λ-DEFINABILITY OF BUILT-IN McCARTHY FUNCTIONS AS
FUNCTIONS WITH INDETERMINATE VALUES OF ARGUMENTS

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The built-in functions of programming languages are functions with indeterminate values of arguments. The built-in McCarthy functions car, cdr, cons, null, atom, if, eq, not, and, or, are used in all functional programming languages. In this paper we show the λ-definability of the built-in McCarthy functions as functions with indeterminate values of arguments. This result is necessary when translating typed functional programming languages into untyped functional programming languages.


Keywords: built-in McCarthy functions, indeterminate values of arguments, λ-definability.

Built-in McCarthy Functions, as Functions with Indeterminate Values of Arguments. Let \( N = \{0, 1, 2, \ldots \} \) be the set of natural numbers, each natural number \( n \in N \) will be called an atom. Let us define the set of symbolic expressions, which we denote by S-expressions [1].

1. \( n \in N \Rightarrow n \in \text{S-expressions} \),
2. \( m_1, \ldots, m_k \in \text{S-expressions}, k \geq 0 \Rightarrow (m_1 \ldots m_k) \in \text{S-expressions} \) and is called a list.

If \( k = 0 \), than the list ( ) will be called the empty list. We denote the empty list by the atom 0. Thus the empty list will be both an atom and a list.

Let \( M = \text{S-expressions} \cup \{ \perp \} \), where \( \perp \) is the element which corresponds to indeterminate value. A mapping \( \phi : M^k \rightarrow M, k \geq 1 \), is said to be function with indeterminate values of arguments. Let us define built-in McCarthy functions car, cdr, cons, null, atom, if, eq, not, and, or, as functions with indeterminate values of arguments.

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car, cdr, null, atom, not: M → M, and for any m ∈ M we have:

\[ \text{car}(m) = \begin{cases} m_1, & \text{if } m = (m_1 \ldots m_k), \text{ where } m_1, \ldots, m_k \in \text{S-expressions}, k \geq 1, \\ \bot, & \text{otherwise.} \end{cases} \]

\[ \text{cdr}(m) = \begin{cases} (m_2 \ldots m_k), & \text{if } m = (m_1 \ldots m_k), \text{ where } m_1, \ldots, m_k \in \text{S-expressions}, k \geq 1, \\ \bot, & \text{otherwise.} \end{cases} \]

\[ \text{null}(m) = \begin{cases} 1, & \text{if } m = 0, \\ 0, & \text{if } m \in \text{S-expressions and } m \neq 0, \\ \bot, & \text{if } m = \bot. \end{cases} \]

\[ \text{atom}(m) = \begin{cases} 1, & \text{if } m \in \mathbb{N}, \\ 0, & \text{if } m \in \text{S-expressions and } m \notin \mathbb{N}, \\ \bot, & \text{if } m = \bot. \end{cases} \]

\[ \text{not}(m) = \text{null}(m). \]

\[ \text{cons}: M^2 \to M, \text{ and for any } m_0, m \in M \text{ we have:} \]

\[ \text{cons}(m_0, m) = \begin{cases} (m_0 m_1 \ldots m_k), & \text{if } m_0 \in \text{S-expressions, } m = (m_1 \ldots m_k), \\ \bot, & \text{where } m_1, \ldots, m_k \in \text{S-expressions, } k \geq 0, \text{ otherwise.} \end{cases} \]

\[ \text{and}, \text{ or}, \text{ eq}: M^2 \to M, \text{ and for any } m_1, m_2 \in M \text{ we have:} \]

\[ \text{and}(m_1, m_2) = \begin{cases} m_1, & \text{if } m_1 = 0, \\ m_2, & \text{if } m_1 \in \text{S-expressions and } m_1 \neq 0, \\ \bot, & \text{if } m_1 = \bot. \end{cases} \]

\[ \text{or}(m_1, m_2) = \begin{cases} m_2, & \text{if } m_1 = 0, \\ m_1, & \text{if } m_1 \in \text{S-expressions and } m_1 \neq 0, \\ \bot, & \text{if } m_1 = \bot. \end{cases} \]

\[ \text{eq}(m_1, m_2) = \begin{cases} 1, & \text{if } m_1, m_2 \in \mathbb{N} \text{ and } m_1 = m_2, \\ 0, & \text{if } m_1, m_2 \in \mathbb{N} \text{ and } m_1 \neq m_2, \\ \bot, & \text{otherwise.} \end{cases} \]

\[ \text{if}: M^3 \to M, \text{ and for any } m_1, m_2, m_3 \in M \text{ we have:} \]

\[ \text{if}(m_1, m_2, m_3) = \begin{cases} m_2, & \text{if } m_1 \in \text{S-expressions and } m_1 \neq 0, \\ m_3, & \text{if } m_1 = 0, \\ \bot, & \text{if } m_1 = \bot. \end{cases} \]

**Untyped \( \lambda \)-Terms, \( \beta \)-Reduction, \( \beta \)-Equality.** The definitions of this section can be found in [2]. Let us fix a countable set of variables \( V \). The set \( \Lambda \) of terms is defined as follows:
1. If \( x \in \mathcal{V} \), then \( x \in \Lambda \),
2. If \( t_1, t_2 \in \Lambda \), then \( (t_1 t_2) \in \Lambda \) (the operation of application),
3. If \( x \in \mathcal{V} \) and \( t \in \Lambda \), then \( (\lambda x.t) \in \Lambda \) (the operation of abstraction).

