PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2019, **53**(2), p. 138–141

P h y s i c s

INVARIANT SOLUTION OF THE DIRAC EQUATION IN THE CROSSED ELECTRIC AND MAGNETIC FIELDS ($H > E$)

R. G. PETROSYAN, M. A. DAVTYAN *

Chair of Theoretical Physics YSU, Armenia

In the article exact analytical and invariant solutions for both spinless and half-spin relativistic charged particles in crossed constant electric and magnetic fields, when $H > E$ have been found. It is shown that in both cases the problem reduces to that of quantum harmonic oscillator.

MSC2010: 81V10.

***Keywords*:** invariant solution, crossed electric and magnetic fields, Klein–Gordon equation, Dirac equation.

Introduction. There are very few problems in quantum mechanics, the exact solutions of which have been found in an analytical and invariant form. The most notable ones are Gordon [1] and Volkov [2] solutions of the problem of quantum relativistic particles with zero and half spin in the field of electromagnetic plane wave. Afterwards, many authors examined the problem of relativistic electron in different electromagnetic field configurations. Particularly, the configuration with non equal absolute values of electric and magnetic fields was examined in [3] and [4], but the solutions in both cases were not represented in invariant form. The invariant solution of the Dirac equation has many applications in some modern problems.

In this paper the dynamics of a particle in crossed constant electric and magnetic fields in case of $H > E$ is examined and the exact invariant solutions of the Klein–Gordon and Dirac equations are found.

The crossed constant electric and magnetic fields configuration is described by the four-vector potential of the form

$$A^{\mu} = a^{\mu}\phi, \tag{1}$$

where $\phi = k^{\nu}x_{\nu}$ ($k^2 > 0$). Lorentz gauge condition results in $k^{\nu}a_{\nu} = 0$.

* E-mail: mher.davtyan@ysumail.am

Scalar Particles. Taking into account the Lorentz gauge condition, Klein–Gordon equation which describes the dynamics of a spinless particle can be written as

$$\left[-h^2\delta_\mu\delta^\mu - \frac{2ie\hbar}{c}A^\mu\delta_\mu + \frac{e^2}{c^2}A^2 - m^2c^2\right]\Phi = 0. \tag{2}$$

Seek the solution of Eq. (2) in the form

$$\Phi = \exp\left(\frac{ip^\mu k_\mu}{\hbar k^2}\phi - \frac{ip^\mu x_\mu}{\hbar}\right)U(\phi)\,, \tag{3}$$

where p^μ is for the time being an arbitrary constant four-vector. Substituting (3) in Eq. (2) Klein–Gordon equation gets the form

$$U'' + \left[\frac{1}{\hbar^2k^2}\left(\frac{(a^\mu p_\mu)^2}{a^2} + \frac{(p^\mu k_\mu)^2}{k^2} - p^2 + m^2c^2\right) - \frac{e^2a^2}{\hbar^2k^2c^2}\left(\phi - \frac{ca^\mu p_\mu}{ea^2}\right)^2\right]U = 0. \tag{4}$$

After changing the variable

$$\left(\frac{e^2a^2}{\hbar^2k^2c^2}\right)^{1/2}\left(\phi - \frac{ca^\mu p_\mu}{ea^2}\right) \to z \tag{5}$$

and introducing dimensionless quantity

$$\gamma = \left(\frac{(a^\mu p_\mu)^2}{a^2} + \frac{(p^\mu k_\mu)^2}{k^2} - p^2 + m^2c^2\right)\left(\frac{e^2a^2}{\hbar^2k^2c^2}\right)^{-1/2}, \tag{6}$$

we get the following equation

$$U'' + (\gamma - z^2)U = 0, \tag{7}$$

which is nothing but the Schrödinger equation of linear quantum harmonic oscillator. Its physical solutions exist under the condition

$$\gamma = 2n+1\,,\ n = 0,1,2\ldots \tag{8}$$

and have the form

$$U_n(z) = \frac{1}{(2^n n!\sqrt{\pi})^{1/2}}\exp(-z^2/2)H_n(z), \tag{9}$$

where $H_n(z) = (-1)^n\exp(z^2/2)(d/dz)^n\exp(-z^2/2)$ are the Hermite polynomials.

Particles with $\frac{1}{2}$ Spin. We will proceed from iterated Dirac equation, which has the form:

$$\left[\left(p^\mu - \frac{e}{c}A^\mu\right)^2 - m^2c^2 - \frac{\hbar e}{2c}F_{\mu\nu}\sigma^{\mu\nu}\right]\Psi = 0, \tag{10}$$

where $\sigma^{\mu\nu} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$. γ is the set of Dirac matrices, $F_{\mu\nu}$ is the electromagnetic field tensor. Seek the solution of this equation in the form

$$\Psi_s = \exp\left(\frac{ip^\mu k_\mu}{\hbar k^2}\phi - \frac{ip^\mu x_\mu}{\hbar}\right) u(\phi) \begin{pmatrix} \varphi \\ \kappa \end{pmatrix}_s , \tag{11}$$

where $\begin{pmatrix} \varphi \\ \kappa \end{pmatrix}_s$ $(s = \pm 1)$ are two eigenvectors of the operator $\frac{\hbar e}{2c} F_{\mu\nu}\sigma^{\mu\nu}$ with eigenvalues

