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On weighted Dirichlet spaces of monogenic functions in $\mathbb{R}^3$

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In a recent paper of ours, we proved that a special "harmonic conjugation" operator is not bounded in weighted Bergman spaces L^2_α of quaternion-valued functions in the 3D ball. In the present paper, we prove that, in contrast to the Bergman spaces case, the same operator is bounded in weighted Dirichlet spaces $\mathcal{D}^2_\alpha$ of quaternion-valued functions in the 3D ball. Furthermore, applying another approach for a construction of harmonic conjugates, we extend the result to weighted Dirichlet spaces $\mathcal{D}^p_\alpha$ with $1 < p < \infty$.

KEYWORDS

Bergman space, Dirichlet space, harmonic conjugates, monogenic function, reduced quaternion

1 | INTRODUCTION

In theorem 4.4 of Morais et al,[1] we have proved that a "harmonic conjugation" operator constructed by special systems of homogeneous harmonic and monogenic polynomials is not bounded in weighted Bergman spaces L^2_α of reduced quaternion-valued functions in the 3D ball. In the present paper, we prove that, in contrast to the Bergman spaces case, the same operator is bounded in weighted Dirichlet spaces $\mathcal{D}^2_\alpha$ of reduced quaternion-valued functions in the 3D ball.

Let $\mathbb{D} := \{z = x + iy \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane and $H(\mathbb{D})$ be the collection of holomorphic functions on $\mathbb{D}$. Classical weighted Dirichlet space $\mathcal{D}^2_\alpha(\mathbb{D})$, $\alpha > -1$, of holomorphic functions on the disk $\mathbb{D}$ is defined by

$$\mathcal{D}^2_\alpha(\mathbb{D}) := \left\{ f(z) \in H(\mathbb{D}) : \|f\|^2_{\mathcal{D}^2_\alpha(\mathbb{D})} := \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^\alpha dA(z) < +\infty \right\},$$

where $dA(z) = \frac{1}{\pi} dxdy$ is the normalized area measure. The space $\mathcal{D}^2_\alpha(\mathbb{D})$ is Banach and equipped with the norm $|f(0)| + \|f\|_{\mathcal{D}^2_\alpha}$. If a holomorphic function is expressed by its Taylor series $f(z) = \sum_{n=0}^\infty a_n z^n$ in $\mathbb{D}$, we can rewrite the seminorm $\|\cdot\|_{\mathcal{D}^2_\alpha}$ in the equivalent form

$$\|f\|^2_{\mathcal{D}^2_\alpha} = \sum_{n=1}^\infty n^2 |a_n|^2 \frac{\Gamma(n)\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} \approx \sum_{n=1}^\infty n^{1-\alpha} |a_n|^2 \quad (\alpha > -1). \tag{1}$$

The notation $A \approx B$ for some $A, B > 0$ denotes the two-sided estimate $c_1 A \leq B \leq c_2 A$ with some positive constants c_1 and c_2 independent of the variable involved.

The weighted (Hilbert) Lebesgue space $L^2_\alpha(\mathbb{D})$, $\alpha > -1$, is defined by

$$L^2_\alpha(\mathbb{D}) := \left\{ f(z) \text{ measurable in } \mathbb{D} : \|f\|^2_{L^2_\alpha(\mathbb{D})} := \int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^\alpha dA(z) < +\infty \right\}.$$

*On weighted Dirichlet spaces.

Weighted Bergman spaces of holomorphic functions are denoted by $H^2_\alpha(\mathbb{D}) := H(\mathbb{D}) \cap L^2_\alpha(\mathbb{D})$. Evidently, $\|f\|_{\mathcal{D}^2_\alpha} = \|f'\|_{L^2_\alpha}$. As is well known, the Dirichlet space $\mathcal{D}^2_1(\mathbb{D})$ coincides with $H^2(\mathbb{D})$, the classical Hardy space in $\mathbb{D}$, while the general Dirichlet space $\mathcal{D}^2_\alpha(\mathbb{D})$, $\alpha > 1$, coincides with the weighted Bergman space $H^2_{\alpha-2}(\mathbb{D})$ of holomorphic functions in $\mathbb{D}$.

For more information on the weighted Bergman and Dirichlet spaces on the disk, we refer to literature.[2-6] Weighted Dirichlet spaces of harmonic functions on the unit ball of $\mathbb{R}^n$ are also well known, see, eg, previous studies.[7-11] Let $B_n := \{x \in \mathbb{R}^n : |x| < 1\}$ be the open unit ball in $\mathbb{R}^n$ and $h(B_n)$ be the collection of harmonic functions on B_n. Weighted Dirichlet space $h\mathcal{D}^2_\alpha(B_n)$, $\alpha > -1$, of harmonic functions in B_n is

$$h\mathcal{D}^2_\alpha(B_n) := \left\{ u(x) \in h(B_n) : \|u\|^2_{\mathcal{D}^2_\alpha(B_n)} := \int_{B_n} |\nabla u(x)|^2 (1-|x|^2)^\alpha \, dV(x) < +\infty \right\},$$

where $dV(x)$ stands for the normalized Lebesgue volume measure on B_n. The space $h\mathcal{D}^2_\alpha(B_n)$ is Banach and equipped with the norm $|u(0)| + \|u\|_{\mathcal{D}^2_\alpha}$. Let $u(x) = \sum_{k=0}^\infty p_k(x)$ be the homogeneous expansion of a harmonic function $u(x)$, where p_k is a spherical harmonic of degree k, see, eg, Axler et al.[12] Then we can rewrite the seminorm $\|\cdot\|_{\mathcal{D}^2_\alpha}$ in the equivalent form analogous to (1), see Bernstein et al,[11(thm.2.6)]

$$\|u\|^2_{\mathcal{D}^2_\alpha} \approx \sum_{k=1}^\infty k^{1-\alpha} \|p_k\|^2_{L^2(S,d\sigma)} \qquad (\alpha > -1), \tag{2}$$

where $d\sigma$ is the normalized surface measure on the unit sphere $S = \partial B_n$.

Similarly, the harmonic Dirichlet space $h\mathcal{D}^2_1(B_n)$ coincides with $h^2(B_n)$, the harmonic Hardy space in B_n, while the general harmonic Dirichlet space $h\mathcal{D}^2_\alpha(B_n)$, $\alpha > 1$, coincides with $h^2_{\alpha-2}(B_n)$, the weighted Bergman space of harmonic functions in B_n,

$$h\mathcal{D}^2_1(B_n) = h^2(B_n), \quad h\mathcal{D}^2_\alpha(B_n) = h^2_{\alpha-2}(B_n), \quad \alpha > 1. \tag{3}$$

In $\mathbb{R}^3$, holomorphic functions can be replaced by monogenic ones. It would be of interest to study weighted Dirichlet spaces of monogenic functions in $\mathbb{R}^3$ with values in reduced quaternions, see previous works[11,13,14] for Dirichlet spaces of quaternion- or Clifford-valued functions. For general theory of quaternionic and Clifford analysis, we refer to previous studies.[15-18]

In the present paper, we are interested in the problem of harmonic conjugation in the framework of quaternionic analysis. Beginning from the paper of Sudbery,[19] the problem of harmonic conjugates in the framework of quaternionic and Clifford analysis was studied by several authors, see literature. [1,15,20-28] In Avetisyan et al,[26] for functions with values in full quaternions, we mainly applied Sudbery's formula[19] for construction of harmonic conjugates. However, since our interest in this paper is the study of monogenic functions with values in the reduced quaternions, Sudbery's formula is not well adapted to our case.

The collection of monogenic functions on the unit ball $B = B_3$ of $\mathbb{R}^3$ will be denoted by $\mathcal{M}(B)$ or $\ker D$, see Section 2. By analogy, we define monogenic Dirichlet spaces $\mathcal{D}^2_\alpha(B)$, $\alpha > -1$, on the 3D unit ball B, as well as monogenic Bergman spaces $\mathcal{H}^2_\alpha(B) = \mathcal{M}(B) \cap L^2_\alpha(B)$ and the monogenic Hardy space $\mathcal{H}^2(B) = \mathcal{M}(B) \cap h^2(B)$. Their definitions are in Section 2.

The first important result of this paper asserts that a special harmonic conjugation operator is bounded in weighted harmonic Dirichlet spaces $h\mathcal{D}^2_\alpha(B)$ on the 3D ball.

Theorem 1. *Let U be a scalar-valued square-integrable harmonic function in B. If $U \in h\mathcal{D}^2_\alpha(B)$ for some $\alpha > -1$, then there exists a monogenic function $\mathbf{f} \in \mathcal{D}^2_\alpha(B)$ such that $\mathbf{Sc}f = U$ in B and*

$$\|\mathbf{f}\|_{\mathcal{D}^2_\alpha} \le C_\alpha \|U\|_{\mathcal{D}^2_\alpha}.$$

*In other words, the Dirichlet $\mathcal{D}^2_\alpha$-seminorms of this monogenic function **f** and its scalar part are equivalent,*

$$\|\mathbf{f}\|_{\mathcal{D}^2_\alpha} \approx \|U\|_{\mathcal{D}^2_\alpha}. \tag{4}$$

Unless otherwise stated, throughout this paper, the letters $C(\alpha, \beta, \dots), C_\alpha$ etc stand for positive different constants depending only on the parameters indicated not necessarily the same in each instance.

Our second important theorem (Theorem 3) concerns the relations of type (3). Unexpectedly, it turns out that the relations of type (3) are no longer true for the monogenic Dirichlet spaces $\mathcal{D}_\alpha^2(B)$. Moreover, in contrast to the classical cases, no one monogenic Dirichlet space coincides with the Hardy or the weighted Bergman space. The precise formulations are in the next sections.

In the final section of the paper, we extend Theorem 1 to the Dirichlet spaces $\mathcal{D}_\alpha^p(B)$ with the range $1 < p < \infty$, see Theorem 5, though in that case, we construct a monogenic function by its scalar part by a quite different method.

2 | NOTATION AND DEFINITIONS

Let $\mathbb{H} := \{\mathbf{x} = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k},\ x_0, x_1, x_2, x_3 \in \mathbb{R}\}$ be the real quaternion algebra with the basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ where the imaginary units $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ are subject to the multiplication rules:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \quad \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \quad \mathbf{ki} = \mathbf{j} = -\mathbf{ik}.$$

Evidently, the real vector space $\mathbb{R}^4$ may be embedded in $\mathbb{H}$ by identifying the element $x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4$ with $\mathbf{x} = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in \mathbb{H}$. Consider the subset

$$\mathcal{A} := \text{span}_{\mathbb{R}}\{\mathbf{1}, \mathbf{i}, \mathbf{j}\} \subset \mathbb{H},$$

then the real vector space $\mathbb{R}^3$ may be embedded in $\mathcal{A}$ via the identification of $x := (x_0, x_1, x_2) = (x_0, \underline{x}) \in \mathbb{R}^3$ with the reduced quaternion $\mathbf{x} := x_0 + x_1\mathbf{i} + x_2\mathbf{j} \in \mathcal{A}$. It should be noted, however, that $\mathcal{A}$ is a real vectorial subspace, but not a subalgebra of $\mathbb{H}$. Like in the complex case, $\mathbf{Sc}(\mathbf{x}) = x_0$ and $\mathbf{Vec}(\mathbf{x}) = x_1\mathbf{i} + x_2\mathbf{j}$ define the scalar and vector parts of $\mathbf{x}$. The conjugate of $\mathbf{x}$ is the reduced quaternion $\overline{\mathbf{x}} = x_0 - x_1\mathbf{i} - x_2\mathbf{j}$, and the norm $|\mathbf{x}|$ of $\mathbf{x}$ is defined by $|\mathbf{x}| = \sqrt{\mathbf{x}\overline{\mathbf{x}}} = \sqrt{\overline{\mathbf{x}}\mathbf{x}} = \sqrt{x_0^2 + x_1^2 + x_2^2}$, and it coincides with its corresponding Euclidean norm as a vector in $\mathbb{R}^3$.