We use the following shorthand notations: a term \( (\ldots (t_1 t_2) \ldots t_k) \), where \( t_i \in \Lambda \), \( i = 1, \ldots, k, \ k \geq 2 \), is denoted by \( t_1 t_2 \ldots t_k \), and a term \( (\lambda x_1 (\lambda x_2 (\ldots (\lambda x_n t) \ldots ))) \), where \( x_j \in \mathcal{V}, j = 1, \ldots, n, \ n \geq 1, \ t \in \Lambda \), is denoted by \( \lambda x_1 x_2 \ldots x_n t \).

The notions of free and bound occurrences of variables in terms as well as the notion of free variable are introduced in the conventional way. The set of all free variables of a term \( t \) is denoted by \( \text{FV}(t) \). A term which does not contain free variables is called a closed term. Terms \( t_1 \) and \( t_2 \) are said to be congruent (which is denoted by \( t_1 \equiv t_2 \)) if one term can be obtained from the other by renaming bound variables. In what follows, congruent terms are considered identical.

To show a variable \( x \) of interest of a term \( t \), the notation \( t [x] \) is used. The notation \( t [\tau] \) denotes the term obtained by the simultaneous substitution of the term \( \tau \) for all free occurrences of the variable \( x \). A substitution is said to be admissible if all free variables of the term being substituted remain free after the substitution. We will consider only admissible substitutions.

A term of the form \( (\lambda x.t[x]) \), where \( x \in \mathcal{V}, t, \tau \in \Lambda \) is called a \( \beta \)-redex and the term \( t [\tau] \) is called its convolution. A one-step \( \beta \)-reduction \( (\rightarrow_{\beta}) \) and \( \beta \)-equality \( (=_{\beta}) \) are defined in a standard way (see [2]). A term containing no \( \beta \)-redexes is called a normal form. The set of all normal forms is denoted by \( \text{NF} \), and the set of all closed normal forms is denoted by \( \text{NF}^0 \). A term \( t \) is said to have a normal form if there exists a term \( \tau \), such that \( \tau \in \text{NF} \) and \( t =_{\beta} \tau \). Of the Church-Rosser theorem \([2]\) it follows:

1. \( t =_{\beta} \tau \) and \( \tau \in \text{NF} \) \( \Rightarrow t \rightarrow_{\beta} \tau \).
2. \( t =_{\beta} \tau_1, t =_{\beta} \tau_2 \) and \( \tau_1, \tau_2 \in \text{NF} \) \( \Rightarrow \tau_1 \equiv \tau_2 \).

If a term has a form \( \lambda x_1 \ldots x_k. x t_1 \ldots t_n \), where \( x_1, \ldots, x_k, x \in \mathcal{V}, t_1, \ldots, t_n \in \Lambda \), \( k, n \geq 0 \), it is called a head normal form and \( x \) is called its head variable. The set of all head normal forms is denoted by \( \text{HNF} \). A term \( t \) is said to have a head normal form if there exists a term \( \tau \), such that \( \tau \in \text{HNF} \) and \( t =_{\beta} \tau \). It is known, that \( \text{NF} \subseteq \text{HNF} \), but \( \text{HNF} \nsubseteq \text{NF} \) (see [2]).

**Proposition 1** \([2]\). Let \( t \in \Lambda \), then: \( t \) does not have a head normal form \( \Rightarrow \) for any \( \tau \in \Lambda \) the term \( t \tau \) will not have a head normal form.

A term \( t \) with a fixed occurrence of a subterm \( \tau_1 \) is denoted by \( t_{\tau_1} \), and a term with this occurrence of \( \tau_1 \) replaced by a term \( \tau_2 \) is denoted by \( t_{\tau_2} \).

**Proposition 2** \([2]\). Let \( t, \tau_1, \tau_2 \in \Lambda \), then: \( \tau_1 =_{\beta} \tau_2 \) \( \Rightarrow t_{\tau_1} =_{\beta} t_{\tau_2} \).

Consider the following equation: \( f = t[f] \), where \( f \in \mathcal{V}, t[f] \in \Lambda \), \( \text{FV}(t[f]) \subseteq \{ f \} \). The term \( \tau \) is a solution of this equation if \( \tau =_{\beta} t[\tau] \).
Proposition 3 \[2\]. The term $\lambda f . f[f]$ is a solution of equation $f = \text{if} \[f\].$

We introduce notations for some terms: $I = \lambda x . x$, $T = \lambda x y . x$, $F = \lambda x y z . x$, $\Omega = (\lambda x y . x)(\lambda x y . x)$, $[ ] = I$, $[t_1, \ldots, t_k] = \lambda x . x t_1 t_2 \cdots t_k$, where $x, y \in V$, $t_i \in \Lambda$, $i = 1, \ldots, k$, $k \geq 1$. It is easy to see that: $\beta \rightarrow t_1$, $T t_1 t_2 \rightarrow \beta t_1$, $F t_1 t_2 \rightarrow \beta t_2$, $t_1, t_2 \in \Lambda$, the term $\Omega$ does not have a head normal form.

Let $M = \{ S\text{-expressions} \}$. To each $m \in M$ we associate the term $m' \in \Lambda$ as follows: $0' = I$, $(n + 1)' = \lambda x . x \lambda m . m'$, where $n \in N$; $(m_1 \ldots m_k)' \equiv [m_1', \ldots, m_k']$, where $m_i \in S\text{-expressions}$, $i = 1, \ldots, k$, $k \geq 0$; $\bot' = \Omega$. It is easy to see that, if $m \in S\text{-expressions}$, then $m' \in \text{NF}^0$, and if $m_1 m_2 \in S\text{-expressions}$, $m_1 \neq m_2$, then $m_1'$ and $m_2'$ are not congruent closed normal forms.

Definition. We say that term $\Phi \in \Lambda \lambda$-defines $\{ \| m \| \}$ the function $\phi : M^k \rightarrow M$ $(k \geq 1)$ as a function with indeterminate values of arguments, if for all $m_1, \ldots, m_k \in M$ we have:

- $\phi(m_1, \ldots, m_k) = m' \land m' \neq \bot' \Rightarrow \phi m_1' \ldots m_k' \equiv \beta m'$.
- $\phi(m_1, \ldots, m_k) = \bot' \Rightarrow \phi m_1' \ldots m_k'$ does not have a head normal form.

Theorem. For functions $\text{car}$, $\text{cdr}$, $\text{cons}$, $\text{null}$, $\text{atom}$, $\text{if}$, $\text{eq}$, $\text{not}$, $\text{and}$, $\text{or}$, respectively, which $\lambda$-define them.

Proof. Let us show that the term $\text{null} \equiv \lambda x . x(\lambda x z . z) T 0' 1'$ $\lambda$-defines the function $\text{null}$. To do this, show the following:

a) $\text{null} 0' \rightarrow \beta 1'$.

b) $\text{null} (n + 1)' \rightarrow \beta 0'$, where $n \in N$.

c) $\text{null} [m_1', \ldots, m_k'] \rightarrow \beta 0'$, where $m_i \in S\text{-expressions}$, $i = 1, \ldots, k$, $k \geq 1$.

d) $\text{null} \Omega$ does not have a head normal form.

(a) $\text{null} 0' \equiv (\lambda x . x(\lambda x z . z) T 0' 1') I \rightarrow \beta I(\lambda x z . z) T 0' 1' \rightarrow \beta (\lambda x z . z) T 0' 1' \rightarrow \beta 1'$.

(b) $\text{null} (n + 1)' \equiv (\lambda x . x(\lambda x z . z) T 0' 1')(n + 1)' \rightarrow \beta (n + 1)'(\lambda x z . z) T 0' 1' \equiv (\lambda x . F n')(\lambda x z . z) T 0' 1' \rightarrow \beta (\lambda x . x F n') T 0' 1' \rightarrow \beta 0'$.

c) $\text{null} [m_1', \ldots, m_k' \equiv (\lambda x . x(\lambda x z . z) T 0' 1')(m_1', \ldots, m_k') \rightarrow \beta (\lambda x z . z) m_1'[m_1', \ldots, m_k'] T 0' 1' \rightarrow \beta 0'$.

(d) $\text{null} \Omega \equiv (\lambda x . x(\lambda x z . z) T 0' 1') \Omega \rightarrow \beta \Omega(\lambda x z . z) T 0' 1'$ and, according to Proposition 1, the term $\text{null} \Omega$ does not have a head normal form.

Let us show that the term $\text{if} \equiv \lambda x y z . (\text{null} x) T y z \lambda$-defines the function $\text{if}$.

To do this, show the following:

a) $\text{if} 0' m_1' \rightarrow \beta m_1'$, where $m_2, m_3 \in M$.

b) $\text{if}(n + 1)' m_2' \rightarrow \beta m_2'$, where $n \in N$, $m_2, m_3 \in M$. 

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c) If \( \mu_1, \ldots, \mu_k \mid m_2' m_3' \rightarrow_T \beta m_2' \), where \( \mu_i \in S\text{-expressions}, i = 1, \ldots, k, k \geq 1 \), \( m_2, m_3 \in M \).

d) If \( \Omega m_2' m_3' \), where \( m_2, m_3 \in M \), does not have a head normal form.

(a) If\( 0' \mid m_2' m_3' \equiv (\lambda x. (\text{Null} x) T y z) 0' m_2' m_3' \rightarrow_T (\text{Null} 0') T m_2' m_3' \rightarrow_T \beta \)
\( 1' T m_2' m_3' \equiv (\lambda x. F 0') T m_2' m_3' \rightarrow_T (T F 0') m_2' m_3' \rightarrow_T \beta m_2' m_3' \).

(b) If\( (n+1)' \mid m_2' m_3' \equiv (\lambda x. (\text{Null} x) T y z) (n+1)' m_2' m_3' \rightarrow_T \beta \)
\( (\text{Null} (n+1)') T m_2' m_3' \rightarrow_T 0' T m_2' m_3' \rightarrow_T \beta T m_2' m_3' \rightarrow_T \beta m_2' m_3' \).

(c) If\( \mu_1, \ldots, \mu_k \mid m_2' m_3' \equiv (\lambda x. (\text{Null} x) T y z) \mu_1, \ldots, \mu_k \mid m_2' m_3' \rightarrow_T \beta \)
\( \text{Null} [\mu_1, \ldots, \mu_k] T m_2' m_3' \rightarrow_T 0' T m_2' m_3' \rightarrow_T \beta T m_2' m_3' \rightarrow_T \beta m_2' m_3' \).

(d) If\( \Omega m_2' m_3' \equiv (\lambda x. (\text{Null} x) T y z) \Omega m_2' m_3' \rightarrow_T \beta (\text{Null} \Omega) T m_2' m_3' \) and, according to Proposition 1, the term \( \Omega m_2' m_3' \) does not have a head normal form.

Let us show that the term \( \text{Atom} \equiv \lambda x. \text{If}(\text{Null} x) 1'(\text{False}(x T)) \), where \( \text{False} \equiv \lambda x. \lambda (\lambda x. x. 0') T 1' F \), \( \lambda \)-defines the function \( \text{atom} \). To do this, show the following:

a) \( \text{Atom} 0' \rightarrow_T \beta 1' \).

b) \( \text{Atom} (n+1)' \rightarrow_T \beta 1' \), where \( n \in N \).

c) \( \text{Atom} [m_2', \ldots, m_k'] \rightarrow_T \beta 0' \), where \( m_i \in S\text{-expressions}, i = 1, \ldots, k, k \geq 1 \).

d) \( \text{Atom} \Omega \) does not have a head normal form.

(a) \( \text{Atom} 0' \equiv (\lambda x. \text{If}(\text{Null} x) 1'(\text{False}(x T))) 0' \rightarrow_T \beta \text{If}(\text{Null} 0') 1'(\text{False}(0' T)) \rightarrow_T \beta \)
\( 1' 1'(\text{False}(0' T)) \rightarrow_T \beta 1' \).

(b) \( \text{Atom} (n+1)' \equiv (\lambda x. \text{If}(\text{Null} x) 1'(\text{False}(x T))) (n+1)' \rightarrow_T \beta \)
\( \text{If}(\text{Null} (n+1)') 1'(\text{False}(n+1) T) \rightarrow_T \beta \text{If}(n+1)' 1'(\text{False}(n+1) T) \rightarrow_T \beta \text{False} (n+1) T \equiv \)
\( \lambda x. x. (\lambda x. x. 0') T 1' F \rightarrow_T \beta F (\lambda x. x. 0') T 1' F \rightarrow_T \beta 1' \).

(c) \( \text{Atom} [m_1', \ldots, m_k'] \equiv (\lambda x. \text{If}(\text{Null} x) 1'(\text{False}(x T))) [m_1', \ldots, m_k'] \rightarrow_T \beta \)
\( \text{If}(\text{Null} [m_1', \ldots, m_k'] T) \rightarrow_T \beta \text{If} [m_1', \ldots, m_k'] T \equiv \beta \text{False} [m_1', \ldots, m_k'] T \rightarrow_T \beta \text{False} [m_1', \ldots, m_k'] T \).

There are three possible cases: c1) \( m_i' \equiv 0' \), c2) \( m_i' \equiv (n+1)' \), where \( n \in N \), c3) \( m_i' \equiv [\mu_1, \ldots, \mu_s] \), where \( \mu_i \in S\text{-expressions}, i = 1, \ldots, s, s \geq 1 \).

(c1) \( \text{False} 0' \equiv (\lambda x. x. (\lambda x. x. 0') T 1' F) 1' \rightarrow_T \beta (\lambda x. x. 0') T 1' F \rightarrow_T \beta \)
\( (\lambda x. y. 0') T 1' F \rightarrow_T \beta 0' \).

(c2) \( \text{False} (n+1)' \equiv (\lambda x. x. (\lambda x. x. 0') T 1' F) (n+1)' \rightarrow_T \beta (n+1)' (\lambda x. x. 0') T 1' F \equiv \)
\( \lambda x. x. F n' (\lambda x. x. 0') T 1' F \rightarrow_T \beta (\lambda x. x. 0') F n' T 1' F \rightarrow_T \beta 0' 1' F \rightarrow_T \beta 1' F \equiv \)
\( \lambda x. x. F 0' F \rightarrow_T \beta F F 0' \rightarrow_T \beta 0' \).

(c3) \( \text{False} [\mu_1, \ldots, \mu_s] \equiv (\lambda x. x. (\lambda x. x. 0') T 1' F) [\mu_1, \ldots, \mu_s] \rightarrow_T \beta \)
\( [\mu_1, \ldots, \mu_s] (\lambda x. x. 0') T 1' F \equiv (\lambda x. x. [\mu_1, \ldots, \mu_s]) (\lambda x. x. 0') T 1' F \rightarrow_T \beta \)
\( (\lambda x. x. 0') \mu_1 [\mu_2, \ldots, \mu_s] T 1' F \rightarrow_T \beta 0' 1' F \rightarrow_T \beta 1' F \equiv \)
(\lambda x. F' o) \rightarrow \rightarrow \beta FF' \rightarrow \rightarrow \beta'.

(d) \text{Atom}\Omega \equiv (\lambda x. If(Null\Omega)x'y')(\lambda y. (Null\Omega)\rightarrow \rightarrow \beta If(Null\Omega)')\Omega \rightarrow \rightarrow \beta If(Null\Omega)'

\equiv (\lambda x.\lambda y.\lambda z. T\Omega')\rightarrow \rightarrow \beta T\Omega'\rightarrow \rightarrow \beta.

Null\Omega \equiv (\lambda x.\lambda y.\lambda z. T\Omega')\rightarrow \rightarrow \beta T\Omega'\rightarrow \rightarrow \beta.

And, according to Proposition 1, the term \text{Atom}\Omega does not have a head normal form.

Since not=Null, then Not\equiv Null.

Let us show that the term \text{And} \equiv \lambda xy. If(Null\Omega)xy \lambda \text{-defines the function and.}

To do this, show the following:

a) And'0'm2' \rightarrow \rightarrow \beta 0', where m2 \in M.

b) And'(n+1)'m2' \rightarrow \rightarrow \beta m2', where n \in N, m2 \in M.

c) And[\mu_1', \ldots, \mu_k']m2' \rightarrow \rightarrow \beta m2', where \mu_i \in S-expressions, i=1, \ldots, k, k \geq 1, m2 \in M.

d) And\Omega m2', where m2 \in M, does not have a head normal form.

