On idempotent algebras with hyperidentities of associativity

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According to [5, 6, 7, 8], a hyperidentity (or $\forall(\forall)$-identity) is a universal second-order formula of the following type:

$$\forall X_1, \ldots, X_m \forall x_1, \ldots, x_n \; (w_1 = w_2),$$

where $X_1, \ldots, X_m$ are functional variables, and $x_1, \ldots, x_n$ are object variables in the words (terms): w_1, w_2. Hyperidentities are usually written without quantifiers: $w_1 = w_2$. We say that the hyperidentity $w_1 = w_2$ is satisfied in the algebra $(Q; \Sigma)$ (or the algebra $(Q; \Sigma)$ satisfies the hyperidentity $w_1 = w_2$) if this equality is valid, whenever every object variable x_i and every functional variable X_j in it is replaced by any element from Q and by any operation of the corresponding arity from Σ, respectively (supposing the possibility of such replacement). If $m > 1$, the hyperidentity is called non-trivial. The number m is called functional rank of the hyperidentity. (For the second order formulas see [2, 3].)

The binary algebra $(Q; \Sigma)$ is called:

a) idempotent if every its operation is idempotent, i.e. $A(x, x) = x$ for all $A \in \Sigma$ and $x \in Q$;

b) q-algebra, if $Q(A)$ is a quasigroup for some $A \in \Sigma$;

c) invertible algebra (or system of quasigroups), if $Q(A)$ is a quasigroup for any $A \in \Sigma$;

d) e-algebra, if $Q(A)$ is a groupoid with an identity element for some $A \in \Sigma$;

e) functionally non-trivial, if the cardinality of Σ is: $|\Sigma| > 1$.

Theorem 1 ([4, 5, 6, 7, 8]). *If a non-trivial hyperidentity of associativity is satisfied in a functionally non-trivial q-algebra (or e-algebra), then it can be only of functional rank 2 and of one of the following forms:*

$$X(x, Y(y, z)) = Y(X(x, y), z), \qquad (ass)_1$$

$$X(x, Y(y, z)) = X(Y(x, y), z), \qquad (ass)_2$$

$$X(x, X(y, z)) = Y(Y(x, y), z). \qquad (ass)_3$$

Moreover, in the class of all q-algebras (e-algebras) from the hyperidentity $(ass)_3$ it follows the hyperidentity $(ass)_2$, and from the hyperidentity $(ass)_2$ it follows the hyperidentity $(ass)_1$. Every q-algebra with a non-trivial hyperidentity of associativity is an algebra with group operations.

Corollary 1 ([1]). *If a non-trivial hyperidentity of associativity is satisfied in a functionally non-trivial invertible algebra, then it can be only of functional rank 2 and of one of the following forms:*$(ass)_1$, $(ass)_2$,$(ass)_3$.

Theorem 2. *If* $(Q;\Sigma)$ *is a functionally non-trivial idempotent algebra with hyperidentity of associativity* $(ass)_1$, *then the cardinality of* Q *is* $|Q| \geq 4$. *Moreover, there exist 24 functionally non-trivial idempotent algebras* $Q(+,\cdot)$, $|Q| = 4$, *with the hyperidentity of associativity* $(ass)_1$.

Theorem 3. *There is not a functionally non-trivial idempotent algebra with hyperidentity of associativity* $(ass)_2$.

Theorem 4. *There is not a functionally non-trivial idempotent algebra with hyperidentity of associativity* $(ass)_3$.

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