Throughout the paper, only functions with values in $\mathcal{A}$ are considered,

$$\mathbf{f} : B \to \mathcal{A}, \quad B \subset \mathbb{R}^3, \quad \mathbf{f}(x) = U(x) + V_1(x)\mathbf{i} + V_2(x)\mathbf{j},$$

that is, reduced quaternion-valued functions, or $\mathcal{A}$-valued functions for short, where U, V_1, V_2 are real-valued functions defined in B. Properties (like integrability, continuity, or differentiability) of $\mathbf{f}$ are defined componentwise. For a real-differentiable $\mathcal{A}$-valued function $\mathbf{f}$ that has continuous first partial derivatives, the (reduced) quaternionic operators

$$D\mathbf{f} = \frac{\partial \mathbf{f}}{\partial x_0} + \mathbf{i}\frac{\partial \mathbf{f}}{\partial x_1} + \mathbf{j}\frac{\partial \mathbf{f}}{\partial x_2} \quad \text{and} \quad \overline{D}\mathbf{f} = \frac{\partial \mathbf{f}}{\partial x_0} - \mathbf{i}\frac{\partial \mathbf{f}}{\partial x_1} - \mathbf{j}\frac{\partial \mathbf{f}}{\partial x_2}$$

are called, respectively, generalized and conjugate generalized Cauchy-Riemann operators on $\mathbb{R}^3$.

For a continuously real-differentiable scalar-valued function, the operator D coincides with the usual gradient ∇. A continuously real-differentiable $\mathcal{A}$-valued function $\mathbf{f}$ is said to be monogenic if $D\mathbf{f} = 0$, which is equivalent to the system

$$(R) \qquad \begin{cases} \frac{\partial U}{\partial x_0} - \frac{\partial V_1}{\partial x_1} - \frac{\partial V_2}{\partial x_2} = 0 \\ \frac{\partial U}{\partial x_1} + \frac{\partial V_1}{\partial x_0} = 0, \quad \frac{\partial U}{\partial x_2} + \frac{\partial V_2}{\partial x_0} = 0, \quad \frac{\partial V_1}{\partial x_2} - \frac{\partial V_2}{\partial x_1} = 0. \end{cases}$$

Any monogenic $\mathcal{A}$-valued function is two-sided monogenic. This means that it satisfies simultaneously the equations $D\mathbf{f} = \mathbf{f}D = 0$. The same condition $D\mathbf{f} = 0$ defines (left) monogenic $\mathbb{H}$-valued functions.

We may point out that the 3-tuple $\overline{\mathbf{f}}$ is said to be a system of conjugate harmonic functions in the sense of Stein and Weiss,[29] and system (R) is called the Riesz system. Following Leutwiler,[30] the solutions of the system (R) are customary called (R)-solutions. The subspace of polynomial (R)-solutions of degree n will be denoted by $\mathcal{R}^+(B; \mathcal{A}; n)$. In Leutwiler,[30] it is shown that the space $\mathcal{R}^+(B; \mathcal{A}; n)$ has dimension $2n + 3$. We further introduce the real-linear Hilbert space of square-integrable $\mathcal{A}$-valued functions defined in B, which we denote by $L^2(B; \mathcal{A}; \mathbb{R})$. Also, $\mathcal{R}^+(B) = \mathcal{R}^+(B; \mathcal{A}) := L^2(B; \mathcal{A}; \mathbb{R}) \cap \ker D$ will denote the space of square-integrable $\mathcal{A}$-valued monogenic functions defined in B.

In the next section, we review a suitable set of special monogenic polynomials, which forms a complete orthogonal system in $\mathcal{R}^{+}(B;\mathcal{A})$ in the sense of the scalar inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle_{L^2(B;\mathcal{A};\mathbb{R})} := \int_B \mathbf{Sc}(\overline{\mathbf{f}}\,\mathbf{g})\, dx. \tag{5}$$

For a function $\mathbf{f}(x) = \mathbf{f}(r\zeta)$ in $B\,(0 \leq r < 1, \zeta \in S)$, its integral means of order $p \geq 1$ are defined by

$$M_p(\mathbf{f}; r) := \left(\int_S |\mathbf{f}(r\zeta)|^p \, d\sigma(\zeta) \right)^{1/p} = \left(\frac{1}{|S_r|} \int_{|x|=r} |\mathbf{f}(x)|^p \, ds(x) \right)^{1/p}, \quad 0 < r < 1,$$

where $|S_r| = 4\pi r^2$ is the surface area of the sphere $S_r = \{x \in \mathbb{R}^n : |x| = r\}$ and $d\sigma = ds/4\pi$ is the surface area measure on the unit sphere $S = \partial B$ normalized so that $\sigma(S) = 1$. We will also denote by $h(B;\mathbb{X})$ the set of harmonic functions on B with values in $\mathbb{X}$ ($\mathbb{X} = \mathbb{R}$ or $\mathcal{A}$ or $\mathbb{H}$). The Hardy spaces of monogenic or harmonic functions are defined as follows:

$$h^p(B) = \left\{ u \in h(B;\mathbb{X}) : \|u\|_{h^p(B)} < +\infty \right\},$$
$$\mathcal{H}^p(B) = h^p(B) \cap \ker D.$$

The norm in the Hardy space of $\mathbf{f}$ in B is defined by

$$\|\mathbf{f}\|_{h^p(B)} := \sup_{0<r<1} M_p(\mathbf{f}; r).$$

As shown in Gürlebeck and Malonek,[31] $(\frac{1}{2}\overline{D})\mathbf{f}$ defines a derivative of monogenic functions by analogy with the complex one-dimensional case.

Definition 1. (see Sudbery[19] and Gürlebeck and Malonek[31]). Let $\mathbf{f}$ be a continuously real-differentiable $\mathcal{A}$-valued monogenic function; $(\frac{1}{2}\overline{D})\mathbf{f}$ is called hypercomplex derivative of $\mathbf{f}$.

Definition 2. An $\mathcal{A}$-valued monogenic function with an identically vanishing hypercomplex derivative is called a hyperholomorphic constant.

We set the weighted Bergman space of $\mathbf{f}$ on B by

$$L^p_\alpha(B) := \left\{ \mathbf{f} \text{ measurable in } B : \|\mathbf{f}\|^p_{L^p_\alpha(B)} := \int_B |\mathbf{f}(x)|^p (1 - |x|^2)^\alpha \, dV(x) < \infty \right\}, \quad 1 \leq p < \infty, \ \alpha > -1.$$

Let the subspaces of $L^p_\alpha(B)$ consisting of harmonic or monogenic functions be

$$h^p_\alpha(B) = L^p_\alpha(B) \cap h(B) \quad \text{and} \quad \mathcal{H}^p_\alpha(B) = L^p_\alpha(B) \cap \ker D.$$

In polar coordinates, we have $dV(x) = 3r^2 dr d\sigma(\zeta)$. Therefore,

$$\|\mathbf{f}\|_{L^p_\alpha(B)} = \left(3 \int_0^1 M^p_p(\mathbf{f}; r)(1 - r^2)^\alpha \, r^2 \, dr \right)^{1/p}.$$

We set weighted Dirichlet spaces of monogenic or harmonic functions by

$$\mathcal{D}^p_\alpha(B) := \left\{ \mathbf{f} : \overline{D}\mathbf{f} \in L^p_\alpha(B) \right\}, \qquad 1 \leq p < \infty, \ \alpha > -1,$$

under the seminorm

$$\|\mathbf{f}\|_{\mathcal{D}^p_\alpha(B)} := \|\overline{D}\mathbf{f}\|_{L^p_\alpha(B)} = \left(\int_B |\overline{D}\mathbf{f}(x)|^p (1 - |x|^2)^\alpha \, dV(x) \right)^{1/p}.$$

3 | SPECIAL SYSTEMS OF HOMOGENEOUS HARMONIC AND MONOGENIC POLYNOMIALS

The material of this section is mainly inspired by the works of Cação[32,33] and Morais and Gürlebeck.[34] The treatment is only introductory since we have not attempted to cover all the ongoing research. For more detailed information, we refer to Gürlebeck and Morais[27] and Morais.[35]

The following constructions are based on the introduction of a standard system of spherical harmonics as considered, eg, in Sansone.[36] We use spherical coordinates, $x_0 = r\cos\theta, x_1 = r\sin\theta\cos\varphi,\ x_2 = r\sin\theta\sin\varphi$, where $0 \le r < \infty, 0 \le \theta \le \pi$, and $0 \le \varphi < 2\pi$. We start by recalling a suitable set of homogeneous harmonic polynomials,

$$\{r^{n+1}U^l_{n+1}, r^{n+1}V^m_{n+1},\ l = 0, 1, \dots, n+1,\ m = 1, \dots, n+1\}_{n\in\mathbb{N}_0} \tag{6}$$

($\mathbb{N}_0 = \{0, 1, 2, \dots\}$) formed by the extensions in the ball of the spherical harmonics

$$U^l_{n+1}(\theta,\varphi) = P^l_{n+1}(\cos\theta)\cos(l\varphi) \quad (l = 0, \dots, n+1),$$
$$V^m_{n+1}(\theta,\varphi) = P^m_{n+1}(\cos\theta)\sin(m\varphi) \quad (m = 1, \dots, n+1).$$

Here, P_{n+1} stands for the Legendre polynomial of degree $n+1$ and the functions P^l_{n+1}, where $l = 0, \dots, n+1$ are the associated Legendre functions. In Cação[32] and Cação et al,[37] a special $\mathbb{R}$-linear complete orthonormal system of $\mathcal{A}$-valued monogenic polynomials in the unit ball of $\mathbb{R}^3$ is explicitly constructed by applying the operator $\frac{1}{2}\overline{D}$ to the system (6). Restricting the resulting solid spherical monogenics to the surface of B, we do obtain a system of spherical monogenics, denoted by

$$\{\mathbf{X}^l_n, \mathbf{Y}^m_n :\ l = 0, \dots, n+1,\ m = 1, \dots, n+1\}_{n\in\mathbb{N}_0}.$$

For the fundamental references for the preceding arguments and explicit expressions of these special polynomials, see Gürlebeck,[28] Morais,[35] and Morais and Gürlebeck.[34] We recall from previous studies[27,32,34,38] the following properties:

1. The functions $\mathbf{X}^{l,\dagger}_n := r^n\mathbf{X}^l_n$ and $\mathbf{Y}^{m,\dagger}_n := r^n\mathbf{Y}^m_n$ are homogeneous monogenic polynomials.
2. For each $n = 0, 1, \dots$, the polynomials $\mathbf{X}^{l,\dagger}_n$ $(l = 0, \dots, n+1)$ and $\mathbf{Y}^{m,\dagger}_n$ $(m = 1, \dots, n+1)$ form a complete orthogonal system in $\mathcal{R}^+(B;\mathcal{A})$, and their norms are explicitly given by

$$\begin{aligned}\left\|\mathbf{X}^{l,\dagger}_n\right\|^2_{L^2(B;\mathcal{A};\mathbb{R})} &= (1+\delta_{l,0})\frac{\pi}{2}\frac{(n+1)}{(2n+3)}\frac{(n+1+l)!}{(n+1-l)!},\\ \left\|\mathbf{Y}^{m,\dagger}_n\right\|^2_{L^2(B;\mathcal{A};\mathbb{R})} &= \frac{\pi}{2}\frac{(n+1)}{(2n+3)}\frac{(n+1+m)!}{(n+1-m)!},\end{aligned} \tag{7}$$

   where $\delta_{l,0}$ denotes the Kronecker delta.
3. For each $n = 0, 1, \dots$, the scalar parts of the polynomials $\mathbf{X}^{l,\dagger}_n$ and $\mathbf{Y}^{m,\dagger}_n$ form a complete orthogonal system in $L^2(B)$, and their norms are explicitly given by

$$\begin{aligned}\left\|\mathbf{Sc}(\mathbf{X}^{l,\dagger}_n)\right\|^2_{L^2(B)} &= \frac{(n+1+l)^2}{2n+3}(1+\delta_{l,0})\frac{\pi}{2}\frac{1}{(2n+1)}\frac{(n+l)!}{(n-l)!},\\ \left\|\mathbf{Sc}(\mathbf{Y}^{m,\dagger}_n)\right\|^2_{L^2(B)} &= \frac{(n+1+m)^2}{2n+3}\frac{\pi}{2}\frac{1}{(2n+1)}\frac{(n+m)!}{(n-m)!}.\end{aligned} \tag{8}$$

4. For $n \ge 1$, we have

$$\begin{aligned}\left(\frac{1}{2}\overline{D}\right)\mathbf{X}^{l,\dagger}_n &= (n+l+1)\mathbf{X}^{l,\dagger}_{n-1} \quad (l = 0, \dots, n),\\ \left(\frac{1}{2}\overline{D}\right)\mathbf{Y}^{m,\dagger}_n &= (n+m+1)\mathbf{Y}^{m,\dagger}_{n-1} \quad (m = 1, \dots, n),\end{aligned} \tag{9}$$

   ie, the hypercomplex differentiation of a basis function delivers a multiple of another basis function one degree lower.
5. The polynomials $\mathbf{X}^{n+1,\dagger}_n$ and $\mathbf{Y}^{n+1,\dagger}_n$ are hyperholomorphic constants.

6. Each $\mathbf{f} \in \mathcal{R}^+(B;\mathcal{A})$ can be decomposed in an orthogonal sum of a monogenic "main part" $\mathbf{g}$ and a hyperholomorphic constant $\mathbf{h}$:

$$\mathbf{f}(x) = \mathbf{f}(0) + \mathbf{g}(x) + \mathbf{h}(\underline{x}), \tag{10}$$

where the functions $\mathbf{g}$ and $\mathbf{h}$ have the Fourier series ($\alpha_n^0,\ \alpha_n^m,\ \beta_n^m \in \mathbb{R},\ 1 \le m \le n+1$)

$$\mathbf{g}(x) = \sum_{n=1}^{\infty}\left[\frac{\mathbf{X}_n^{0,\dagger}}{\|\mathbf{X}_n^{0,\dagger}\|_{L^2(B)}}\alpha_n^0 + \sum_{m=1}^{n}\left(\frac{\mathbf{X}_n^{m,\dagger}}{\|\mathbf{X}_n^{m,\dagger}\|_{L^2(B)}}\alpha_n^m + \frac{\mathbf{Y}_n^{m,\dagger}}{\|\mathbf{Y}_n^{m,\dagger}\|_{L^2(B)}}\beta_n^m\right)\right],$$

$$\mathbf{h}(\underline{x}) = \sum_{n=1}^{\infty}\left(\frac{\mathbf{X}_n^{n+1,\dagger}}{\|\mathbf{X}_n^{n+1,\dagger}\|_{L^2(B)}}\alpha_n^{n+1} + \frac{\mathbf{Y}_n^{n+1,\dagger}}{\|\mathbf{Y}_n^{n+1,\dagger}\|_{L^2(B)}}\beta_n^{n+1}\right).$$

4 | GENERATION OF A-VALUED MONOGENIC FUNCTIONS BY CONJUGATE HARMONICS

We begin by recalling the notion of harmonic conjugates in the context of quaternionic analysis.

Definition 3. (Conjugate harmonic functions). Let U be a harmonic function defined in an open subset Ω of $\mathbb{R}^3$. A vector-valued harmonic function V in Ω is called conjugate harmonic to U if $\mathbf{f} := U + V$ is monogenic in Ω. The pair $(U;V)$ is called a pair of conjugate harmonic functions in Ω.

We recall from Gürlebeck and Morais[27] an algorithm for the calculation of any $f \in \mathcal{R}^+(B;\mathcal{A})$ via conjugate harmonics.

Theorem 2. (Gürlebeck and Morais;[27] Construction of a harmonic conjugate). *Let $U(x)$ be harmonic and square integrable in B given by*

$$U = \sum_{n=0}^{\infty}\left[\frac{\boldsymbol{Sc}(\mathbf{X}_n^{0,\dagger})}{\|\boldsymbol{Sc}(\mathbf{X}_n^{0,\dagger})\|_{L^2(B)}}a_n^0 + \sum_{m=1}^{n}\left(\frac{\boldsymbol{Sc}(\mathbf{X}_n^{m,\dagger})}{\|\boldsymbol{Sc}(\mathbf{X}_n^{m,\dagger})\|_{L^2(B)}}a_n^m + \frac{\boldsymbol{Sc}(\mathbf{Y}_n^{m,\dagger})}{\|\boldsymbol{Sc}(\mathbf{Y}_n^{m,\dagger})\|_{L^2(B)}}b_n^m\right)\right], \tag{11}$$

where for each $n \in \mathbb{N}_0$, a_n^l $(0 \le l \le n)$ and b_n^m $(1 \le m \le n)$ are the associated Fourier coefficients. If, additionally, the Fourier coefficients satisfy the condition

$$\sum_{n=0}^{\infty}\left(\frac{2n+1}{n+1}(a_n^0)^2 + \sum_{m=1}^{n}\frac{(n+1)(2n+1)}{(n+1)^2-m^2}\left[(a_n^m)^2 + (b_n^m)^2\right]\right) < \infty, \tag{12}$$

then the series

$$V := \sum_{n=0}^{\infty}\left[\frac{\boldsymbol{Vec}(\mathbf{X}_n^{0,\dagger})}{\|\boldsymbol{Sc}(\mathbf{X}_n^{0,\dagger})\|_{L^2(B)}}a_n^0 + \sum_{m=1}^{n}\left(\frac{\boldsymbol{Vec}(\mathbf{X}_n^{m,\dagger})}{\|\boldsymbol{Sc}(\mathbf{X}_n^{m,\dagger})\|_{L^2(B)}}a_n^m + \frac{\boldsymbol{Vec}(\mathbf{Y}_n^{m,\dagger})}{\|\boldsymbol{Sc}(\mathbf{Y}_n^{m,\dagger})\|_{L^2(B)}}b_n^m\right)\right] \tag{13}$$

is convergent and defines a square-integrable vector-valued harmonic function V conjugate to U such that $\boldsymbol{f}(x) := U(x) + V(x)$ is an $\mathcal{A}$-valued monogenic function.

Observe that by adding any hyperholomorphic constant $\boldsymbol{\varphi}$ to V, the resulting function $\tilde{V} := V + \boldsymbol{\varphi}$ is harmonic conjugate to U also. On the other hand, each monogenic $\mathcal{A}$-valued function with vanishing scalar part must be a hyperholomorphic constant.

By the direct construction of formula (13), we only get $2n+1$ homogeneous monogenic polynomials of degree n (ie, the monogenic "main part" of $\mathbf{f}$). However, since $\dim \mathcal{R}^+(B;\mathcal{A};n) = 2n+3$, adding two hyperholomorphic constants, the necessary number of independent polynomials is achieved. Such a result will allow the definition of a continuous operator between spaces of harmonic and monogenic functions given by the construction of harmonic conjugates.

We will consider the "main parts" of monogenic functions in $\mathcal{R}^+(B;\mathcal{A})$ as a separate (orthogonal) subspace.

Definition 4. Let $\mathbf{f}$ be a square-integrable $\mathcal{A}$-valued monogenic functions defined in B, ie, $\mathbf{f} \in \mathcal{R}^+(B;\mathcal{A})$. We will write $\mathbf{f} \in \mathcal{R}^+_m(B;\mathcal{A})$, if $\mathbf{f}$ is represented in the form

$$\mathbf{f}(x) = \sum_{n=1}^{\infty} \mathbf{P}_n(x), \quad \text{with} \tag{14}$$

$$\mathbf{P}_n(x) = \frac{\mathbf{X}_n^{0,\dagger}}{\|\mathbf{Sc}(\mathbf{X}_n^{0,\dagger})\|_{L^2(B)}} a_n^0 + \sum_{m=1}^{n} \left(\frac{\mathbf{X}_n^{m,\dagger}}{\|\mathbf{Sc}(\mathbf{X}_n^{m,\dagger})\|_{L^2(B)}} a_n^m + \frac{\mathbf{Y}_n^{m,\dagger}}{\|\mathbf{Sc}(\mathbf{Y}_n^{m,\dagger})\|_{L^2(B)}} b_n^m \right). \tag{15}$$

In fact, we drop hyperholomorphic constants $\mathbf{h}$ and $\mathbf{f}(0)$ in the decomposition (10) of the monogenic function $\mathbf{f}$.

5 | PROOF OF THEOREM 1

Before the proof of Theorem 1, we need some preparations. We compute the norms and equivalent norms in various spaces of monogenic and harmonic functions.

Lemma 1. (Norm in the monogenic Hardy space $\mathcal{H}^2$). *Suppose that* $\mathbf{f}$ *is an arbitrary* $\mathcal{A}$*-valued monogenic function on* B*, having homogeneous expansion (14)-(15). If* $\mathbf{f} \in \mathcal{H}^2(B)$*, then*

$$\|\mathbf{f}\|^2_{\mathcal{H}^2} = \|\mathbf{f}\|^2_{L^2(S)} = \sum_{n=1}^{\infty} \left(\frac{(2n+1)(2n+3)}{n+1} (a_n^0)^2 + \sum_{m=1}^{n} \frac{(n+1)(2n+1)(2n+3)}{(n+1)^2 - m^2} \left[(a_n^m)^2 + (b_n^m)^2 \right] \right)$$
$$\approx \sum_{n=1}^{\infty} \left(n (a_n^0)^2 + n^3 \sum_{m=1}^{n} \frac{1}{(n+1)^2 - m^2} \left[(a_n^m)^2 + (b_n^m)^2 \right] \right).$$

Proof. By virtue of the orthogonality of the polynomials $\mathbf{P}_n$, clearly $\|\mathbf{f}\|^2_{\mathcal{H}^2} = \|\mathbf{f}\|^2_{L^2(S;\mathcal{A};\mathbb{R})} = \sum_{n=0}^{\infty} \|\mathbf{P}_n\|^2_{L^2(S;\mathcal{A};\mathbb{R})}$, so we should only compute $L^2(S)$-norms of $\mathbf{P}_n$. By Cação[32, (p91)] or Gürlebeck and Morais,[39, (p641)]

$$\|\mathbf{X}_n^0\|^2_{L^2(S;\mathcal{A};\mathbb{R})} = \pi(n+1), \quad \|\mathbf{X}_n^m\|^2_{L^2(S;\mathcal{A};\mathbb{R})} = \|\mathbf{Y}_n^m\|^2_{L^2(S;\mathcal{A};\mathbb{R})} = \frac{\pi}{2}(n+1)\frac{(n+1+m)!}{(n+1-m)!} \tag{16}$$

for all $1 \leq m \leq n$. Norms (16) together with (8) yield

$$\frac{\|\mathbf{X}_n^0\|^2_{L^2(S;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(\mathbf{X}_n^{0,\dagger})\|^2_{L^2(B)}} = \frac{(2n+1)(2n+3)}{n+1}, \quad \frac{\|\mathbf{X}_n^m\|^2_{L^2(S;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(\mathbf{X}_n^{m,\dagger})\|^2_{L^2(B)}} = \frac{\|\mathbf{Y}_n^m\|^2_{L^2(S;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(\mathbf{Y}_n^{m,\dagger})\|^2_{L^2(B)}} = \frac{(n+1)(2n+1)(2n+3)}{(n+1+m)(n+1-m)}.$$

Hence,

$$\begin{aligned}\|\mathbf{P}_n\|^2_{L^2(S;\mathcal{A};\mathbb{R})} &= \frac{\|\mathbf{X}_n^0\|^2_{L^2(S;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(\mathbf{X}_n^{0,\dagger})\|^2_{L^2(B)}} (a_n^0)^2 + \sum_{m=1}^{n} \frac{\|\mathbf{X}_n^m\|^2_{L^2(S;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(\mathbf{X}_n^{m,\dagger})\|^2_{L^2(B)}} (a_n^m)^2 + \frac{\|\mathbf{Y}_n^m\|^2_{L^2(S;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(\mathbf{Y}_n^{m,\dagger})\|^2_{L^2(B)}} (b_n^m)^2 \\ &= \frac{(2n+1)(2n+3)}{n+1} (a_n^0)^2 + \sum_{m=1}^{n} \frac{(n+1)(2n+1)(2n+3)}{(n+1)^2 - m^2} \left[(a_n^m)^2 + (b_n^m)^2 \right].\end{aligned} \tag{17}$$

□

Lemma 2. (Norm in the harmonic Hardy space h^2). *Let* U *be a scalar-valued harmonic function in* B *with homogeneous expansion (11)*$U(x) = \sum_{n=0}^{\infty} \mathbf{Sc}\,\mathbf{P}_n(x)$*, where* $\mathbf{P}_n$ *are monogenic polynomials (15). If* $U \in h^2(B)$*, then*

$$\|U\|^2_{h^2(B)} = \|U\|^2_{L^2(S)} = \sum_{n=0}^{\infty} (2n+3) \left[(a_n^0)^2 + \sum_{m=1}^{n} (a_n^m)^2 + (b_n^m)^2 \right].$$

Proof. By the orthogonality of the polynomials $\mathbf{ScP}_n$, clearly $\|U\|^2_{h^2(B)} = \|U\|^2_{L^2(S)} = \sum_{n=0}^{\infty} \|\mathbf{Sc}\,\mathbf{P}_n\|^2_{L^2(S)}$, so we should only compute the $L^2(S)$-norms of $\mathbf{ScP}_n$. By Gürlebeck and Morais,[39, (prop.5, p643)]

$$\|\mathbf{Sc}(\mathbf{X}_n^0)\|^2_{L^2(S)} = \frac{\pi(n+1)^2}{2n+1}, \quad \|\mathbf{ScX}_n^m\|^2_{L^2(S)} = \|\mathbf{ScY}_n^m\|^2_{L^2(S)} = (n+1+m)^2 \frac{\pi}{2} \frac{1}{2n+1} \frac{(n+m)!}{(n-m)!}. \tag{18}$$

Norms (18) together with (8) yield

$$\frac{\|\mathbf{Sc}(\mathbf{X}_n^0)\|^2_{L^2(S)}}{\|\mathbf{Sc}(\mathbf{X}_n^{0,\dagger})\|^2_{L^2(B)}} = \frac{\|\mathbf{ScX}_n^m\|^2_{L^2(S)}}{\|\mathbf{Sc}(\mathbf{X}_n^{m,\dagger})\|^2_{L^2(B)}} = \frac{\|\mathbf{ScY}_n^m\|^2_{L^2(S)}}{\|\mathbf{Sc}(\mathbf{Y}_n^{m,\dagger})\|^2_{L^2(B)}} = 2n+3, \quad 1 \le m \le n. \tag{19}$$

Consequently,

$$\begin{aligned}
\|\mathbf{Sc}\,\mathbf{P}_n(\zeta)\|^2_{L^2(S)} &= \frac{\|\mathbf{Sc}(\mathbf{X}_n^0)\|^2_{L^2(S)}}{\|\mathbf{Sc}(\mathbf{X}_n^{0,\dagger})\|^2_{L^2(B)}} (a_n^0)^2 + \sum_{m=1}^{n} \left(\frac{\|\mathbf{Sc}(\mathbf{X}_n^m)\|^2_{L^2(S)}}{\|\mathbf{Sc}(\mathbf{X}_n^{m,\dagger})\|^2_{L^2(B)}} (a_n^m)^2 + \frac{\|\mathbf{Sc}(\mathbf{Y}_n^m)\|^2_{L^2(S)}}{\|\mathbf{Sc}(\mathbf{Y}_n^{m,\dagger})\|^2_{L^2(B)}} (b_n^m)^2 \right) \\
&= (2n+3) \left[(a_n^0)^2 + \sum_{m=1}^{n} (a_n^m)^2 + (b_n^m)^2 \right].
\end{aligned}$$

□

Lemma 3. (Norm in the monogenic Bergman space $\mathcal{H}^2_{\boldsymbol{\alpha}}$). *Suppose that* $\mathbf{f} \in \mathcal{R}_m^+(B; \mathcal{A})$ *having homogeneous expansion ((14)-(15)). If* $\mathbf{f} \in \mathcal{H}^2_\alpha(B)$ *for some* $\alpha > -1$, *then*

$$\|\mathbf{f}\|^2_{L^2_\alpha} \approx \sum_{n=0}^{\infty} \frac{1}{(n+1)^{\alpha+1}} \|\mathbf{P}_n\|^2_{L^2(S;\mathcal{A};\mathbb{R})} \tag{20}$$

$$\approx \sum_{n=1}^{\infty} \left(\frac{1}{n^\alpha} (a_n^0)^2 + \frac{1}{n^{\alpha-2}} \sum_{m=1}^{n} \frac{1}{(n+1)^2 - m^2} \left[(a_n^m)^2 + (b_n^m)^2 \right] \right), \qquad \alpha > -1, \tag{21}$$

Proof. The equivalence (20) is proved in lemma 4.1 of Morais et al.[1] Then by making use of (17), we obtain

$$\begin{aligned}
\|\mathbf{f}\|^2_{L^2_\alpha} &\approx \sum_{n=0}^{\infty} \frac{(2n+1)(2n+3)}{(n+1)^{\alpha+1}(n+1)} (a_n^0)^2 + \sum_{m=1}^{n} \frac{(n+1)(2n+1)(2n+3)}{(n+1)^{\alpha+1}((n+1)^2 - m^2)} \left[(a_n^m)^2 + (b_n^m)^2 \right] \\
&\approx \sum_{n=1}^{\infty} \left(\frac{1}{n^\alpha} (a_n^0)^2 + \frac{1}{n^{\alpha-2}} \sum_{m=1}^{n} \frac{1}{(n+1)^2 - m^2} \left[(a_n^m)^2 + (b_n^m)^2 \right] \right).
\end{aligned}$$

□

Lemma 4. (Norm in the monogenic Dirichlet space $\mathcal{D}^2_{\boldsymbol{\alpha}}$). *Suppose that* $\mathbf{f} \in \mathcal{R}_m^+(B; \mathcal{A})$ *having homogeneous expansion (14)-(15). If* $\mathbf{f} \in \mathcal{D}^2_\alpha(B)$ *for some* $\alpha > -1$, *then*

$$\|\mathbf{f}\|^2_{\mathcal{D}^2_\alpha} \approx \sum_{n=1}^{\infty} n^{2-\alpha} \left[(a_n^0)^2 + \sum_{m=1}^{n} (a_n^m)^2 + (b_n^m)^2 \right].$$

Proof. The Dirichlet seminorm can be written by means of polar coordinates, and after that by the orthogonality of the polynomials $\overline{D}\mathbf{P}_n$, we get

$$\begin{aligned}
\|\mathbf{f}\|^2_{\mathcal{D}^2_\alpha} &= 3 \int_0^1 \|\overline{D}\mathbf{f}\|^2_{L^2(S)} (1-r^2)^\alpha r^2 \, dr = 3 \int_0^1 \sum_{n=1}^{\infty} r^{2(n-1)} \|\overline{D}\mathbf{P}_n\|^2_{L^2(S)} (1-r)^\alpha r^2 \, dr \\
&= 3 \sum_{n=1}^{\infty} \|\overline{D}\mathbf{P}_n\|^2_{L^2(S)} \int_0^1 (1-r^2)^\alpha r^{2n} \, dr = \frac{3}{2} \sum_{n=1}^{\infty} \|\overline{D}\mathbf{P}_n\|^2_{L^2(S)} \frac{\Gamma(\alpha+1)\Gamma(n+1/2)}{\Gamma(\alpha+n+3/2)} \\
&\approx \sum_{n=1}^{\infty} \frac{1}{n^{\alpha+1}} \|\overline{D}\mathbf{P}_n\|^2_{L^2(S)}.
\end{aligned}$$

So we should only compute the $L^2(S)$-norms of the polynomials $\overline{D}\mathbf{P}_n$. Formulae (16) and (8) lead to

$$\frac{\|\mathbf{X}_{n-1}^{0}\|_{L^2(S)}^2}{\|\mathbf{Sc}(\mathbf{X}_n^{0,\dagger})\|_{L^2(B)}^2} = \frac{n(2n+1)(2n+3)}{(n+1)^2}, \quad \frac{\|\mathbf{X}_{n-1}^{m}\|_{L^2(S)}^2}{\|\mathbf{Sc}(\mathbf{X}_n^{m,\dagger})\|_{L^2(B)}^2} = \frac{\|\mathbf{Y}_{n-1}^{m}\|_{L^2(S)}^2}{\|\mathbf{Sc}(\mathbf{Y}_n^{m,\dagger})\|_{L^2(B)}^2} = \frac{n(2n+1)(2n+3)}{(n+1+m)^2}.$$

In view of (9), we obtain

$$\begin{aligned}
\left\|\frac{1}{2}\overline{D}\mathbf{P}_n\right\|_{L^2(S)}^2 &= \sum_{n=1}^{\infty}\Bigg[\frac{(n+1)^2\|\mathbf{X}_{n-1}^{0}\|_{L^2(S)}^2}{\|\mathbf{Sc}(\mathbf{X}_n^{0,\dagger})\|_{L^2(B)}^2}(a_n^0)^2 + \\
&\quad + \sum_{m=1}^{n}\frac{(n+m+1)^2\|\mathbf{X}_{n-1}^{m}\|_{L^2(S)}^2}{\|\mathbf{Sc}(\mathbf{X}_n^{m,\dagger})\|_{L^2(B)}^2}(a_n^m)^2 + \frac{(n+m+1)^2\|\mathbf{Y}_{n-1}^{m}\|_{L^2(S)}^2}{\|\mathbf{Sc}(\mathbf{Y}_n^{m,\dagger})\|_{L^2(B)}^2}(b_n^m)^2\Bigg] \\
&= \sum_{n=1}^{\infty} n(2n+1)(2n+3)\left[(a_n^0)^2 + \sum_{m=1}^{n}(a_n^m)^2 + (b_n^m)^2\right].
\end{aligned}$$

Thus,

$$\|\mathbf{f}\|_{\mathcal{D}_\alpha^2}^2 \approx \sum_{n=1}^{\infty}\frac{n(2n+1)(2n+3)}{n^{\alpha+1}}\left[(a_n^0)^2 + \sum_{m=1}^{n}(a_n^m)^2 + (b_n^m)^2\right] \approx \sum_{n=1}^{\infty} n^{2-\alpha}\left[(a_n^0)^2 + \sum_{m=1}^{n}(a_n^m)^2 + (b_n^m)^2\right].$$

□

Lemma 5. (Norm in the harmonic Dirichlet space $h\mathcal{D}_{\alpha}^{2}$). *Let U be a square-integrable scalar-valued harmonic function in B with homogeneous expansion (11)$U(x) = \sum_{n=0}^{\infty}\mathbf{Sc}\,\mathbf{P}_n(x)$, where $\mathbf{P}_n$ are monogenic polynomials (15). If $U \in h\mathcal{D}_\alpha^2(B)$ for some $\alpha > -1$, then*

$$\|U\|_{\mathcal{D}_\alpha^2}^2 \approx \sum_{n=1}^{\infty} n^{2-\alpha}\left[(a_n^0)^2 + \sum_{m=1}^{n}(a_n^m)^2 + (b_n^m)^2\right].$$

Proof. Taking into account (2) and (19), we get

$$\begin{aligned}
\|U\|_{\mathcal{D}_\alpha^2}^2 &\approx \sum_{n=1}^{\infty} n^{1-\alpha}\|\mathbf{Sc}\,\mathbf{P}_n\|_{L^2(S)}^2 \\
&= \sum_{n=1}^{\infty} n^{1-\alpha}\left[\frac{\|\mathbf{Sc}\mathbf{X}_n^0\|_{L^2(S)}^2}{\|\mathbf{Sc}(\mathbf{X}_n^{0,\dagger})\|_{L^2(B)}^2}(a_n^0)^2 + \sum_{m=1}^{n}\frac{\|\mathbf{Sc}\mathbf{X}_n^m\|_{L^2(S)}^2}{\|\mathbf{Sc}(\mathbf{X}_n^{m,\dagger})\|_{L^2(B)}^2}(a_n^m)^2 + \frac{\|\mathbf{Sc}\mathbf{Y}_n^m\|_{L^2(S)}^2}{\|\mathbf{Sc}(\mathbf{Y}_n^{m,\dagger})\|_{L^2(B)}^2}(b_n^m)^2\right] \\
&= \sum_{n=1}^{\infty} n^{1-\alpha}(2n+3)\left[(a_n^0)^2 + \sum_{m=1}^{n}(a_n^m)^2 + (b_n^m)^2\right] \approx \sum_{n=1}^{\infty} n^{2-\alpha}\left[(a_n^0)^2 + \sum_{m=1}^{n}(a_n^m)^2 + (b_n^m)^2\right].
\end{aligned}$$

□

Proof of Theorem. For a square-integrable harmonic function $U(x)$ in B, by Theorem 2, we recover the monogenic function $\mathbf{f}$ with $\mathbf{Sc}\mathbf{f} = U$ and series expansion $\mathbf{f}(x) = \sum_{n=0}^{\infty}\mathbf{P}_n(x)$, where $\mathbf{P}_n$ are spherical monogenics (15). An application of Lemmas 4 and 5 immediately implies the required equivalence (4),

$$\|\mathbf{f}\|_{\mathcal{D}_\alpha^2} \approx \|U\|_{\mathcal{D}_\alpha^2} \approx \sum_{n=1}^{\infty} n^{2-\alpha}\left[(a_n^0)^2 + \sum_{m=1}^{n}(a_n^m)^2 + (b_n^m)^2\right].$$

This means that the harmonic conjugation operator $U \mapsto \mathbf{f}$ given by formulae (11), (13), (14), and (15) is bounded in the weighted Dirichlet spaces $\mathcal{D}_\alpha^2(B)$. □

6 | SHARP INCLUSIONS IN THE DIRICHLET, BERGMAN, AND HARDY SPACES

Our second main theorem concerns relations of type (3). Unexpectedly, it turns out that the relations of type (3) are no longer true for the monogenic Dirichlet spaces. Moreover, no one monogenic Dirichlet space coincides with the Hardy or the weighted Bergman space.

Theorem 3.

(i) While the harmonic Dirichlet space $h\mathcal{D}_1^2(B)$ coincides with the harmonic Hardy space $h^2(B)$, this fact fails for their monogenic subspaces:

$$h\mathcal{D}_1^2 = h^2, \quad \mathcal{D}_1^2 \neq \mathcal{H}^2.$$

The monogenic Dirichlet space $\mathcal{D}_1^2(B)$ in fact lies between the monogenic Hardy space $\mathcal{H}^2$ and the monogenic (unweighted) Bergman space $\mathcal{H}_0^2(B) = \mathcal{M}(B) \cap L_0^2(B)$:

$$\mathcal{D}_0^2 \cap \mathcal{R}_m^+(B) \subset \mathcal{H}^2, \quad \mathcal{H}^2 \subset \mathcal{D}_1^2, \quad \mathcal{D}_1^2 \cap \mathcal{R}_m^+(B) \subset \mathcal{H}_0^2, \tag{22}$$

where all the inclusions are continuous and strict.

(ii) More generally, although the harmonic Dirichlet space $h\mathcal{D}_\alpha^2(B)$, $\alpha > 1$, coincides with the harmonic Bergman space $h_{\alpha-2}^2(B) = h(B) \cap L_{\alpha-2}^2(B)$, this fact fails for their monogenic subspaces:

$$h\mathcal{D}_\alpha^2 = h_{\alpha-2}^2, \quad \mathcal{D}_\alpha^2 \neq \mathcal{H}_{\alpha-2}^2, \quad \alpha > 1.$$

The precise relations are

$$\mathcal{H}_{\alpha-2}^2 \subset \mathcal{D}_\alpha^2, \quad \mathcal{D}_\alpha^2 \cap \mathcal{R}_m^+(B) \subset \mathcal{H}_{\alpha-1}^2, \quad \alpha > 1, \tag{23}$$

where both inclusions are continuous and strict. Moreover, the inclusions are sharp in the sense that the subscripts $\alpha - 2, \alpha - 1, \alpha$ in (23), as well as the subscripts 0 and 1 in (22), cannot be improved.

Proof. The continuity of inclusions (23), ie, $C_1\|\mathbf{f}\|_{L_{\alpha-1}^2} \leq \|\mathbf{f}\|_{\mathcal{D}_\alpha^2} \leq C_2\|\mathbf{f}\|_{L_{\alpha-2}^2}$, $\alpha > 1$, follows from a comparison of the Dirichlet and Bergman norms

$$\|\mathbf{f}\|_{\mathcal{D}_\alpha^2}^2 \approx \sum_{n=1}^{\infty} n^{2-\alpha} \left[(a_n^0)^2 + \sum_{m=1}^{n} (a_n^m)^2 + (b_n^m)^2 \right],$$

$$\|\mathbf{f}\|_{L_{\alpha-1}^2}^2 \approx \sum_{n=1}^{\infty} \left(\frac{1}{n^{\alpha-1}} (a_n^0)^2 + \frac{1}{n^{\alpha-3}} \sum_{m=1}^{n} \frac{1}{(n+1)^2 - m^2} \left[(a_n^m)^2 + (b_n^m)^2 \right] \right),$$

$$\|\mathbf{f}\|_{L_{\alpha-2}^2}^2 \approx \sum_{n=1}^{\infty} \left(\frac{1}{n^{\alpha-2}} (a_n^0)^2 + \frac{1}{n^{\alpha-4}} \sum_{m=1}^{n} \frac{1}{(n+1)^2 - m^2} \left[(a_n^m)^2 + (b_n^m)^2 \right] \right).$$

An estimation

$$\frac{1}{n^{\alpha-1}} (a_n^0)^2 + \frac{1}{n^{\alpha-3}} \sum_{m=1}^{n} \frac{1}{(n+1)^2 - m^2} \left[(a_n^m)^2 + (b_n^m)^2 \right] \leq \frac{1}{n^{\alpha-1}} (a_n^0)^2 + \frac{1}{n^{\alpha-3}(2n+1)} \sum_{m=1}^{n} \left[(a_n^m)^2 + (b_n^m)^2 \right]$$
$$\leq \frac{1}{n^{\alpha-2}} \left(\frac{1}{n} (a_n^0)^2 + \sum_{m=1}^{n} \left[(a_n^m)^2 + (b_n^m)^2 \right] \right)$$

shows that for functions $\mathbf{f} \in \mathcal{R}_m^+(B; \mathcal{A})$, one obtains $\|\mathbf{f}\|_{L_{\alpha-1}^2} \leq C\|\mathbf{f}\|_{\mathcal{D}_\alpha^2}$ with $C = 1$.

On the other hand,

$$\frac{1}{n^{\alpha-2}} (a_n^0)^2 + \frac{1}{n^{\alpha-4}} \sum_{m=1}^{n} \frac{1}{(n+1)^2 - m^2} \left[(a_n^m)^2 + (b_n^m)^2 \right] \geq \frac{1}{n^{\alpha-2}} (a_n^0)^2 + \frac{1}{n^{\alpha-4} n(n+2)} \sum_{m=1}^{n} \left[(a_n^m)^2 + (b_n^m)^2 \right]$$
$$\geq \frac{1}{2n^{\alpha-2}} \left((a_n^0)^2 + \sum_{m=1}^{n} \left[(a_n^m)^2 + (b_n^m)^2 \right] \right).$$

So, for functions $\mathbf{f} \in \mathcal{R}^+(B;\mathcal{A})$, we come to $\|\mathbf{f}\|_{\mathcal{D}^2_\alpha} \le C\|\mathbf{f}\|_{L^2_{\alpha-2}}$ with $C = \sqrt{2}$. Thus, inclusions (23) are continuous. Inclusions (22) can be proved in the same manner.

Now, let us show the strictness and sharpness of the inclusions. Define the following typical monogenic functions as possible counterexamples,

$$\mathbf{g}(x) := \sum_{n=1}^{\infty} \frac{\mathbf{X}_n^{n,\dagger}}{\|\mathbf{Sc}(\mathbf{X}_n^{n,\dagger})\|L^2(B)} a_n^n, \quad \mathbf{h}(x) := \sum_{n=1}^{\infty} \frac{\mathbf{X}_n^{0,\dagger}}{\|\mathbf{Sc}(\mathbf{X}_n^{0,\dagger})\|_{L^2(B)}} a_n^0.$$

The exact values for the coefficients a_n^n and a_n^0 will be defined below. We have to compute several norms of $\mathbf{g}$ and $\mathbf{h}$, by Lemmas 1 to 4. No one Dirichlet space $\mathcal{D}^2_\alpha$ coincides with the Hardy space $\mathcal{H}^2$, indeed, $\mathcal{D}^2_\alpha \not\subset \mathcal{H}^2$ for any $0 < \alpha \le 1$, since

$$\exists\, \mathbf{g} \in \mathcal{D}^2_\alpha, \quad \mathbf{g} \notin \mathcal{H}^2, \quad \|\mathbf{g}\|^2_{\mathcal{D}^2_\alpha} \approx \sum_{n=1}^{\infty} n^{2-\alpha}(a_n^n)^2 < +\infty, \quad \|\mathbf{g}\|^2_{\mathcal{H}^2} \approx \sum_{n=1}^{\infty} n^2 (a_n^n)^2 = +\infty. \tag{24}$$

We can choose in (24), $a_n^n = n^{-3/2}$, for example. Conversely, $\mathcal{H}^2 \not\subset \mathcal{D}^2_\beta$ for any $0 \le \beta < 1$, since

$$\exists\, \mathbf{h} \in \mathcal{H}^2, \quad \mathbf{h} \notin \mathcal{D}_\beta, \quad \|\mathbf{h}\|^2_{\mathcal{H}^2} \approx \sum_{n=1}^{\infty} n(a_n^0)^2 < +\infty, \quad \|\mathbf{h}\|^2_{\mathcal{D}^2_\beta} \approx \sum_{n=1}^{\infty} n^{2-\beta}(a_n^0)^2 = +\infty. \tag{25}$$

We can choose in (25), $(a_n^0)^2 = n^{\beta-3}$, for example. So, the inclusions $\mathcal{D}^2_0 \cap \mathcal{R}^+_m(B) \subset \mathcal{H}^2 \subset \mathcal{D}^2_1$ are sharp.

Further, no one Dirichlet space $\mathcal{D}^2_\alpha$ coincides with the Bergman space $\mathcal{H}^2_\alpha$. We have already proved inclusions (23). One can show that these inclusions are strict and sharp in the sense that the subscripts in (23) cannot be improved. Indeed, $\mathcal{D}^2_\beta \cap \mathcal{R}^+_m(B;\mathcal{A}) \not\subset \mathcal{H}^2_{\alpha-1}$ for any $\beta > \alpha > 1$, since

$$\exists\, \mathbf{g} \in \mathcal{D}^2_\beta, \quad \mathbf{g} \notin \mathcal{H}^2_{\alpha-1}, \quad \|\mathbf{g}\|^2_{\mathcal{D}^2_\beta} \approx \sum_{n=1}^{\infty} n^{2-\beta}(a_n^n)^2 < +\infty, \quad \|\mathbf{g}\|^2_{L^2_{\alpha-1}} \approx \sum_{n=1}^{\infty} n^{2-\alpha}(a_n^n)^2 = +\infty. \tag{26}$$

We can choose in (26), $(a_n^n)^2 = n^{\alpha-3}$, for example.

Next, $\mathcal{H}^2_{\alpha-2} \not\subset \mathcal{D}^2_\gamma$ for any $\gamma < \alpha$, since

$$\exists\, \mathbf{h} \in \mathcal{H}^2_{\alpha-2}, \quad \mathbf{h} \notin \mathcal{D}^2_\gamma, \quad \|\mathbf{h}\|^2_{L^2_{\alpha-2}} \approx \sum_{n=1}^{\infty} n^{2-\alpha}(a_n^0)^2 < +\infty, \quad \|\mathbf{h}\|^2_{\mathcal{D}^2_\gamma} \approx \sum_{n=1}^{\infty} n^{2-\gamma}(a_n^0)^2 = +\infty. \tag{27}$$

We can choose in (27), $(a_n^0)^2 = n^{\gamma-3}$, for example. Thus, inclusions (23) are best possible.

This completes the proof of Theorem 3. □

7 | HYPERCOMPLEX DERIVATIVES AND PRIMITIVES IN THE BERGMAN AND HARDY SPACES

The lemma below shows how partial derivatives act in harmonic Bergman spaces.

Lemma 6. (eg, Avetisyan et al[26,(lemma 5)]). *Let $1 \le p < \infty$, $\alpha > -1$, m be a positive integer, and $\lambda = (\lambda_0, \lambda_1, \lambda_2) \in \mathbb{N}_0^3$. Then for all $\mathcal{A}$-valued harmonic functions f,*

$$\|f\|_{L^p_\alpha} \approx \sum_{|\lambda|<m} \left|\partial^\lambda f(0)\right| + \sum_{|\lambda|=m} \left\|\partial^\lambda f\right\|_{L^p_{\alpha+pm}}, \tag{28}$$

where ∂^λ denotes the partial differential operator of the order $|\lambda| = \lambda_0 + \lambda_1 + \lambda_2$ with respect to x_0, x_1, x_2.

In particular,

$$\|f\|_{L^p_\alpha} \approx |f(0)| + \|\nabla f\|_{L^p_{\alpha+p}} \approx |f(0)| + |\nabla f(0)| + \left\|\nabla^2 f\right\|_{L^p_{\alpha+2p}}. \tag{29}$$

The involved constants depend on the parameters p, α, m only.

It is natural here to ask whether the gradient in Lemma 6 can be replaced by the hypercomplex derivative $\overline{D}$ of monogenic functions. Corollary 2 below states that the answer is negative. Besides the hypercomplex derivative $\overline{D}$, we are interested in monogenic primitives in the mentioned spaces.

Definition 5. For $\mathbf{g} \in C^1(B; \mathbb{H})$, we say that $\mathbf{f} \in \mathcal{M}(B; \mathcal{A})$ is a (monogenic) primitive of $\mathbf{g}$ if $\overline{D}\mathbf{f} = \mathbf{g}$ in B.

We immediately obtain that if $\mathbf{g}$ admits a primitive $\mathbf{f}$, then $\mathbf{g}$ is necessarily left monogenic, $\mathbf{g} \in \mathcal{M}(B; \mathbb{H})$, because $D\mathbf{g} = D\overline{D}\mathbf{f} = \Delta \mathbf{f} = 0$. If $\mathbf{f}$ is a primitive of $\mathbf{g}$, then $\{\mathbf{f} + \boldsymbol{\varphi}\}$ is the set of primitives of $\mathbf{g}$, where $\boldsymbol{\varphi}$ is an arbitrary hyperholomorphic constant, $\overline{D}\boldsymbol{\varphi} = 0$.

Sharp inclusions (22) and (23) lead to the following corollaries of Theorem 3.

Corollary 1. *For any* $\mathbf{f} \in \mathcal{R}^+(B; \mathcal{A})$,

$$\|\overline{D}\mathbf{f}\|_{L_1^2} \le C \|\mathbf{f}\|_{\mathcal{H}^2},$$
$$\|\overline{D}\mathbf{f}\|_{L^2_{\alpha+2}} \le C_\alpha \|\mathbf{f}\|_{L^2_\alpha}, \qquad \alpha > -1.$$

Corollary 2. *For any* $\mathbf{f} \in \mathcal{R}^+_m(B; \mathcal{A})$,

$$\|\mathbf{f}\|_{\mathcal{H}^2} \le C \|\overline{D}\mathbf{f}\|_{L_0^2},$$
$$\|\mathbf{f}\|_{L^2_{\alpha+1}} \le C_\alpha \|\overline{D}\mathbf{f}\|_{L^2_{\alpha+2}}, \qquad \alpha > -2.$$

Remark 1. The results of Corollary 1 correspond to the classical case in Lemma 6, while in the opposite direction in Corollary 2, we have stated somewhat weaker assertion. Namely, we have obtained in our setting that the monogenic primitive $(\overline{D})^{-1}$ in the weighted Bergman spaces scale $\mathcal{H}^2_\alpha$ behaves one degree worse than a classical one.

8 | CONSTRUCTION OF A GENERAL MONOGENIC FUNCTION BY ITS FIRST COMPONENT AND FURTHER LEMMAS

In this section, we apply an alternative algorithm for the explicit construction of a "unique" pair of conjugate harmonic functions in $\mathbb{R}^3$.

Theorem 4. (Morais et al;[1] Construction of a harmonic conjugate). *Let U be a scalar-valued harmonic function defined in B. Define*

$$V_1(x) := -x_0 \int_0^1 \frac{\partial U(\rho x_0, x_1, x_2)}{\partial x_1} \, d\rho + W(x_1, x_2), \tag{30}$$

where the function $W(x_1, x_2)$ *is chosen so that*

$$\Delta_{(x_1,x_2)} W = \frac{\partial^2 U(0, x_1, x_2)}{\partial x_0 \, \partial x_1} \tag{31}$$

and

$$V_2(x) := \int_0^1 \left[-\begin{vmatrix} x_0 & x_2 \\ \frac{\partial U(tx)}{\partial x_0} & \frac{\partial U(tx)}{\partial x_2} \end{vmatrix} + \begin{vmatrix} x_1 & x_2 \\ \frac{\partial V_1(tx)}{\partial x_1} & \frac{\partial V_1(tx)}{\partial x_2} \end{vmatrix} \right] dt. \tag{32}$$

Then the function $\boldsymbol{f} := U + V_1 \boldsymbol{i} + V_2 \boldsymbol{j}$ *is monogenic in B. Moreover, the most general monogenic function* $\mathbf{g}$ *having U as its scalar part is given by*

$$\mathbf{g}(x) = \mathbf{f}(x) + \boldsymbol{\varphi}(x_1, x_2),$$

where $\boldsymbol{\varphi}(x_1, x_2)$ *is a hyperholomorphic constant.*

We need some more lemmas.

Lemma 7. (Flett[40]). *Let* $w(x)$ *be a nonnegative subharmonic function in B and* $M_1(w; r)$ *be its integral mean of order one. If* $M_1(w; r)$ *is bounded on* $0 \le r < 1$, *then* $w(x)$ *has a harmonic majorant* $h(x) \in h^1(B)$ *on B so that*

$$w(x) \le h(x), \; x \in B, \quad \text{and} \quad \|h\|_{h^1(B)} \le C \sup_{0<r<1} M_1(w; r),$$

and as a consequence, $M_1(h; r) \le C M_1(w; r), 0 \le r < 1$.

For any fixed $\rho, r \in (0,1)$, we also consider the following bounded domain in $\mathbb{R}^3$:

$$E_{\rho,r} := \left\{ x = (x_0, x_1, x_2) \in \mathbb{R}^3 : \frac{x_0^2}{\rho^2 r^2} + \frac{x_1^2}{r^2} + \frac{x_2^2}{r^2} < 1 \right\},$$

which denotes the inner domain of the oblate spheroid $\partial E_{\rho,r}$.