(a) And'0'm2' \equiv (\lambda x.\lambda y. If(Null\Omega)x'y')0' \rightarrow \rightarrow \beta If(Null\Omega)'0'm2' \rightarrow \rightarrow \beta If'0'm2' \rightarrow \rightarrow \beta .

(b) And'(n+1)'m2' \equiv (\lambda x.\lambda y. If(Null\Omega)x'y')(n+1)' \rightarrow \rightarrow \beta If(Null(n+1))'(n+1)' \rightarrow \rightarrow \beta .

(c) And[\mu_1', \ldots, \mu_k']m2' \equiv (\lambda x.\lambda y. If(Null\Omega)x'y')[\mu_1', \ldots, \mu_k'] \rightarrow \rightarrow \beta .

\text{If}(Null[\mu_1', \ldots, \mu_k'])m2' \rightarrow \rightarrow \beta If'0'[\mu_1', \ldots, \mu_k']m2' \rightarrow \rightarrow \beta m2'.

d) And\Omega m2' \equiv (\lambda x.\lambda y. If(Null\Omega)x'y')\Omega m2' \rightarrow \rightarrow \beta If(Null\Omega)\Omega m2' \equiv (\lambda x.\lambda y. If(Null\Omega)x'y')\Omega m2' \rightarrow \rightarrow \beta Null(Null\Omega)T\Omega m2' \equiv (\lambda x.\lambda y.\lambda z. T\Omega'1')\rightarrow \rightarrow \beta Null(Null\Omega)T\Omega m2' \rightarrow \rightarrow \beta (Null\Omega)(\lambda x.\lambda y.\lambda z. T\Omega'1')T\Omega m2' and, according to Proposition 1, the term And\Omega m2' does not have a head normal form.

Let us show that the term \text{Or} \equiv \lambda xy. If(Null\Omega)xy \lambda \text{-defines the function or.} To do this, show the following:

a) Or'0'm2' \rightarrow \rightarrow \beta m2', where m2 \in M.

b) Or'(n+1)'m2' \rightarrow \rightarrow \beta (n+1)', where n \in N, m2 \in M.

c) Or[\mu_1', \ldots, \mu_k']m2' \rightarrow \rightarrow \beta [\mu_1', \ldots, \mu_k'], where \mu_i \in S-expressions, i=1, \ldots, k, k \geq 1, m2 \in M.

d) Or\Omega m2', where m2 \in M, does not have a head normal form.

(a) Or'0'm2' \equiv (\lambda x.\lambda y. If(Null\Omega)x'y')0' \rightarrow \rightarrow \beta If(Null\Omega)'0'm2' \rightarrow \rightarrow \beta If'0'm2' \rightarrow \rightarrow \beta .

(b) Or'(n+1)'m2' \equiv (\lambda x.\lambda y. If(Null\Omega)x'y')(n+1)' \rightarrow \rightarrow \beta If(Null(n+1))'(n+1)' \rightarrow \rightarrow \beta .

(c) Or[\mu_1', \ldots, \mu_k']m2' \equiv (\lambda x.\lambda y. If(Null\Omega)x'y')[\mu_1', \ldots, \mu_k'] \rightarrow \rightarrow \beta .

\text{If}(Null[\mu_1', \ldots, \mu_k'])m2' \rightarrow \rightarrow \beta If'0'[\mu_1', \ldots, \mu_k'] \rightarrow \rightarrow \beta .

[\mu_1', \ldots, \mu_k'].
Or \(\Omega m_2' \equiv (\lambda x. If(Null)x)\Omega m_2' \rightarrow_\beta If(Null\Omega)\Omega m_2' \Omega \equiv (\lambda x. x)(\Omega m_2') \Omega \rightarrow_\beta Null(Null\Omega)\Omega m_2' \Omega \equiv (\lambda x. x)(\lambda x y z)T_0' 1'(Null\Omega)\Omega m_2' \Omega \rightarrow_\beta (Null\Omega)(\lambda x y z)T_0' 1' T_0' \Omega m_2' \Omega\) and, according to Proposition 1, the term \(Or\Omega m_2'\) does not have a head normal form.

Let us show that the term \(Car \equiv \lambda x. If(Atom x)\Omega(xT)\) \(\lambda\)-defines the function \(car\). To do this, show the following:

a) \(Car[m_1', \ldots , m_k'] \rightarrow_\beta m_1'\), where \(m_i\in S\)-expressions, \(i = 1, \ldots , k\), \(k \geq 1\).

b) \(Car'n\), where \(n\in N\), does not have a head normal form.

c) \(Car\Omega\) does not have a head normal form.

(a) \(Car[m_1', \ldots , m_k'] \equiv (\lambda x. If(Atom x)\Omega(xT))[m_1', \ldots , m_k'] \rightarrow_\beta\)

\(If(Atom[m_1', \ldots , m_k'])\Omega(m_1', \ldots , m_k')T \rightarrow_\beta If(Atom'[m_1', \ldots , m_k']T) \rightarrow_\beta m_1'\), where \(m_i\in S\)-expressions, \(i = 1, \ldots , k\), \(k \geq 1\).

\(Car'n \equiv (\lambda x. If(Atom x)\Omega(xT))n' \rightarrow_\beta If(Atom'n)\Omega(n'T) \rightarrow_\beta \Omega\)

and the term \(Car'n\) does not have a head normal form.

