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# ARTIN THEOREM FOR SEMIGROUPS

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In this paper, we prove an Artin type theorem for semigroups. Namely, we consider the concepts of hyperalternative and hyperassociative semigroups and prove that every two elements in any hyperalternative semigroup generate a hyperassociative subsemigroup. As a consequence, we characterize all hyperalternative semigroups, and prove that these semigroups form a variety of semigroups with four identities.

*Keywords*: alternative hyperidentity; associative hyperidentity; hyperalternative semigroup; hyperassociative semigroup; polynomial satisfiability; semigroup identity; variety of semigroups.

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## 1. Introduction

In non-associative ring theory, the Artin theorem states that in an alternative algebra the subalgebra generated by any two elements is associative (see [1]). In this paper, we consider *hyperalternative* (cf. [2]) and *hyperassociative* (see [3,4,5,6,7,8,9]) semigroups and prove that any two elements in a hyperalternative semigroup generate a hyperassociative subsemigroup.

About the second order formulas (and languages) see [10,11]. Recall ([12,13,14,15,16,17]) that a *hyperidentity* is a second order formula of the following form:

$$\forall X_1, \ldots, X_m \forall x_1, \ldots, x_n \quad (\omega_1 = \omega_2), \qquad (*)$$

where $\omega_1$, $\omega_2$ are words(terms) in the alphabet of functional variables $X_1, \dots, X_m$ and of object variables $x_1, \dots, x_n$. However, hyperidentities are usually presented without universal quantifiers, i.e. in the form: $\omega_1 = \omega_2$. A hyperidentity $\omega_1 = \omega_2$ is said to be *satisfied* in an algebra $(Q; \Sigma)$ (or the algebra $(Q; \Sigma)$ *satisfies* the hyperidentity $\omega_1 = \omega_2$) if this equality is true when any functional variable $X_i$ is replaced by any operation of the same arity from $\Sigma$ (the possibility of such replacements is assumed) and any object variable $x_j$ is replaced by any element of $Q$. This concept was first considered in [18] for algebras with binary quasigroup operations.

For example ([19,20,21]), in the term algebra of any Boolean algebra the following hyperidentity is satisfied:

$$X(x_1, \dots, x_{n-1}, X(x_1, \dots, x_{n-1}, X(x_1, \dots, x_n))) = X(x_1, \dots, x_n),$$

for any positive integer $n$. For the two-element Boolean algebra this hyperidentity means the equivalence of the corresponding two switching circuits. Varieties of varieties are characterized by hyperidentities ([22]). For applications of hyperidentities see [23,24,25,26,27,28,29].

The variety $V$ satisfies the given hyperidentity if every algebra of the variety $V$ satisfies the same hyperidentity. This hyperidentity is called hyperidentity of the variety $V$.

The hyperidentity $(*)$ is said to be *non-trivial* if $m > 1$, and it is *trivial* if $m = 1$. The number $m$ is called the *functional rank* of the hyperidentity $(*)$.

A binary algebra $(Q; \Sigma)$ is said to be a $q$-algebra ($e$-algebra) if there is an operation $A \in \Sigma$ such that $Q(A)$ is a quasigroup (a groupoid with a unit). A binary algebra $(Q; \Sigma)$ is called *functionally non-trivial* if $|\Sigma| > 1$. It is known ([12,13]) (see also [21,14]) that if an associative non-trivial hyperidentity is satisfied in a functionally non-trivial $q$-algebra ($e$-algebra), then this hyperidentity can only be of functional rank 2 and of one of the following forms:

$$X(x, Y(y, z)) = Y(X(x, y), z), \tag{1.1}$$

$$X(x, Y(y, z)) = X(Y(x, y), z), \tag{1.2}$$

$$Y(x, Y(y, z)) = X(X(x, y), z). \tag{1.3}$$

Moreover, in the class of $q$-algebras ($e$-algebras) the hyperidentity (1.3) implies the hyperidentity (1.2), and the hyperidentity (1.2) implies the hyperidentity (1.1).

Let $Q(\cdot)$ be a semigroup. The following function is said to be its *binary polynomial* (term):

$$F(x, y) = z_1^{\varepsilon_1} z_2^{\varepsilon_2} \dots z_n^{\varepsilon_n}\,, \tag{1.4}$$

where $n \in \mathbb{N}, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \in \mathbb{N}$, $z_1, z_2, \dots, z_n \in \{x, y\}$ and $z_i \neq z_{i+1}$. The number $n$ is called the length of this representation of the polynomial $F(x, y)$. However, due to the identities in the semigroup $Q(\cdot)$, the same polynomial $F(x, y)$ can have different representations of the form (1.4).

By $Q^2_{pol}$ we denote the collection of all binary polynomials of the semigroup $Q(\cdot)$.

We say that the hyperidentity $(*)$ is *polynomially satisfied* (valid) in the semigroup $Q(\cdot)$ if this hyperidentity is satisfied in the binary algebra $\left(Q; Q^2_{pol}\right)$. In [3], it is proved that the class of all semigroups polynomially satisfying the trivial associative hyperidentity:

$$X(x, X(y, z)) = X(X(x, y), z)\,, \tag{$*,*$}$$

forms a finitely based variety of semigroups, and the basis of this variety contains about 1000 identities. These semigroups are called *hyperassociative*. For example, any two-element semigroup is hyperassociative [3]. In [4] (see also [5]), a basis of the identities of the same variety is given, which contains the following four identities:

$$\begin{gathered} x^4 = x^2, \\ xyxzxyx = xyzyx, \\ xy^2z^2 = xyz^2yz^2, \\ x^2y^2z = x^2yx^2yz, \end{gathered}$$

i.e. a semigroup is hyperassociative iff it satisfies these four identities (see also [6,7,8,30,31]).