Lemma 8. (Krantz[41]). *Let $P_{E_{\rho,r}}(x,y)$ be the Poisson kernel for $E_{\rho,r}$. Then*

$$P_{E_{\rho,r}}(x,y) \approx \frac{dist(x, \partial E_{\rho,r})}{|x-y|^3}, \qquad x \in E_{\rho,r},\ y \in \partial E_{\rho,r},$$

in particular,

$$P_{E_{\rho,r}}(0,y) \approx \frac{\rho r}{|y|^3}, \qquad y \in \partial E_{\rho,r}.$$

Lemma 9. (Hardy's inequalities; Flett[42, (p490)] and Flett[43, (p758)]). *If $1 \le p < \infty, \gamma < -1 < \alpha$, and $h(r) \ge 0$, then*

$$\int_0^1 (1-r)^\alpha r^\gamma \left(\int_0^r h(t)dt \right)^p dr \le C(p,\alpha,\gamma) \int_0^1 (1-r)^{\alpha+p} r^{\gamma+p} h^p(r) dr, \tag{33}$$

$$\int_0^1 (1-r)^\alpha \left(\int_0^r h(t)dt \right)^p dr \le C(p,\alpha) \int_0^1 (1-r)^{\alpha+p} h^p(r) dr. \tag{34}$$

Lemma 10. (Zhao[44]). *Let $w = w(x_1,x_2)$ be a nonnegative superharmonic function in the unit disk $\mathbb{D} := \{x_1^2 + x_2^2 < 1\}$ and $\beta > -1, 0 < p < 2 + \beta$. Then for any point $a \in \mathbb{D}$,*

$$\|w\|_{L^p_\beta(\mathbb{D})} \le C(p,\beta,a) w(a).$$

9 | HARMONIC CONJUGATES IN WEIGHTED DIRICHLET SPACES $\mathcal{D}^p_\alpha$

Our last theorem extends Theorem 3 to the range $1 < p < \infty$ and generalizes it by using the method of Theorem 4 for construction of quaternion-valued harmonic conjugates.

Theorem 5. *Let U be a real-valued harmonic function in B. If $U \in h\mathcal{D}^p_\alpha(B)$ for some $1 < p < \infty$ and $\alpha > -1$, then there exists a monogenic function $\mathbf{f} \in \mathcal{D}^p_\alpha(B)$ such that $\mathbf{Sc} f = U$ in B.*

Moreover, if $\widetilde{W}(x_1,x_2) := G[\psi](x_1,x_2)$ is the Green potential of the function

$$\psi(x_1,x_2) := -\left| \nabla_{(x_1,x_2)} \frac{\partial^2 U(0,x_1,x_2)}{\partial x_0 \partial x_1} \right| \tag{35}$$

in $\mathbb{D}$, then

$$\|\mathbf{f}\|_{\mathcal{D}^p_\alpha(B)} \le C(p,\alpha) \|U\|_{\mathcal{D}^p_\alpha(B)} + C(p,\alpha,a) \widetilde{W}(a)$$

for any point $a = (a_1,a_2)$, $a_1^2 + a_2^2 < 1$, such that $\widetilde{W}(a) < +\infty$.

Proof. Given a real-valued harmonic function U, we use Theorem 4 to construct $\mathbf{f} = U + V_1 \mathbf{i} + V_2 \mathbf{j}$ where the coordinates V_1 and V_2 are defined by (30) and (32). For any point $x = r\zeta \in B$, we can differentiate and then estimate the first component V_1 in the vector part of $\mathbf{f}$,

$$\left| \frac{\partial V_1(x)}{\partial x_j} \right| \le |x_0| \int_0^1 \left| \frac{\partial^2 U(\rho x_0, x_1, x_2)}{\partial x_1 \partial x_j} \right| d\rho + \left| \frac{\partial W(x_1,x_2)}{\partial x_j} \right| =: \widetilde{\partial V_1}(x) + \left| \frac{\partial W(x_1,x_2)}{\partial x_j} \right|. \tag{36}$$

An application of Minkowski's inequality implies

$$M_p(\partial\widetilde{V}_1;r) \le \int_0^1 \left(\int_{|x|=r} |x_0|^p \left| \frac{\partial^2 U(\rho x_0, x_1, x_2)}{\partial x_1 \partial x_j} \right|^p \frac{ds(x)}{4\pi r^2} \right)^{1/p} d\rho.$$

Denote by $h(x)$ the smallest harmonic majorant of the subharmonic function $\left|\frac{\partial^2 U(x)}{\partial x_1 \partial x_j}\right|^p$ in the ball $B_{\sqrt{r}} = \{y \in \mathbb{R}^3 : |y| < \sqrt{r}\}$, then

$$\left| \frac{\partial^2 U(y)}{\partial x_1 \partial x_j} \right|^p \le h(y), \qquad y \in B_{\sqrt{r}}.$$

A direct evaluation shows that

$$M_p(\partial\widetilde{V}_1;r) \le \int_0^1 \left(\int_{|x|=r} |x_0|^p\, h(\rho x_0, x_1, x_2) \frac{ds(x)}{4\pi r^2} \right)^{1/p} d\rho \le r \int_0^1 \left(\frac{1}{4\pi r^2} \int_{\partial E_{\rho,r}} h(y)\, ds(y) \right)^{1/p} d\rho.$$

We now need the Poisson integral representation of h in the spheroid $E_{\rho,r} \subset B_{\sqrt{r}}$, and then, we can estimate it at the origin by using Lemma 8:

$$h(x) = \int_{\partial E_{\rho,r}} P_{E_{\rho,r}}(x,y)\, h(y)\, d\sigma(y),$$

$$h(0) = \int_{\partial E_{\rho,r}} P_{E_{\rho,r}}(0,y)\, h(y)\, d\sigma(y) \ \ge\ C \int_{\partial E_{\rho,r}} \frac{\rho r}{|y|^3} h(y)\, d\sigma(y) \ \ge\ C \frac{\rho}{r^2} \int_{\partial E_{\rho,r}} h(y)\, d\sigma(y).$$

With these calculations at hand, we get

$$M_p(\partial\widetilde{V}_1;r) \le C r \int_0^1 \left(\frac{r^2}{\rho} h(0) \right)^{1/p} d\rho = C_p\, r^{1+2/p} (h(0))^{1/p}.$$

By the mean value equality for harmonic functions and using Lemma 7, we obtain

$$\begin{aligned} M_p(\partial\widetilde{V}_1;r) &\le C_p\, r^{1+2/p} \int_0^1 \left(\frac{1}{|S_{\sqrt{\rho r}}|} \int_{S_{\sqrt{\rho r}}} h(\xi)\, ds(\xi) \right)^{1/p} d\rho \\ &= C_p\, r^{1+2/p} \int_0^1 M_1^{1/p}(h; \sqrt{\rho r})\, d\rho \\ &\le C_p\, r^{1+1/p} \int_0^1 M_p\left(\frac{\partial^2 U}{\partial x_1 \partial x_j}; \sqrt{\rho r} \right) d\rho \\ &= C_p\, r^{1/p} \int_0^r M_p\left(\frac{\partial^2 U}{\partial x_1 \partial x_j}; \sqrt{t} \right) dt. \end{aligned} \tag{37}$$

Raising both sides of (37) to the pth power and integrating and then by using Lemmas 9 and 6, we obtain

$$\begin{aligned} \|\partial\widetilde{V}_1\|^p_{L^p_\alpha(B)} &\le C \int_0^1 (1-r)^\alpha M_p^p(\partial\widetilde{V}_1;r)\, dr \\ &\le C \int_0^1 (1-r)^\alpha r \left(\int_0^r M_p\left(\frac{\partial^2 U}{\partial x_1 \partial x_j}; \sqrt{t} \right) dt \right)^p dr \\ &\le C \int_0^1 (1-r)^{\alpha+p} M_p^p\left(\frac{\partial^2 U}{\partial x_1 \partial x_j}; \sqrt{r} \right) dr \le C(p,\alpha) \left\| \frac{\partial^2 U}{\partial x_1 \partial x_j} \right\|^p_{L^p_{\alpha+p}(B)} \\ &\le C(p,\alpha) \left\| \nabla^2 U \right\|^p_{L^p_{\alpha+p}(B)} \le C(p,\alpha) \|\nabla U\|^p_{L^p_\alpha(B)}. \end{aligned} \tag{38}$$

The last term in (36) can be estimated by means of Lemma 10 as follows. From (31) by a differentiation with respect to $x_j, j = 1, 2$, we get

$$\Delta_{(x_1,x_2)}\left[\frac{\partial}{\partial x_j}W(x_1,x_2)\right] = \frac{\partial^3 U(0,x_1,x_2)}{\partial x_0\,\partial x_1\,\partial x_j}, \quad j = 1, 2, \tag{39}$$

As is well known (see, eg, Gilbarg and Trudinger[45]), the solution $\frac{\partial}{\partial x_j}W(x_1,x_2)$ of the Poisson equation (39) in $\mathbb{D}$ with vanishing boundary values on the unit circle $\partial\mathbb{D}$ is unique and is the Green potential of $\frac{\partial^3 U(0,x_1,x_2)}{\partial x_0\,\partial x_1\,\partial x_j}$,

$$\frac{\partial}{\partial x_j}W(x_1,x_2) = \int_{\mathbb{D}} G(x_1,x_2,y_1,y_2)\frac{\partial^3 U(0,y_1,y_2)}{\partial y_0\,\partial y_1\,\partial y_j}\,dy_1\,dy_2 =: G\left[\partial^3_{01j}\,U(0,\cdot,\cdot)\right](x_1,x_2), \quad (x_1,x_2) \in \mathbb{D},$$

where $G(x,y) = \log\left|\frac{x-y}{1-\bar{y}x}\right| \le 0$ (in the complex notation) is the (subharmonic) Green function for the disk. Denoting

$$v^+ := \max\{v,0\}, \quad v^- := \max\{-v,0\}, \quad \text{so that} \quad v = v^+ - v^- = -v^- - (-v^+), \quad |v| = v^+ + v^-,$$

for a real-valued function v, we split the function $\frac{\partial^3 U(0,x_1,x_2)}{\partial x_0\,\partial x_1\,\partial x_j}$ into its positive and negative parts,

$$\frac{\partial^3 U(0,x_1,x_2)}{\partial x_0\,\partial x_1\,\partial x_j} = -\left(\frac{\partial^3 U(0,x_1,x_2)}{\partial x_0\,\partial x_1\,\partial x_j}\right)^- - \left[-\left(\frac{\partial^3 U(0,x_1,x_2)}{\partial x_0\,\partial x_1\,\partial x_j}\right)^+\right],$$

and come to

$$\frac{\partial}{\partial x_j}W(x_1,x_2) = W_{j1}(x_1,x_2) - W_{j2}(x_1,x_2), \tag{40}$$

where

$$0 \le W_{j1}(x_1,x_2) := -\int_{\mathbb{D}} G(x,y)\left(\frac{\partial^3 U(0,y_1,y_2)}{\partial y_0\,\partial y_1\,\partial y_j}\right)^- dy, \qquad \Delta W_{j1}(x_1,x_2) = -\left(\frac{\partial^3 U(0,y_1,y_2)}{\partial y_0\,\partial y_1\,\partial y_j}\right)^- \le 0,$$

$$0 \le W_{j2}(x_1,x_2) := -\int_{\mathbb{D}} G(x,y)\left(\frac{\partial^3 U(0,y_1,y_2)}{\partial y_0\,\partial y_1\,\partial y_j}\right)^+ dy, \qquad \Delta W_{j2}(x_1,x_2) = -\left(\frac{\partial^3 U(0,y_1,y_2)}{\partial y_0\,\partial y_1\,\partial y_j}\right)^+ \le 0,$$

that is, W_{j1} and W_{j2} are nonnegative superharmonic functions in $\mathbb{D}$. Consequently, the function $W_{j1} + W_{j2}$,

$$0 \le W_{j1}(x_1,x_2) + W_{j2}(x_1,x_2) = -\int_{\mathbb{D}} G(x,y)\left|\frac{\partial^3 U(0,y_1,y_2)}{\partial y_0\,\partial y_1\,\partial y_j}\right| dy \equiv G\left[-\left|\partial^3_{01j}U(0,\cdot,\cdot)\right|\right](x_1,x_2), \qquad j = 1, 2, \tag{41}$$

is nonnegative and superharmonic, too. Summing two formulae (41) for $j = 1$ and $j = 2$, we get

$$0 \le W_{11}(x_1,x_2) + W_{12}(x_1,x_2) + W_{21}(x_1,x_2) + W_{22}(x_1,x_2) \le \sqrt{2}\int_{\mathbb{D}} -G(x,y)\left|\nabla_{(y_1,y_2)}\frac{\partial^2 U(0,y_1,y_2)}{\partial y_0\,\partial y_1}\right| dy.$$