\(Car\Omega \equiv (\lambda x. If(Atom x)\Omega(xT))\Omega \rightarrow_\beta If(Atom x)\Omega(xT)\Omega \Omega\)

\(\Omega x.y.z.(\lambda x. x)(\Omega x y z)T_0' 1'\Omega(xT)\Omega \Omega\)

\(\rightarrow_\beta (Atom x)\Omega(xT)\Omega(xT)\Omega\) and, according to Proposition 1, the term \(Car\Omega\) does not have a head normal form.

Let us show that the term \(Cdr \equiv \lambda x. If((Atom x)\Omega(xF))\) \(\lambda\)-defines the function \(cdr\). To do this, show the following:

a) \(Cdr[m_1', \ldots , m_k'] \rightarrow_\beta [m_1', \ldots , m_k']\), where \(m_i\in S\)-expressions, \(i = 1, \ldots , k\), \(k \geq 1\).

b) \(Cdr'n\), where \(n\in N\), does not have a head normal form.

c) \(Cdr\Omega\) does not have a head normal form.

(a) \(Cdr[m_1', \ldots , m_k'] \equiv (\lambda x. If((Atom x)\Omega(xF))[m_1', \ldots , m_k'] \rightarrow_\beta\)

\(If(Atom[m_1', \ldots , m_k']F)[m_1', \ldots , m_k']F \rightarrow_\beta Fm_1'[m_1', \ldots , m_k'] \rightarrow_\beta [m_1', \ldots , m_k']\) and, according to Proposition 1, the term \(Cdr\Omega\) does not have a head normal form.

\(Cdr\Omega \equiv (\lambda x. If(Atom x)\Omega(xF))\Omega \rightarrow_\beta If(Atom x)\Omega(xF)\Omega\)

\(\Omega x.y.z.(\lambda x. x)(\Omega x y z)T_0' 1'(Null\Omega)\Omega m_2' \Omega \rightarrow_\beta (Null\Omega)(\lambda x y z)T_0' 1' T_0' \Omega m_2' \Omega\)

\(\rightarrow_\beta (Null\Omega)(\lambda x y z)T_0' 1' T_0' \Omega m_2' \Omega\) and, according to Proposition 1, the term \(Cdr\Omega\) does not have a head normal form.

Let us show that the term

\(Cons \equiv \lambda x.y.If((\And(Atom x))(Null))(\Null x)(\lambda z.x z)(\lambda z.x z))\)

\(\lambda\)-defines the function \(cons\). To do this, show the following:
a) \( \text{Cons}m_0[m_1', \ldots, m_k'] \rightarrow_\beta [m_0', m_1', \ldots, m_k'] \), where \( m_i \in S\)-expressions, \( i = 0, \ldots, k, k \geq 0 \).

b) \( \text{Cons}m'(n+1)' \), where \( m \in M, n \in N \), does not have a head normal form.

c) \( \text{Cons}m\Omega \), where \( m \in M \), does not have a head normal form.

d) \( \text{Cons}\Omega[m_1', \ldots, m_k'] \), where \( m_i \in S\)-expressions, \( i = 1, \ldots, k, k \geq 0 \), does not have a head normal form.

(a) There are two possible cases: a1) \( k = 0 \), a2) \( k > 0 \).

(a1) \( \text{Cons}m'_0[ ] = \) 
\[
(\lambda xy.\text{If}(\text{And}(\text{Atom})(\text{Not}(\text{Nully})))\Omega(\text{If}x(\lambda z.\text{zxy})(\lambda z.\text{zxy}))m'_0[ ] \rightarrow_\beta \text{If}(\text{And}(\text{Atom}[1])(\text{Not}(\text{Null}[1])))\Omega(\text{If}m'_0(\lambda z.\text{zm}_0[1])(\lambda z.\text{zm}_0[1])) \rightarrow_\beta \text{If}(\text{And}1'0')\Omega[m'_0] \rightarrow_\beta \text{If}0'\Omega[m'_0] \rightarrow_\beta [m'_0].
\]

(a2) \( \text{Cons}m'_0[m_1', \ldots, m_k'] = (\lambda xy.\text{If}(\text{And}(\text{Atom})(\text{Not}(\text{Nully})))\Omega(\text{If}x(\lambda z.\text{zxy})(\lambda z.\text{zxy}))m'_0[m_1', \ldots, m_k'] \rightarrow_\beta \text{If}(\text{And}(\text{Atom}[m_1', \ldots, m_k'])(\text{Not}(\text{Null}[m_1', \ldots, m_k'])))\Omega(\text{If}m'_0(\lambda z.\text{zm}_0[m_1', \ldots, m_k'])(\lambda z.\text{zm}_0[m_1', \ldots, m_k'])) \rightarrow_\beta \text{If}(\text{And}1'0')\Omega(\text{If}m'_0(\lambda z.\text{zm}_0[m_1', \ldots, m_k'])(\lambda z.\text{zm}_0[m_1', \ldots, m_k'])) \rightarrow_\beta \text{If}0'\Omega[m'_0, m_1', \ldots, m_k'] \rightarrow_\beta [m'_0, m_1', \ldots, m_k'].$

(b) \( \text{Cons}m'(n+1)' = \) 
\[
(\lambda xy.\text{If}(\text{And}(\text{Atom})(\text{Not}(\text{Nully})))\Omega(\text{If}x(\lambda z.\text{zxy})(\lambda z.\text{zxy}))m'(n+1)' \rightarrow_\beta \text{If}(\text{And}(\text{Atom}[n+1])(\text{Not}(\text{Null}[n+1])))\Omega(\text{If}m'(\lambda z.\text{zm}'(n+1)')(\lambda z.\text{zm}'(n+1)')) \rightarrow_\beta \text{If}(\text{And}1'1')\Omega(\text{If}m'(\lambda z.\text{zm}'(n+1)')(\lambda z.\text{zm}'(n+1)')) \rightarrow_\beta \text{If}1'\Omega(\text{If}m'(\lambda z.\text{zm}'(n+1)')(\lambda z.\text{zm}'(n+1)')) \rightarrow_\beta \Omega \quad \text{and the term } \text{Cons}m'(n+1)' \text{ does not have a head normal form.}
\]

(c) \( \text{Cons}m\Omega \equiv (\lambda xy.\text{If}(\text{And}(\text{Atom})(\text{Not}(\text{Nully})))\Omega(\text{If}x(\lambda z.\text{zxy})(\lambda z.\text{zxy}))m' \Omega \rightarrow_\beta \text{If}(\text{And}(\text{Atom}\Omega)(\text{Not}(\text{Null}\Omega)))(\lambda z.\text{zm}'(n+1)'(\lambda z.\text{zm}'(n+1)')) \rightarrow_\beta \text{If}1'\Omega(\text{If}m'(\lambda z.\text{zm}'(n+1)'(\lambda z.\text{zm}'(n+1)'))(\lambda z.\text{zm}'(n+1)')) \rightarrow_\beta \Omega \quad \text{and the term } \text{Cons}m\Omega \text{ does not have a head normal form.}
\]
There are two possible cases: d1) \( k = 0 \), d2) \( k > 0 \).

(d1) \( \text{Cons}\Omega[\ ] \equiv (\text{Zero} \cdot \text{If}(\text{Atom}(\text{Not}(\text{Null}))\text{Null}))\text{Null} \) \( \rightarrow \beta \)

(b) \( \text{Zero}(n + 1)' \rightarrow \beta 0' \).

We define the auxiliary term Zero \( \equiv \lambda x.\text{If}(\text{Atom}(\text{Null}))\text{Null}x\text{Null}\text{Null} \) and show the following:

(a) \( \text{Zero}0' \rightarrow \beta 1' \).

(b) \( \text{Zero}(n + 1)' \rightarrow \beta 0' \).

(c) \( \text{Zero}[m_1', \ldots, m_k'] \), where \( m_i \in \text{S}\text{-expressions}, i = 1, \ldots, k, k \geq 1 \), does not have a head normal form.

(d) \( \text{Zero}\Omega \) does not have a head normal form.
The proof is by induction on \(n\) following: Where \(m\) is a solution of this equation and \(f,x,y\) according to Proposition 2, we have:

\[
\text{Consider the equation:}
\[
\lambda x y. \text{If}(\text{Zero}(x)) (\text{If}(\text{Zero}(y)) \lambda x y. \text{If}(\text{Zero}(x))(\text{If}(\text{Zero}(y)) f(xF)(yF))),
\]

where \(f,x,y \in V\). According to Proposition 3, the term

\[
\text{Eq} = \lambda x y. \text{If}(\text{Zero}(x)) (\text{If}(\text{Zero}(y)) \lambda x y. \text{If}(\text{Zero}(x))(\text{If}(\text{Zero}(y)) f(xF)(yF)))
\]

is a solution of this equation and

\[
\text{Eq} = \text{If}(\text{Zero}(x)) (\text{If}(\text{Zero}(y)) \lambda x y. \text{If}(\text{Zero}(x))(\text{If}(\text{Zero}(y)) f(xF)(yF)))
\]

According to Proposition 2, we have:

\[
\text{Eqm}_1 m_2' = \text{If}(\text{Zero}(x)) (\text{If}(\text{Zero}(y)) \lambda x y. \text{If}(\text{Zero}(x))(\text{If}(\text{Zero}(y)) f(xF)(yF))) m_1' m_2',
\]

where \(m_1, m_2 \in M\).

Let us show that the term Eq \(\lambda\)-defines the function \(eq\). To do this, show the following:

a) Eqm_1 n_2' = \text{If}(\text{Zero}(x)) (\text{If}(\text{Zero}(y)) \lambda x y. \text{If}(\text{Zero}(x))(\text{If}(\text{Zero}(y)) f(xF)(yF))) n_1 = n_2; and Eqm_1 n_2' = \text{If}(\text{Zero}(x)) (\text{If}(\text{Zero}(y)) f(xF)(yF))) = 0', where \(n_1, n_2 \in N\) and \(n_1 \neq n_2\).

b) Eqm_1 m_2', where \(m_1 \in M\) and \(m_1 \notin N\), or \(m_2 \in M\) and \(m_2 \notin N\), does not have a head normal form.

(a) The proof is by induction on \(n_2\).

Let \(n_2 = 0\), we show that i) Eq0' = \text{If}(\text{Zero}(x)) (\text{If}(\text{Zero}(y)) f(xF)(yF))) = \lambda x y. \text{If}(\text{Zero}(x))(\text{If}(\text{Zero}(y)) f(xF)(yF))) = 0' and ii) Eqn_1' = \text{If}(\text{Zero}(x)) (\text{If}(\text{Zero}(y)) f(xF)(yF))) = 0' if \(n_1 > 0\).

i) Eq0' = \text{If}(\text{Zero}(x)) (\text{If}(\text{Zero}(y)) f(xF)(yF))) = \lambda x y. \text{If}(\text{Zero}(x))(\text{If}(\text{Zero}(y)) f(xF)(yF))) = 0' = \text{If}(\text{Zero}(x)) f(xF)(yF))) = 0' = \text{If}(\text{Zero}(x)) (\text{If}(\text{Zero}(y)) f(xF)(yF))) = 0'.

ii) Eqn_1' = \text{If}(\text{Zero}(x)) (\text{If}(\text{Zero}(y)) f(xF)(yF))) = \lambda x y. \text{If}(\text{Zero}(x))(\text{If}(\text{Zero}(y)) f(xF)(yF))) = 0' = \text{If}(\text{Zero}(x)) f(xF)(yF))) = 0' = \text{If}(\text{Zero}(x)) (\text{If}(\text{Zero}(y)) f(xF)(yF))) = 0'.

Let \(n_2 > 0\). Assuming that the assertion is true for \(n_2 = 1\), we prove it for \(n_2\). We show that i) Eqn_1' n_2' = \text{If}(\text{Zero}(x)) (\text{If}(\text{Zero}(y)) f(xF)(yF))) = 0' and ii) Eqn_1' n_2' = \text{If}(\text{Zero}(x)) (\text{If}(\text{Zero}(y)) f(xF)(yF))) = 0' if \(n_1 > 0\) and \(n_1 \neq n_2\); Eqn_1' n_2' = \text{If}(\text{Zero}(x)) (\text{If}(\text{Zero}(y)) f(xF)(yF))) = 0' if \(n_1 > 0\) and \(n_1 = n_2\).
(b) There are two possible cases: b1) \( m_1 \in \mathbb{M} \) and \( m_1 \notin \mathbb{N} \), b2) \( m_1 \in \mathbb{N} \), \( m_2 \in \mathbb{M} \) and \( m_2 \notin \mathbb{N} \).

(b1) \( \text{Eqm}_1' m_2' = \beta \)
\[
(\lambda x.\text{If}(\text{Zero}\text{r})(\text{If}(\text{Zero}\text{r})1'0')(\text{If}(\text{Zero}\text{r})0'(\text{Eq}(\text{xF})(\text{yF}))))m_1'm_2' \rightarrow \beta
\]
\[
\text{Eq}(\text{Zero}\text{r})'0'(\text{If}(\text{Zero}\text{r})'1'0')(\text{If}(\text{Zero}\text{r})'0'(\text{Eq}(\text{m}_1'\text{F})(\text{m}_2'\text{F})))) \rightarrow \beta
\]
\[
\text{null}(\text{Zero}\text{r})'0'(\text{If}(\text{Zero}\text{r})'0'(\text{Eq}(\text{m}_1'\text{F})(\text{m}_2'\text{F})))) \rightarrow \beta
\]
\[
(\lambda x.y.z.\text{Tyz}(\text{Zerom}_1')(\text{If}(\text{Zerom}_2')1'0'))
\]
\[
(\text{null}(\text{Zerom}_2')0'(\text{Eq}(\text{m}_1'\text{F})(\text{m}_2'\text{F})))) \rightarrow \beta
\]
\[
(\text{null}(\text{Zerom}_2')0'(\text{Eq}(\text{m}_1'\text{F})(\text{m}_2'\text{F})))) \rightarrow \beta
\]
\[
(\text{Zerom}_1'(\lambda x.y.z.\text{Tyz}(\text{Zerom}_2')1'0'))
\]

and, according to Proposition 1, the term \( \text{Eqm}_1' m_2' \) does not have a head normal form.

(b2) We show that i) \( \text{Eq}0'm_2' \) and ii) \( \text{Eq}(n+1)'m_2' \), where \( n \in \mathbb{N} \), do not have a head normal form.

(i) \( \text{Eq}0'm_2' = \beta \)
\[
(\lambda x.\text{If}(\text{Zero}\text{r})(\text{If}(\text{Zero}\text{r})1'0')(\text{If}(\text{Zero}\text{r})0'(\text{Eq}(\text{xF})(\text{yF}))))0'm_2' \rightarrow \beta
\]
\[
(\text{Zerom}_2')1'0'(\text{If}(\text{Zerom}_2')0'(\text{Eq}(\text{m}_2'\text{F})))) \rightarrow \beta
\]
\[
(\text{null}(\text{Zerom}_2')T1'0' = (\lambda x.\text{Tyz}(\text{Zerom}_2')1'0' \rightarrow \beta
\]
\[
(\text{Zerom}_2')(\lambda x.y.z.\text{Tyz}(\text{Zerom}_2')1'0' \rightarrow \beta
\]

and, according to Proposition 1, the term \( \text{Eq}0'm_2' \) does not have a head normal form.
(\lambda xyz.(Nullx)Tyz)(Zero_{m^2'})(Eq((n+1)'F)(m_2'F)) \rightarrow \beta
Null(Zero_{m^2'})(Eq((n+1)'F)(m_2'F)) \equiv
(\lambda x.(\lambda xyz.z)T0'(Eq((n+1)'F)(m_2'F))\rightarrow \beta
(Zero_{m^2'})(\lambda xyz.z)T0'T0'(Eq((n+1)'F)(m_2'F)) and, according to Proposition 1, the term \( Eq(n+1)'m_2' \) does not have a head normal form.

\[
\begin{align*}
R E F E R E N C E S
\end{align*}
\]