We say that two hyperidentities are of *equivalence*, if they simultaneously are either polynomially satisfied or none of them is polynomially satisfied in any semigroup $Q(\cdot)$.

The following hyperidentities are consequences of the trivial associative hyperidentity $(*,*)$:

$$X(x, X(x, z)) = X(X(x, x), z)\,,$$

$$X(x, X(y, y)) = X(X(x, y), y)\,,$$

which are called *left* and *right alternative* hyperidentities, respectively.

The equivalence of the left and right alternative hyperidentities is evident. Indeed, the binary polynomial $F$ of the semigroup $Q(\cdot)$ satisfies the identity:

$$F(F(x, x), y) = F(x, F(x, y))$$

iff the binary polynomial $F^*(x, y) = F(y, x)$ satisfies the identity

$$F^*(F^*(y, x), x) = F^*(y, F^*(x, x)).$$

Moreover, $F = (F^*)^*$.

Thus, we call a semigroup *hyperalternative* if the left (or right) alternative hyperidentity is polynomially satisfied in this semigroup. There exists an hyperalternative semigroup, which is not hyperassociative.

First we note that every idempotent semigroup is hyperalternative, since all the binary terms of an idempotent semigroup are the following: $x$, $y$, $x \cdot y$, $y \cdot x$, $x \cdot y \cdot x$ and $y \cdot x \cdot y$ (also see Corollary 3.1).

An example of an idempotent semigroup with 5 elements, which is hyperalternative but not hyperassociative, is given by its Cayley table below:

| $\cdot$ | **1** | **2** | **3** | **4** | **5** |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 1 | 4 | 5 |
| 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 1 | 2 | 5 | 4 | 5 |
| 5 | 5 | 5 | 5 | 5 | 5 |

As the semigroup having this Cayley table is idempotent, hence it is hyperalternative. On the other hand, the identity $xyxzxyx = xyzyx$ is not satisfied for $x = 2$, $y = 4$ and $z = 3$ (as $xyxzxyx = 1$ and $xyzyx = 5$). Thus, this semigroup is not hyperassociative according to the above Polak theorem.

In the main result of the present paper, we prove that every two elements generate a hyperassociative subsemigroup in any hyperalternative semigroup. As a consequence, we characterize all hyperalternative semigroups, and prove that these semigroups form a variety of semigroups with the four identities: the semigroup $Q(\cdot)$ polynomially satisfies the left alternative hyperidentity iff it satisfies the identities:

$$x^4 = x^2,$$
$$x^3yx^3 = x^2yx^2,$$
$$x^2y^2x^2y^2 = x^2y^2,$$
$$yx^3y = yxyxyxy.$$

## 2. Auxiliary results

Consider the following system of identities:

$$x^4 = x^2, \tag{2.1}$$
$$x^3yx^3 = x^2yx^2, \tag{2.2}$$
$$x^2y^2x^2y^2 = x^2y^2, \tag{2.3}$$
$$yx^3y = yxyxyxy. \tag{2.4}$$

For instance, the identities (2.1)-(2.4) are satisfied in any semilattice (even in any idempotent semigroup).

**Lemma 2.1.** *In the semigroup $Q(\cdot)$ with the identities* (2.1)-(2.4)*, the following identities hold:*

$$xy^axy^bx = xy^cxy^dx, \tag{2.5}$$

*where* $a, b, c, d \geqslant 1$ *are natural numbers and* $a + b - c - d$ *is an even integer;*

$$xy^a x^3 y^b x = xy^a xy^b x, \tag{2.6}$$

*where* $a, b \geqslant 1$ *are natural numbers;*

$$xyx^2y^2x = xy^2x^2yx = xy^3x, \tag{2.7}$$

$$xyx^2y^3x = xy^3x^2yx, \tag{2.8}$$

$$x^2y^a x^2 y^b x = x^2 y^c x^2 y^d x, \tag{2.9}$$

*where* $a, b, c, d \geqslant 1$ *are natural numbers and* $a + b - c - d$ *is an even integer;*

$$xy^a x^2 y^b x^2 = xy^c x^2 y^d x^2, \tag{2.10}$$

*where* $a, b, c, d \geqslant 1$ *are natural numbers and* $a + b - c - d$ *is an even integer;*

$$xyx^2yx = xy^2x, \tag{2.11}$$

$$xy^a x^2 y^b x = xy^c x^2 y^d x, \tag{2.12}$$

*where* $a, b, c, d \geqslant 1$ *are natural numbers and* $a + b - c - d$ *is an even integer;*

*Proof.* First, notice that:

$$(xy)^4 = x(yxyxyxy) \overset{(2.4)}{=} xyx^3y,$$

$$(xy)^4 = (xyxyxyx)y \overset{(2.4)}{=} xy^3xy,$$

and therefore:

$$xyxy = (xy)^2 = (xy)^4 = xyx^3y = xy^3xy,$$

i.e.

$$xyxy = xyx^3y = xy^3xy. \tag{2.13}$$

On the other hand,

$$x^3yxy = x^2(xyxy) \overset{(2.13)}{=} x^2(xyx^3y) = x^3yx^3y \overset{(2.2)}{=} x^2yx^2y,$$

$$xyxy^3 = (xyxy)y^2 \overset{(2.13)}{=} (xy^3xy)y^2 = xy^3xy^3 \overset{(2.2)}{=} xy^2xy^2,$$

i.e.

$$x^3yxy = x^2yx^2y, \tag{2.14}$$

$$xyxy^3 = xy^2xy^2. \tag{2.15}$$

Below we also use the following identities:

$$x^3yx^2y = xx^2yx^2y \overset{(2.2)}{=} xx^3yx^3y \overset{(2.1)}{=} x^2yx^3y,$$

$$xy^2xy^3 = xy^2xy^2y \overset{(2.2)}{=} xy^3xy^3y \overset{(2.1)}{=} xy^3xy^2,$$

i.e. with (2.1) we have:

$$x^n yx^m y = x^m yx^n y, \tag{2.16}$$
$$xy^n xy^m = xy^m xy^n, \tag{2.17}$$

for all natural numbers $n, m \geqslant 2$.

Now, we prove (2.5). Note that according to (2.1), it is enough to prove the identity (2.5) in the case when $1 \leqslant a, b, c, d \leqslant 3$. We have:

$$xy^2xy^3x = xy(yxy^3x) \overset{(2.13)}{=} xy(yxyx) = xy^2xyx \overset{(2.1)}{=} xy^4xyx =$$
$$= xy(y^3xyx) \overset{(2.14)}{=} xy(y^2xy^2x) = xy^3xy^2x = (xy^3xy)yx \overset{(2.13)}{=} (xyxy)yx = xyxy^2x.$$

As this identity has all the required identities when the numbers $a+b$ and $c+d$ are both odd and $1 \leqslant a, b, c, d \leqslant 3$, it remains to prove the identity (2.5) in the case when $a+b$ and $c+d$ are both even. But that case follows from the following identities:

$$xy^2xy^2x = x(y^2xy^2x) \overset{(2.14)}{=} x(y^3xyx) = (xy^3xy)x \overset{(2.13)}{=} xyxyx =$$
$$= x(yxyx) \overset{(2.13)}{=} x(yxy^3x) = xyxy^3x = (xyxy)y^2x \overset{(2.13)}{=} (xy^3xy)y^2x = xy^3xy^3x.$$

Thus, the identity (2.5) is proved. Now, let us prove the identity (2.6). If $a = b$, then

$$xy^ax^3y^ax = (xy^ax^3y^a)x \overset{(2.13)}{=} (xy^axy^a)x = xy^axy^ax.$$

If $a > b$, then

$$xy^ax^3y^bx = xy^{a-b}(y^bx^3y^bx) \overset{(2.13)}{=} xy^{a-b}(y^bxy^bx) = xy^axy^bx.$$

Similarly, if $a < b$, then

$$xy^ax^3y^bx = (xy^ax^3y^a)y^{b-a}x \overset{(2.13)}{=} (xy^axy^a)y^{b-a}x = xy^axy^bx.$$

Therefore, we proved (2.6). For the identity (2.7), we have:

$$xyx^2y^2x \overset{(2.3)}{=} xyx^2y^2x^2y^2x = xy(x^2y^2x^2y^2)x \overset{(2.14)}{=} xy(x^3y^2xy^2)x =$$
$$= xyx^2(xy^2xy^2)x \overset{(2.15)}{=} xyx^2(xyxy^3)x = xyx^3yxy^3x =$$
$$= x(yx^3yx)y^3x \overset{(2.13)}{=} x(yxyx)y^3x = xyx(yxy^3x) \overset{(2.13)}{=} xyx(yxyx) \overset{(2.4)}{=} xy^3x.$$

Similarly,

$$xy^2x^2yx = xy^2x^2y^2x^2yx = xy^3xyx^3yx = xyxyxyx = xy^3x,$$

and the identity (2.7) is also proved.

Now, we prove (2.8):

$$xyx^2y^3x = xyx(xy^3x) \overset{(2.4)}{=} xyx(xyxyxyx) = x(yx^2yxy)xyx \overset{(2.5)}{=} x(yxyx^2y)xyx =$$
$$= xyx(yx^2yxy)x = xyx(yxyx^2y)x = (xyxyxyx)xyx \overset{(2.4)}{=} (xy^3x)xyx = xy^3x^2yx.$$

As above, it is sufficient to prove the identities (2.9) and (2.10) with the conditions $1 \leqslant a, b, c, d \leqslant 3$. We have:

$$x^2y^2x^2yx = x(xy^2x^2yx) \overset{(2.7)}{=} x(xy^3x) = x^2y^3x =$$
$$= (x^2y^2)yx \overset{(2.3)}{=} (x^2y^2x^2y^2)yx = x^2y^2x^2y^3x,$$

On the other hand, we have:

$$x^2y^2x^2yx = x^2y^3x = x(xy^3x) \overset{(2.7)}{=} x(xyx^2y^2x) = x^2yx^2y^2x.$$

Let us prove the following identity:

$$x^2y^3x^2y^2x = x^2y^3x.$$

Indeed, if in the identity

$$x^2z^3x = x(xz^3x) \overset{(2.7)}{=} x(xzx^2z^2x) = x^2zx^2z^2x$$

we make the following substitution: $z = y^3$, we get:

$$x^2y^3x^2(y^3)^2x = x^2(y^3)^3x,$$

and taking into account (2.1):

$$x^2y^3x^2y^2x = x^2y^3x.$$

Therefore,

$$x^2y^2x^2yx = x^2yx^2y^2x = x^2y^3x^2y^2x = x^2y^2x^2y^3x.$$

Thus, the identity (2.9) is proved in the case when $a+b$ and $c+d$ are odd numbers. It remains to prove the identity (2.9), under the condition of $a+b$ and $c+d$ being even numbers. We have:

$$x^2yx^2yx = (x^2yx^2y)x \overset{(2.13)}{=} (x^2y^3x^2y)x = x^2y^3x^2yx = x(xy^3x^2yx) \overset{(2.8)}{=} x(xyx^2y^3x) =$$
$$= x^2yx^2y^3x = (x^2yx^2y^3)x \overset{(2.15)}{=} (x^2y^2x^2y^2)x = x^2y^2x^2y^2x,$$

Simultaneously,

$$x^2yx^2yx = x^2yx^2y^3x = (x^2yx^2y)y^2x \overset{(2.13)}{=} (x^2y^3x^2y)y^2x = x^2y^3x^2y^3x.$$

Analogously, we can prove the identity (2.10).

Next, we prove (2.11):

$$xyx^2yx \overset{(2.1)}{=} x(yx^6y)x \overset{(2.4)}{=} x(yx^2yx^2yx^2y)x =$$
$$= x(yx^2yx^2y)x^2yx \overset{(2.5)}{=} x(yxyxy)x^2yx = (xyxyxyx)xyx \overset{(2.4)}{=} (xy^3x)xyx =$$
$$= xy^3x^2yx \overset{(2.8)}{=} xyx^2y^3x = xyx(xy^3x) \overset{(2.4)}{=} xyxxyxyxyx =$$
$$= xyx^2y(xyxyx) \overset{(2.5)}{=} xyx^2y(xy^2xy^2x) =$$
$$= xyx^2(yxy^2xy^2)x \overset{(2.10)}{=} xyx^2(yx^2y^2x^2y^2)x \overset{(2.3)}{=} xyx^2(yx^2y^2)x =$$
$$= (xyx^2yx^2)y^2x \overset{(2.10)}{=} (xy^2x^2y^2x^2)y^2x \overset{(2.3)}{=} (xy^2x^2)y^2x \overset{(2.1)}{=}$$
$$\overset{(2.1)}{=} x(y^2)x^2(y^2)^2x \overset{(2.7)}{=} x(y^2)^3x = xy^6x \overset{(2.1)}{=} xy^2x.$$

Now the identity (2.12) is to be proved. According to (2.11) we have:

$$xy^2x^2y^2x = x(y^2)^2x = xy^2x,$$
$$xy^3x^2y^3x = x(y^3)^2x = xy^2x;$$

Moreover,

$$xyx^2y^3x \overset{(2.8)}{=} xy^3x^2yx = (xy^3x)xyx \overset{(2.4)}{=} (xyxyxyx)xyx =$$
$$= xyx(yxyx^2y)x \overset{(2.5)}{=} xyx(yx^2yxy)x = xy(xyx^2yx)yx \overset{(2.11)}{=} xyxy^2xyx =$$
$$= x(yxy^2xy)x \overset{(2.11)}{=} xyx^2yx \overset{(2.11)}{=} xy^2x.$$

Thus, the identity (2.12) is proved when $a+b$ and $c+d$ are even numbers. Note that the identity also holds when $a+b$ and $c+d$ are both odd numbers. Indeed,

$$xyx^2y^2x \overset{(2.7)}{=} xy^3x,$$
$$xy^2x^2yx \overset{(2.7)}{=} xy^3x.$$

Substituting $y^3$ for $y$ in these identities we get:

$$xy^3x^2y^2x = xy^3x,$$
$$xy^2x^2y^3x = xy^3x.$$

Therefore, (2.12) is also proved in the case when $a+b$ and $c+d$ are odd numbers. □

If a binary polynomial in the semigroup $Q(\cdot)$ has the form:

$$F(x,y) = z_1^{\varepsilon_1} \cdot z_2^{\varepsilon_2} \cdots z_n^{\varepsilon_n}, \quad z_1 \neq z_2 \neq \cdots \neq z_n,$$

where $z_i \in \{x,y\}$, $i = 1,2,\dots,n$, then the number $n$ is called length of this representation of the polynomial $F(x,y)$ and is denoted by $\partial(F)$; the variable $z_1$ (resp. $z_n$) is called first (resp. last) variable of this representation of the polynomial $F(x,y)$ and is denoted by $fv(F)$ (resp. $\ell v(F)$). Also, denote by $fd(F) = \varepsilon_1$ and $\ell d(F) = \varepsilon_n$. If $x^{\delta_1}, x^{\delta_2}, \dots, x^{\delta_m}$ are all the occurrences of the variable $x$ in the given representation of the polynomial $F(x, y)$; then the sum: $\delta_1 + \delta_2 \cdots + \delta_m = \sum_{k=1}^{m} \delta_k$ is called degree of $x$ in $F(x, y)$ and is denoted by $\deg_x(F)$. Analogously, one defines the notion of a degree of the variable $y$ in the representation of $F(x, y)$ and denotes it by $\deg_y(F)$. Moreover, we call the representation of the polynomial $F(x, y)$ non-trivial if $\varepsilon_1 > 1$ and $\varepsilon_n > 1$. We call the polynomials $F(x, y)$ and $G(x, y)$ similar, if the following three conditions hold:

1. $fv(F) = fv(G)$, $\ell v(F) = \ell v(G)$;
2. $fd(F) = fd(G) = 1$ or simultaneously $fd(F) \geq 2$ and $fd(G) \geq 2$;
3. $\ell d(F) = \ell d(G) = 1$ or simultaneously $\ell d(F) \geq 2$ and $\ell d(G) \geq 2$.

**Lemma 2.2.** *Let $Q(\cdot)$ be a semigroup satisfying identities* (2.1)-(2.4) *and let the non-trivial polynomials $F(x, y)$ and $G(x, y)$ of the semigroup $Q(\cdot)$ have the length $\geqslant 6$ for some representation:*

$$F(x, y) = z_1^{\varepsilon_1} \cdot z_2^{\varepsilon_2} \cdots z_n^{\varepsilon_n},$$
$$G(x, y) = u_1^{\delta_1} \cdot u_2^{\delta_2} \cdots u_m^{\delta_m}.$$

*If in these representations $fv(F) = fv(G)$, $\ell v(F) = \ell v(G)$ and $\deg_x(F) - \deg_x(G)$, $\deg_y(F) - \deg_y(G)$ are even integers, then the following identity holds in the semigroup $Q(\cdot)$:*

$$F(x, y) = G(x, y).$$

*Proof.* From the conditions $fv(F) = fv(G)$, $\ell v(F) = \ell v(G)$ it implies that $\partial(F)$ and $\partial(G)$ have the same parity.

Suppose $\partial(F)$ and $\partial(G)$ are even numbers. We can assume both $F(x, y)$ and $G(x, y)$ starting with the variable $x$, otherwise we can switch the names of the variables:

$$F(x, y) = x^{\alpha_1} y^{\beta_1} \cdots x^{\alpha_n} y^{\beta_n},$$
$$G(x, y) = x^{\gamma_1} y^{\tau_1} \cdots x^{\gamma_m} y^{\tau_m},$$

where $m, n \geqslant 3$, i.e. $\partial(F) = 2n$, $\partial(G) = 2m$. Then, we have:

$$x^{\alpha_1}y^{\beta_1}x^{\alpha_2}y^{\beta_2}x^{\alpha_3}y^{\beta_3} = x^{\alpha_1}y^{\beta_1-1}(yx^{\alpha_2}y^{\beta_2}x^{\alpha_3}y)y^{\beta_3-1} =$$
$$= x^{\alpha_1}y^{\beta_1-1}(yx^2y^{\beta_2}x^{\alpha_3+\alpha_2+2}y)y^{\beta_3-1} = x^{\alpha_1}y^{\beta_1}x^2y^{\beta_2}x^{\alpha_3+\alpha_2+2}y^{\beta_3} =$$
$$= x^{\alpha_1-1}(xy^{\beta_1}x^2y^{\beta_2}x)x^{\alpha_3+\alpha_2+1}y^{\beta_3} = x^{\alpha_1-1}(xy^{\beta_1+\beta_2+2}x^2y^2x)x^{\alpha_3+\alpha_2+1}y^{\beta_3} =$$
$$= x^{\alpha_1}y^{\beta_1+\beta_2+2}x^2y^2x^{\alpha_3+\alpha_2+2}y^{\beta_3} = x^{\alpha_1}y^{\beta_1+\beta_2}y^2x^2y^2x^2x^{\alpha_2+\alpha_3}y^{\beta_3} =$$
$$= x^{\alpha_1}y^{\beta_1+\beta_2}y^2x^2x^{\alpha_2+\alpha_3}y^{\beta_3} = x^{\alpha_1}y^{\beta_1+\beta_2+2}x^{\alpha_2+\alpha_3+2}y^{\beta_3} = x^{\alpha_1}y^{\beta_1+\beta_2}x^{\alpha_2+\alpha_3}y^{\beta_3},$$

because $z^{\sigma+2} \overset{(2.1)}{=} z^{\sigma}$, where $\sigma > 1$. Therefore,

$$
\begin{aligned}
F(x,y) &= x^{\alpha_1} y^{\beta_1+\beta_2+\cdots+\beta_{n-1}} x^{\alpha_2+\alpha_3+\cdots+\alpha_n} y^{\beta_n} = \\
&= x^{\alpha_1} y^{\beta_1+\cdots+\beta_{n-1}-2} (y^2x^2) x^{\alpha_2+\cdots+\alpha_n-2} y^{\beta_n} \overset{(2.3)}{=} \\
&= x^{\alpha_1} y^{\beta_1+\cdots+\beta_{n-1}-2} (y^2x^2y^2x^2) x^{\alpha_2+\cdots+\alpha_n-2} y^{\beta_n} = \\
&= x^{\alpha_1} y^{\beta_1+\cdots+\beta_{n-1}} (x^2y^2) x^{\alpha_2+\cdots+\alpha_n} y^{\beta_n} \overset{(2.1)}{=} \\
&= x^{\alpha_1} y^{\beta_1+\cdots+\beta_{n-1}} x^2 y^{2(\beta_1+\cdots+\beta_{n-1})} x^{\alpha_2+\cdots+\alpha_n} y^{\beta_n} = \\
&= \left(x^{\alpha_1} y^{\beta_1+\cdots+\beta_{n-1}} x^2 y^{\beta_1+\cdots+\beta_{n-1}}\right) y^{\beta_1+\cdots+\beta_{n-1}} x^{\alpha_2+\cdots+\alpha_n} y^{\beta_n} \overset{(2.16)}{=} \\
&= \left(x^2 y^{\beta_1+\cdots+\beta_{n-1}} x^{\alpha_1} y^{\beta_1+\cdots+\beta_{n-1}}\right) y^{\beta_1+\cdots+\beta_{n-1}} x^{\alpha_2+\cdots+\alpha_n} y^{\beta_n} = \\
&= x^2 y^{\beta_1+\cdots+\beta_{n-1}} x^{\alpha_1} y^2 x^{\alpha_2+\cdots+\alpha_n} y^{\beta_n} = \\
&= x^2 y^{\beta_1+\cdots+\beta_{n-1}-1} \left(y x^{\alpha_1} y^2 x^{\alpha_2+\cdots+\alpha_n} y\right) y^{\beta_n-1} = \\
&= x^2 y^{\beta_1+\cdots+\beta_{n-1}-1} \left(y x^{\alpha_1+\cdots+\alpha_n} y^2 x^2 y\right) y^{\beta_n-1} = \\
&= x^2 y^{\beta_1+\cdots+\beta_{n-1}} x^{\alpha_1+\cdots+\alpha_n-2} \left(x^2y^2x^2y^{\beta_n}\right) \overset{(2.17)}{=} \\
&= x^2 y^{\beta_1+\cdots+\beta_{n-1}} x^{\alpha_1+\cdots+\alpha_n-2} \left(x^2y^{\beta_n}x^2y^2\right) = \\
&= x^2 y^{\beta_1+\cdots+\beta_{n-1}} x^{\alpha_1+\cdots+\alpha_n} y^{\beta_n} x^2 y^2 = x\left(x y^{\beta_1+\cdots+\beta_{n-1}} x^{\alpha_1+\cdots+\alpha_n} y^{\beta_n} x\right) x y^2 = \\
&= x\left(x y^{\beta_1+\cdots+\beta_n} x^{\alpha_1+\cdots+\alpha_n} y^2 x\right) x y^2 = x^2 y^{\beta_1+\cdots+\beta_n} x^{\alpha_1+\cdots+\alpha_n-2} \left(x^2y^2x^2y^2\right) \overset{(2.3)}{=} \\
&= x^2 y^{\beta_1+\cdots+\beta_n} x^{\alpha_1+\cdots+\alpha_n-2} \left(x^2y^2\right) = x^2 y^{\beta_1+\cdots+\beta_n} x^{\alpha_1+\cdots+\alpha_n} y^2.
\end{aligned}
$$

Hence, we have:

$$
\begin{aligned}
F(x,y) &= x^2 y^{\beta_1+\cdots+\beta_n} x^{\alpha_1+\cdots+\alpha_n} y^2, \\
G(x,y) &= x^2 y^{\tau_1+\cdots+\tau_m} x^{\gamma_1+\cdots+\gamma_m} y^2.
\end{aligned}
$$

According to the conditions of the lemma:

$$
\begin{aligned}
x^{\alpha_1+\cdots+\alpha_n} &= x^{\gamma_1+\cdots+\gamma_m}, \\
y^{\beta_1+\cdots+\beta_n} &= y^{\tau_1+\cdots+\tau_m},
\end{aligned}
$$

since $\alpha_1+\cdots+\alpha_n$ and $\gamma_1+\cdots+\gamma_m$, as well as $\beta_1+\cdots+\beta_n$ and $\tau_1+\cdots+\tau_m$ have the same parity. Thus, $F(x,y) = G(x,y)$.

Now, suppose $\partial(F)$ and $\partial(G)$ are odd numbers and let

$$
\begin{aligned}
F(x,y) &= x^{\alpha_1} y^{\beta_1} x^{\alpha_2} y^{\beta_2} \cdots x^{\alpha_n} y^{\beta_n} x^{\alpha_{n+1}}, \\
G(x,y) &= x^{\gamma_1} y^{\tau_1} x^{\gamma_2} y^{\tau_2} \cdots x^{\gamma_m} y^{\tau_m} x^{\gamma_{m+1}},
\end{aligned}
$$

where $m, n \geqslant 3$. As above, we get:

$$\begin{aligned}
F(x,y) &= x^2y^{\beta_1+\dots+\beta_n}x^{\alpha_1+\dots+\alpha_n}y^2x^{\alpha_{n+1}} = \\
&= x^2y^{\beta_1+\dots+\beta_n-2}(y^2x^2)x^{\alpha_1+\dots+\alpha_n-2}y^2x^{\alpha_{n+1}} \overset{(2.3)}{=} \\
&= x^2y^{\beta_1+\dots+\beta_n-2}(y^2x^2y^2x^2)x^{\alpha_1+\dots+\alpha_n-2}y^2x^{\alpha_{n+1}} = \\
&= x^2y^{\beta_1+\dots+\beta_n}(x^2y^2x^{\alpha_1+\dots+\alpha_n}y^2)x^{\alpha_{n+1}} \overset{(2.16)}{=} \\
&= x^2y^{\beta_1+\dots+\beta_n}x^{\alpha_1+\dots+\alpha_n}\left(y^2x^2y^2x^{\alpha_{n+1}}\right) \overset{(2.17)}{=} \\
&= x^2y^{\beta_1+\dots+\beta_n}x^{\alpha_1+\dots+\alpha_n}y^2x^{\alpha_{n+1}}y^2x^2 = \\
&= x^2y^{\beta_1+\dots+\beta_n-1}\left(yx^{\alpha_1+\dots+\alpha_n}y^2x^{\alpha_{n+1}}y\right)yx^2 = \\
&= x^2y^{\beta_1+\dots+\beta_n-1}\left(yx^{\alpha_1+\dots+\alpha_{n+1}}y^2x^2y\right)yx^2 = \\
&= x^2y^{\beta_1+\dots+\beta_n}x^{\alpha_1+\dots+\alpha_{n+1}}\left(y^2x^2y^2x^2\right) \overset{(2.3)}{=} \\
&= x^2y^{\beta_1+\dots+\beta_n}x^{\alpha_1+\dots+\alpha_{n+1}}y^2x^2.
\end{aligned}$$