Therefore, for a point $a = (a_1,a_2) \in \mathbb{D}$,

$$0 \le W_{11}(a) + W_{12}(a) + W_{21}(a) + W_{22}(a) \le \sqrt{2}G\left[-\left|\nabla\,\partial_{01}U(0,\cdot,\cdot)\right|\right](a) = \sqrt{2}G[\psi](a) = \sqrt{2}\widetilde{W}(a) < +\infty,$$

by assumption, where the function ψ is defined by (35). By Lemma 10, for four nonnegative superharmonic functions W_{11}, W_{12}, W_{21}, and W_{22}, we have

$$\|W_{ji}\|_{L^p_p(\mathbb{D})} \le C(p,a)\,W_{ji}(a), \qquad j, i = 1, 2.$$

Since the integral mean M_p of a superharmonic function is decreasing with respect to r,

$$\|W_{ji}\|_{L^p_p(\mathbb{D})} \ge \left(\int_0^r (1-t^2)^p M_p^p\left(W_{ji};t\right)t\,dt\right)^{1/p} \ge C_p M_p\left(W_{ji};r\right)$$

for all $\frac{1}{2} < r < 1$. It follows that for $\alpha > -1$,

$$\left(\int_{1/2}^1 (1-r)^\alpha M_p^p\left(W_{ji};r\right)r\,dr\right)^{1/p} \le C(p,\alpha)\,\|W_{ji}\|_{L^p_p(\mathbb{D})} \le C(p,\alpha,a)\,W_{ji}(a). \tag{42}$$

Moreover, inequality (42) remains valid for an extension $W_{ji}(x_0, x_1, x_2)$ of $W_{ji}(x_1, x_2)$ to the ball B.

According to (36) and (40),

$$\left|\frac{\partial V_1(x)}{\partial x_j}\right| \le \widetilde{\partial V}_1(x) + \left|\frac{\partial W(x_1, x_2)}{\partial x_j}\right| \le \widetilde{\partial V}_1(x) + W_{j1} + W_{j2}. \tag{43}$$

Because of the increasing property of $M_p(u; r)$ in r for a harmonic function u, the cut Bergman norm

$$\|u\|_{L^p_\alpha(1/2<|x|<1)} := \left(\int_{1/2<|x|<1} |u(x)|^p (1 - |x|^2)^\alpha \, dV(x)\right)^{1/p}$$

is equivalent to the full norm $\|u\|_{L^p_\alpha(B)}$. Therefore, by (43), (38), and (42),

$$\begin{aligned}\left\|\frac{\partial V_1}{\partial x_j}\right\|_{L^p_\alpha(B)} &\le C(p,\alpha)\left\|\frac{\partial V_1}{\partial x_j}\right\|_{L^p_\alpha(1/2<|x|<1)} \le C\left\|\widetilde{\partial V}_1\right\|_{L^p_\alpha(1/2<|x|<1)} + C\|W_{j1} + W_{j2}\|_{L^p_\alpha(1/2<|x|<1)} \\ &\le C(p,\alpha)\|\nabla U\|_{L^p_\alpha(B)} + C(p,\alpha,a)\left(W_{j1}(a) + W_{j2}(a)\right), \qquad j = 1, 2.\end{aligned}$$

Summing two norm inequalities, we get

$$\begin{aligned}\left\|\frac{\partial V_1}{\partial x_1}\right\|_{L^p_\alpha(B)} + \left\|\frac{\partial V_1}{\partial x_2}\right\|_{L^p_\alpha(B)} &\le C(p,\alpha)\|\nabla U\|_{L^p_\alpha(B)} + C(p,\alpha,a)\,(W_{11}(a) + W_{12}(a) + W_{21}(a) + W_{22}(a)) \\ &\le C(p,\alpha)\|\nabla U\|_{L^p_\alpha(B)} + C(p,\alpha,a)\,\widetilde{W}(a).\end{aligned}$$

Since $\frac{\partial V_1(x)}{\partial x_0} = \frac{\partial U(x)}{\partial x_1}$ by the generalized Cauchy-Riemann system (R), we conclude that

$$\|\nabla V_1\|_{L^p_\alpha(B)} \le C(p,\alpha)\|\nabla U\|_{L^p_\alpha(B)} + C(p,\alpha,a)\,\widetilde{W}(a). \tag{44}$$

Let us proceed to estimations for $V_2(x)$, (32),

$$\begin{aligned}V_2(x) &= \int_0^1 \left[-x_0\frac{\partial U(tx)}{\partial x_2} + x_2\frac{\partial U(tx)}{\partial x_0} + x_1\frac{\partial V_1(tx)}{\partial x_2} - x_2\frac{\partial V_1(tx)}{\partial x_1}\right] dt, \\ \frac{\partial V_2(x)}{\partial x_1} &= \int_0^1 \left[-x_0\frac{\partial^2 U(tx)}{\partial x_1 \partial x_2} + x_2\frac{\partial^2 U(tx)}{\partial x_0 \partial x_1} + \frac{\partial V_1(tx)}{\partial x_2} + x_1\frac{\partial^2 V_1(tx)}{\partial x_1 \partial x_2} - x_2\frac{\partial^2 V_1(tx)}{\partial x_1^2}\right] dt.\end{aligned}$$

An application of Minkowski's inequality implies

$$\begin{aligned}M_p\left(\frac{\partial V_2}{\partial x_1}; r\right) &\le C\int_0^1 \left(rM_p\left(\nabla^2 U; tr\right) + rM_p\left(\nabla U; tr\right) + M_p\left(\nabla V_1; tr\right) + rM_p\left(\nabla^2 V_1; tr\right)\right) dt \\ &= C\int_0^r M_p\left(\nabla^2 U; t\right) dt + C\int_0^r M_p\left(\nabla U; t\right) dt + C\frac{1}{r}\int_0^r M_p\left(\nabla V_1; t\right) dt + C\int_0^r M_p\left(\nabla^2 V_1; t\right) dt.\end{aligned}$$

Raising to the pth power and integrating with respect to $(1 - r)^\alpha r^2 dr$, we obtain

$$\begin{aligned}\left\|\frac{\partial V_2}{\partial x_1}\right\|^p_{L^p_\alpha(B)} &\le C\int_0^1 (1-r)^\alpha \left(\int_0^r M_p\left(\nabla^2 U; t\right) dt\right)^p r^2\,dr + C\int_0^1 (1-r)^\alpha \left(\int_0^r M_p\left(\nabla U; t\right) dt\right)^p r^2\,dr + \\ &\quad + C\int_0^1 (1-r)^\alpha \frac{1}{r^p}\left(\int_0^r M_p\left(\nabla V_1; t\right) dt\right)^p r^2\,dr + C\int_0^1 (1-r)^\alpha \left(\int_0^r M_p\left(\nabla^2 V_1; t\right) dt\right)^p r^2\,dr \\ &\le C\int_0^1 (1-r)^{\alpha+p} M_p^p\left(\nabla^2 U; r\right) r^2\,dr + C\int_0^1 (1-r)^{\alpha+p} M_p^p\left(\nabla U; r\right) r^2\,dr + \\ &\quad + C\int_0^1 (1-r)^{\alpha+p} \frac{1}{r^p}\, r^p M_p^p\left(\nabla V_1; t\right) r^2\,dr + C\int_0^1 (1-r)^{\alpha+p} M_p^p\left(\nabla^2 V_1; t\right) r^2\,dr.\end{aligned}$$

Here, we have used Hardy's inequalities (33) and (34) of Lemma 9. Finally, by Lemma 6 and (44),

$$\left\| \frac{\partial V_2}{\partial x_1} \right\|^p_{L^p_\alpha(B)} \le C(p,\alpha) \left(\left\| \nabla^2 U \right\|^p_{L^p_{\alpha+p}(B)} + \|\nabla U\|^p_{L^p_{\alpha+p}(B)} + \|\nabla V_1\|^p_{L^p_{\alpha+p}(B)} + \left\| \nabla^2 V_1 \right\|^p_{L^p_{\alpha+p}(B)} \right)$$
$$\le C(p,\alpha) \left(\|\nabla U\|^p_{L^p_\alpha(B)} + \|\nabla V_1\|^p_{L^p_\alpha(B)} \right) \le C(p,\alpha)\, \|\nabla U\|^p_{L^p_\alpha(B)} + C(p,\alpha,a) \left(\widetilde{W}(a) \right)^p.$$

Thus,

$$\left\| \frac{\partial V_2}{\partial x_1} \right\|_{L^p_\alpha(B)} \le C(p,\alpha) \|\nabla U\|_{L^p_\alpha(B)} + C(p,\alpha,a)\, \widetilde{W}(a). \tag{45}$$

The same inequality for $\frac{\partial V_2}{\partial x_2}$ can be proved similarly. By summing (45) for ∇V_2 and (44), we come to

$$\|\mathbf{f}\|_{\mathcal{D}^p_\alpha(B)} \le C(p,\alpha) \|U\|_{\mathcal{D}^p_\alpha(B)} + C(p,\alpha,a)\, \widetilde{W}(a)$$

for the monogenic function $\mathbf{f} = U + V_1 \mathbf{i} + V_2 \mathbf{j}$. This completes the proof of Theorem 5. □

Remark 2. We have proved a quaternion version of "harmonic conjugation theorem" for the weighted Dirichlet spaces $\mathcal{D}^p_\alpha(B)$ with $1 < p < \infty, \alpha > -1$. Earlier in Morais et al,[1], (thm.7.1) we proved a similar theorem for weighted Bergman spaces $\mathcal{H}^p_\alpha(B) = L^p_\alpha(B) \cap \ker D$ again with $1 < p < \infty, \alpha > -1$. Taking into account Lemma 6 and a close connection between $\mathcal{D}^p_\alpha(B)$ and $L^p_\alpha(B)$, we could derive the result by combining Morais et al[1] and Lemma 6. It is possible but provides a result only for $\alpha > p - 1$, which does not include two important cases—unweighted space $\mathcal{D}^p_0(B)$ and the limit space $\mathcal{D}^p_{p-1}(B)$. That is why Theorem 5 deserves an independent proof, which is given above.

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CONFLICT OF INTEREST

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