φ, ψ determine reciprocally inverse isomorphisms of natural 2-transformations. In general case we have not $\varphi \circ \psi = 1$, but only $(\varphi \circ \psi)^2 = \varphi \circ \psi$. The similar remark is correct in S^* -couniversal case.

(b) There is a simple method to reduce the results and notions for 2-categories to 1-categories by interpreting 1-categories as trivial 2-categories, which have only identity transformations.

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S-ունիվերսալությունը 2-կատեգորիաներում

2-կատեգորիաներում ներմուծվում և ուսումնասիրվում է S-ունիվերսալության հասկացողությունը:

С. Г. Далалян

S-универсальность в 2-категориях

В 2-категориях вводится и изучается понятие S-универсальности.