Thus:

$$\begin{aligned}
F(x,y) &= x^2y^{\beta_1+\dots+\beta_n}x^{\alpha_1+\dots+\alpha_{n+1}}y^2x^2, \\
G(x,y) &= x^2y^{\tau_1+\dots+\tau_m}x^{\gamma_1+\dots+\gamma_{m+1}}y^2x^2,
\end{aligned}$$

and again, $F(x,y) = G(x,y)$ because of the conditions of the lemma. □

The following lemma has a similar proof.

**Lemma 2.3.** *Let $Q(\cdot)$ be a semigroup satisfying the identities* (2.1)-(2.4) *and let the similar polynomials $F(x,y)$ and $G(x,y)$ of the semigroup $Q(\cdot)$ have the length $\geqslant 8$ for some representation:*

$$\begin{aligned}
F(x,y) &= z_1^{\varepsilon_1}\cdot z_2^{\varepsilon_2}\cdots z_n^{\varepsilon_n}, \\
G(x,y) &= u_1^{\delta_1}\cdot u_2^{\delta_2}\cdots u_m^{\delta_m}.
\end{aligned}$$

*If in these representations $\deg_x(F) - \deg_x(G)$ and $\deg_y(F) - \deg_y(G)$ are even integers, then the following identity holds in the semigroup $Q(\cdot)$:*

$$F(x,y) = G(x,y).$$

## 3. Main Result

Recall that the semigroup which polynomially satisfies the left alternative hyperidentity:

$$X(X(x,x),y) = X(x,X(x,y)), \tag{3.1}$$

is called hyperalternative.

**Theorem 3.1.** *If the semigroup $Q(\cdot)$ is hyperalternative, then any two elements in $Q(\cdot)$ generate a hyperassociative subsemigroup, i.e. the following identity*

$$X(X(A(x,y),B(x,y)),C(x,y)) = X(A(x,y),X(B(x,y),C(x,y))), \tag{3.2}$$

*holds for any binary polynomials $X, A, B, C$ of $Q(\cdot)$. (Hence, the semigroup $Q(\cdot)$ polynomially satisfies the following hyperidentity of functional rank 4:*

$$X(X(Y(x,y), Z(x,y)), U(x,y)) = X(Y(x,y), X(Z(x,y), U(x,y))).) \tag{3.3}$$

**Corollary 3.1.** *The semigroup $Q(\cdot)$ polynomially satisfies the left alternative hyperidentity* (3.1) *iff it satisfies the identities* (2.1)-(2.4).

*Proof.* The proof is divided into two parts. In the first part, we derive the necessary conditions for the semigroup $Q(\cdot)$ to be hyperalternative. Then, in the second part, we show that these conditions are sufficient for the subsemigroup generated by any 2 elements, to be hyperassociative.

Let us prove that for the semigroup $Q(\cdot)$ to be hyperalternative, it is necessary to show that $Q(\cdot)$ satisfies identities (2.1)-(2.4).

If $X(x,y) = x^2$ then from the identity (3.1) we get $x^4 = x^2$, i.e. identity (2.1). If we substitute $X(x,y) = xyx$ in (3.1) we get $x^3yx^3 = x^2yx^2$, i.e. we also have identity (2.2). Now, substituting $X(x,y) = xy^2$ in (3.1), we get:

$$x^3y^2 = x^2y^2xy^2.$$

If we also take $x = z^2$ then we have $z^2y^2 = z^2y^2z^2y^2$, which is identity (2.3).

Taking $X(x,y) = yxy$ in (3.1), we get: $yx^3y = yxyxyxy$, i.e. identity (2.4).

Next, we prove the sufficiency.

Denote by $D(x,y)$ and $E(x,y)$ respectively the left and right polynomials of the equality (3.3).

Let us prove that the polynomials $D(x,y)$ and $E(x,y)$ are similar. First of all, we prove that $fv(D) = fv(E)$ and $\ell v(D) = \ell v(E)$. If the polynomial $X(x,y)$ starts with $x$, i.e. if $fv(X) = x$, then: $fv(D) = fv(X(A(x,y), B(x,y))) = fv(A(x,y)) = fv(A) = fv(E)$, hence $fv(D) = fv(E)$. Now, if $fv(X) = y$, then: $fv(D) = fv(C) = fv(X(B(x,y), C(x,y))) = fv(E)$. Therefore, in any case, $fv(D) = fv(E)$. Analogously, we prove that $\ell v(D) = \ell v(E)$. In particular, this implies that the lengths of the polynomials $D(x,y)$ and $E(x,y)$ have the same parity. Now, we prove that if $fd(D) = 1$, then $fd(E) = 1$, too. If $fv(X) = x$, then we have: $1 = fd(D) = fd(A) = fd(E)$. If $fv(X) = y$, then we get: $1 = fd(D) = fd(C) = fd(E)$. Moreover, notice that if $fd(D) \geq 2$ then $fd(E) \geq 2$, too. Analogously, we prove that $\ell d(D) = \ell d(E) = 1$ or simultaneously $\ell d(D) \geq 2$ and $\ell d(E) \geq 2$. Thus, the polynomials $D(x,y)$ and $E(x,y)$ are similar.

Next, we prove that $deg_x(D) - deg_x(E)$ and $deg_y(D) - deg_y(E)$ are even integers. Indeed:

$$deg_x(D) = deg_x(X)(deg_x(X)deg_x(A) + deg_y(X)deg_x(B)) + deg_y(X)deg_x(C),$$

$$deg_x(E) = deg_x(X)deg_x(A) + deg_y(X)(deg_x(X)deg_x(B) + deg_y(X)deg_x(C)),$$

and therefore:

$$deg_x(D) - deg_x(E) = deg_x(A)deg_x(X)(deg_x(X) - 1) - deg_x(C)deg_y(X)(deg_y(X) - 1)$$

is an even integer.

In the same way, we prove that $deg_y(D) - deg_y(E)$ is even, too.

Moreover, we get the following results: $deg_x(D) = 0$ iff $deg_x(E) = 0$ and $deg_x(D) = 1$ iff $deg_x(E) = 1$; and analogously, $deg_y(D) = 0$ iff $deg_y(E) = 0$ and $deg_y(D) = 1$ iff $deg_y(E) = 1$. Indeed, if the numbers $a$ and $b$ are non-negative integers, then $ab = 0$ iff $a^2b = 0$ and $ab = 1$ iff $a^2b = 1$.

Therefore, due to Lemma 2.3, if we were able to make the lengths of the polynomials $D(x, y)$ and $E(x, y)$ greater than or equal to 8, then these two polynomials would be equal.

Now, we find the polynomials which have a representation of a length greater than or equal to 8. First of all, notice the following equalities:

$$xy^3x = xyxyxyx = (xy)^3x = (xy)^4x,$$

$$xy^2x = xy^6x = x(y^2)^3x = (xy^2)^4x,$$

$$xyxy = (xy)^2 = (xy)^4,$$

$$x^2y^2 = (x^2y^2)^2 = (x^2y^2)^4.$$

Hence, the only polynomials which we cannot make of a longer length $\geq 8$ using the above four equalities are the following:

1. $F_1(x, y) = x^m, m = 1, 2$ or 3;
2. $F_2(x, y) = y^m, m = 1, 2$ or 3;
3. $F_3(x, y) = x^my^n$, where either $m$ or $n$ is 1;
4. $F_4(x, y) = y^mx^n$, where either $m$ or $n$ is 1;
5. $F_5(x, y) = x^myx^n$, where $m \leq 2$ and $n \leq 3$;
6. $F_6(x, y) = y^mxy^n$, where $m \leq 2$ and $n \leq 3$;

Hence, it is enough to consider the cases, when, at least, one of the polynomials $D(x, y)$ and $E(x, y)$ coincides with one of the six polynomials $F_i(x, y), i = 1, \cdots, 6$.

We consider the cases when the polynomial $D(x, y)$ coincides with one of the above polynomials. The other cases can be handled analogously.

1) If $D(x, y) = F_1(x, y) = x^m$ then $deg_y(D)$ is zero, and hence it is $deg_y(E)$. Hence, $\ell(E) = 1$ as well. If we also have that $m = 1$, then we get $deg_x(D) = 1$; therefore $deg_x(E) = 1$, hence, $D(x, y) = x = E(x, y)$. If $m = 2$ or $m = 3$, then $deg_x(E)$ is bigger than 1 and has the same parity as $deg_x(D)$, and hence, $D(x, y) = x^{deg_x(D)} = x^{deg_x(E)} = E(x, y)$. The case of $D(x, y) = F_2(x, y)$ is done analogously.

2) If $D(x, y) = F_3(x, y) = x^my^n$ where one of the numbers $m$ and $n$ is equal to 1, then either $deg_x(D) = 1$ or $deg_y(D) = 1$. The same is true for the polynomial $E(x, y)$. As both polynomials start and end with the same variables, then the length of $E(x, y)$ also is 2. If both $m$ and $n$ are equal to 1, then $D(x, y) = xy = E(x, y)$. If, for example, $m = 1$ and $n \geq 2$, then $D(x, y) = xy^{deg_y(D)} = xy^{deg_y(E)} = E(x, y)$,

since $deg_y(E) - deg_y(D)$ is an even integer, and both numbers are greater than 1. The case of $F_4(x, y)$ is handled in the same way.

3) Now, if $D(x, y) = F_5(x, y) = x^m yx^n$ then, since we have that $deg_y(D) = 1$, we get $deg_y(E) = 1$. Moreover, we know that the polynomials $D(x, y)$ and $E(x, y)$ start and end with the same variables, hence, $E(x, y)$ has the length 3 and is of the same form as $D(x, y)$. If $m = 1$, then the respective power in the polynomial $E(x, y)$ also is 1. The same is true for $n$ too. Now, because of equalities (2.2) and $x^2yx^3 = x^4yx^3 = x(x^3yx^3) \overset{(2.2)}{=} x(x^2yx^2) = x^3yx^2$, we get the equality of the polynomials $D(x, y)$ and $E(x, y)$, as the numbers $deg_x(D)$ and $deg_x(E)$ have the same parity. The case of $F_6(x, y)$ is handled in the same way.

This completes the proof of the theorem. □

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