$$\lambda_s = s\frac{e\hbar}{c}\sqrt{H^2 - E^2}. \tag{12}$$

By substituting Eq. (11) into Eq. (10) with changing the variable in same manner as in Eq. (5) and introducing dimensionless quantity

$$\widetilde{\gamma} = \left(\frac{(a^\mu p_\mu)^2}{a^2} + \frac{(p^\mu k_\mu)^2}{k^2} - p^2 + m^2c^2 - \lambda\right)\left(\frac{e^2a^2}{\hbar^2k^2c^2}\right)^{-1/2}, \tag{13}$$

we get

$$u^{''} + (\widetilde{\gamma} - z^2)U = 0, \tag{14}$$

which, again, is nothing but the Schrödinger equation of linear quantum harmonic oscillator. Its physical solutions exist under the condition

$$\widetilde{\gamma} = 2n + 1 \ , \ n = 0, 1, 2\ldots \tag{15}$$

and have the form

$$u_n(z) = \frac{1}{(2^n n!\sqrt{\pi})^{1/2}} \exp(-z^2/2) H_n(z), \tag{16}$$

where $H_n(z) = (-1)^n \exp(z^2/2)(d/dz)^n \exp(-z^2/2)$ are the Hermite polynomials.

The energy spectrum is discrete and is implicitly given by Eq. (15). To calculate the exact expression for energy spectrum, we take the frame, where the electric field is directed along the y axis, and the magnetic field is directed along the z axis. With such a specification of the coordinate system, the four-vector potential takes the form

$$A^\mu = a^\mu\phi \ , \ \phi = k^\nu x_\nu = \left(t - \frac{xn}{c}\right), \tag{17}$$

where

$$k^\mu = \frac{1}{c}\left(1, \frac{H}{E}, 0, 0\right) \ , \ a^\mu = (0, 0, -cE, 0). \tag{18}$$

Taking into consideration Eqs. (12), (17) and (18), we get the energy spectrum:

$$\left(\varepsilon_{ns} - \frac{cp_xE}{H}\right)^2 = \frac{H^2 - E^2}{H^2}\left(p_z^2c^2 + m^2c^4 - (2n + s + 1)\, e\hbar c\sqrt{H^2 - E^2}\right), \tag{19}$$

where s is the spin projection number with values $s = +1$ and $s = -1$. For $E = 0$ Eq. (19) reduces to the well known result [5] for the energy spectrum of a half spin particle in a magnetic field:

$$\varepsilon_{ns}^2 = p_z^2c^2 + m^2c^4 - (2n + s + 1)\, e\hbar cH. \tag{20}$$

Main Result. The invariant solution of relativistic particle in crossed electromagnetic field configuration when $H > E$ was exactly found. It is shown that for both spinless particles and for particles with $\frac{1}{2}$ spin the energy spectrum is discrete.

Received 08.04.2019
Reviewed 23.04.2019
Accepted 29.04.2019

REFERENCES

1. Gordon W. Der Comptoneffekt nach der Schrödingerschen Theorie. *Zeitschrift für Physik*, **40**:1–2 (1926), 117–133.
2. Wolkow D.M. Über eine Klasse von Lösungen der Diracschen Gleichung. *Zeitschrift für Physik*, **94**:3 (1935), 250–260.
3. Canuto V., Chiuderi C. Solution of the Dirac Equation in Orthogonal Electric and Magnetic Fields. *Lettere al Nuovo Cimento*, **2**:6 (1969–1970), 223–227.
4. Lam L. Dirac Electron in Orthogonal Electric and Magnetic Fields. *Lettere al Nuovo Cimento*, **3**:9 (1969–1970), 292–294.
5. Berestetskii V.B., Pitaevskii L.P., Lifshitz E.M. *Quantum Electrodynamics*. **Vol. 4** (2nd ed.). Butterworth–Heinemann (1982).
6. Gradshtein I.S., Rijik I.M. *Tables of Integrals, Sums, Series and Products*. M., Nauka (1971) (in Russian).
7. Landau L.D., Lifshitz E.M. *The Classical Theory of Fields*. **Vol. 2** (2nd ed.). Pergamon Press (1959).

Ռ. Գ. ՊԵՏՐՈՍՅԱՆ, Մ. Ա. ԴԱՎԹՅԱՆ

ԴԻՐԱԿԻ ՀԱՎԱՍԱՐՄԱՆ ԻՆՎԱՐԻԱՆՏ ԼՈՒԾՈՒՄԸ ԽԱՉՎՈՂ ԷԼԵԿՏՐԱԿԱՆ ԵՎ ՄԱԳՆԻՍԱԿԱՆ ԴԱՇՏԵՐՈՒՄ ($H > E$)

Աշխատանքում հայտնաբերված են ճշգրիտ անալիտիկ և ինվարիանտ լուծումներ ինչպես զրո, այնպես էլ կես սպինով ռելյատիվիստական լիցքավորված մասնիկների համար, հաստատուն խաչվող էլեկտրական և մագնիսական դաշտերում, $H > E$-ի դեպքում: Ցույց է տրված, որ երկու դեպքում էլ խնդիրը բերվում է քվանտային հարմոնիկ օսցիլյատորի հավասարմանը: