Equational Theory of Algebras with Fuzzy Operations

Sergey Davidov
Yerevan State University
Yerevan, Armenia
e-mail: davidov@ysu.am

Jahangir Hatami
Yerevan State University
Yerevan, Armenia
e-mail: jhatami51@gmail.com

ABSTRACT
In this paper we prove the compactness theorem for equational theory of algebras with fuzzy operations.

Keywords
Fuzzy equivalence relation, Fuzzy equality, Fuzzy function, Fuzzy operation, Complete residuated lattice, Equational theory

1. INTRODUCTION
The problem of development of algebras with fuzzy operations is formulated in ([1], P: 136). Fuzzy approaches to various universal algebraic concepts started with Rosenfeld's fuzzy groups [2]. Since then, many fuzzy algebraic structures have been studied (vector spaces, rings, etc.). Another fuzzy approach to universal algebras was initiated by Bělohl avek and Vychodil [1,3], who studied the so-called algebras with fuzzy equalities and developed a fuzzy equational logic. These structures have two parts: the functional part, which is an ordinary algebra and the relational part, which is the carrier set of the algebra, equipped with a fuzzy equality which is compatible with all of the fundamental operations of the ordering algebra. Algebras with fuzzy equalities are structures for the equational fragment of fuzzy logic and they are only special cases of more general fuzzy structures.
In the fuzzy set theory there were many different approaches to the concept of a fuzzy function. In a number of papers various kinds of fuzzy functions based on fuzzy equivalence relations were studied. In particular, such approach was used in definitions of partial fuzzy functions and fuzzy functions, given by Klawonn [4], strong fuzzy functions and perfect fuzzy functions, given by Demirci [5].

2. PRELIMINARIES
In this paper we will use complete residuated lattices as the structures of truth values. Residuated lattices were introduced by Ward and Dilworth in ring theory. Complete residuated lattices as a structure of truth values were introduced into the context of fuzzy sets and fuzzy logic by Goguen [6].
Definition 2.1 *A complete residuated lattice is an algebra* $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$, *where*

(i) $\mathbf{L} = \langle L, \wedge, \vee, 0, 1\rangle$ *is a complete lattice with the least element 0 and the greatest element 1,*
(ii) $\langle L, \otimes, 1\rangle$ *is a commutative monoid, i.e.,* $\otimes$ *is associative, commutative, and* $a \otimes 1 = a$ *for each* $a \in L$,
(iii) $\otimes$ *and* $\rightarrow$ *satisfy adjointness, i.e.,*

$$a \otimes b \leq c \text{ iff } a \leq b \rightarrow c \tag{1}$$

for each $a, b, c \in L$ ($\leq$ *denotes the lattice ordering).*

An $\mathbf{L}$-fuzzy set of X is a mapping $\mu : X \rightarrow L$ and the set of all $\mathbf{L}$-fuzzy sets of X is denoted by L^X. An n-ary L-relation of X is a mapping $r : X^n \rightarrow L$. An L- fuzzy set r of $X \times Y$ is called an L-fuzzy relation or simply, a fuzzy relation from X to Y.

Definition 2.2. *An* $L-$*equivalence (fuzzy equivalence) relation* E *on a set* X *is a mapping* $E : X \times X \rightarrow L$ *satisfying*

(i) $E(x, x) = 1 \quad (Reflexivity)$,
(ii) $E(x, y) = E(y, x) \quad (Symmetry)$,
(iii) $E(x, y) \otimes E(y, z) \leq E(x, z) \quad (Transitivity)$,
for every $x, y, z \in X$.

An $L-equivalence$ E *on* X, where $E(x, y) = 1$ *implies* $x = y$ will be called an $L-equality$ $(fuzzy\ equality)$. $L-equalities$ will usually be denoted by $\approx$.

Lemma 2.3. *Let* $M = \prod_{i\in I} M_i$ *be a direct product of a family of sets* $\{M_i | i \in I\}$ *and* $\approx^{M_i}$ *be a fuzzy equality on* M_i *for every* $i \in I$. *Also let*

$$(a \approx^M b) = \wedge_{i\in I}(a(i) \approx^{M_i} b(i))$$

for all $a, b \in M$. *Then* $\approx^M$ *is a fuzzy equality on* M.

Definition 2.4. *Let* $\approx^{\mathbf{M}}$ *be a fuzzy equality on* M. *An (n+1)-ary fuzzy relation* ρ *on a set* M *is called an n-ary fuzzy operation w.r.t.* $\approx^{\mathbf{M}}$ *and* $\approx^{\mathbf{M^n}}$ *if we have the following conditions*
Extensionality:
$(p \approx^{\mathbf{M^n}} p') \otimes (y \approx^{\mathbf{M}} y') \otimes \rho(p, y) \leq \rho(p', y') \quad \forall p, p' \in M^n, \ \forall y, y' \in M$,
Functionality:

$\rho(p, y) \otimes \rho(p, y') \leq y \approx^{\mathbf{M}} y' \quad \forall p \in M^n, \ \forall y, y' \in M$,

Fully- defined:

$$\bigvee_{y\in M} \rho(p, y) = 1 \quad \forall p \in M^n,$$

where $(a_1, \cdots, a_n) \approx^{M^n} (b_1, \cdots, b_n) = \bigwedge_{i=1}^{n}(a_i \approx^M b_i)$. *We say that* ρ *is a fuzzy operation on* M *with arity* n.

Definition 2.5. *A type is a triplet* $<\approx, F, \sigma>$ *where* $\approx \notin F$ *and* σ *is a mapping* $\sigma : F \bigcup\{\approx\} \mapsto \mathbf{N}_0$ *with* $\sigma(\approx) = 2$. *Each* $f \in F$ *is called a functional symbol,* $\approx$ *is a relation symbol called a symbol of fuzzy equality. Mapping* σ *assigns an arity* $\sigma(f)$ *to every function* *symbol $f \in F$. Symbol $\approx$ stands for a binary relation symbol.*

Definition 2.6. *A model of the type $<\approx, F, \sigma>$ is a triplet $\mathcal{M} =< M, \approx^M, F^M >$ consisting of*

1. *a non empty set M called a domain or a based set, or the underlying set of the model $\mathcal{M}$;*
2. *for symbol $\approx$ a fuzzy equality $\approx^M$ on the set M;*
3. *for every n-ary functional symbol $f^{(n)}$ of F, a fuzzy n-ary operation $f_M^{(n)}$ on M called the interpretation of the symbol f in the model $\mathcal{M}$.*

The model $\mathcal{M} =< M, \approx^M, F^M >$ is called an algebra with fuzzy operations and fuzzy equality of type $< M, \approx^M, F^M >$ or, simply, F-algebra.

For some properties of F-algebras see [7].

3. EQUATIONAL THEORY OF F-ALGEBRAS

Suppose $\mathcal{F} =<\approx, F, \sigma>$ is a type and $\mathcal{L}$ is a complete residuated lattice. A language of equational logic of type $\mathcal{F} =<\approx, F, \sigma>$ for $\mathcal{L}$ consists of a binary relation symbol $\approx$ called a symbol of fuzzy equality, functional symbols $f \in F$ with their arities $\sigma(f) \in \mathcal{N}_0$, symbol $\overline{a}$ of truth degrees ($a \in L$), (at least denumerable) set X of variables with $X \cap F = \emptyset$, and auxiliary symbols (parentheses, etc.).

Definition 3.1. *Let X be a set of variables. Let $\mathcal{F}$ be a type such that $X \cap F = \emptyset$. The set X of terms of type $\mathcal{F}$ is the smallest set such that*

(i) $X \subset T(X)$;
(ii) If $f \in F$, f is an n-ary and $t_1, \ldots, t_n \in T(X)$, then $f(t_1, \ldots, t_n) \in T(X)$.

Every $t \in T(X)$ is called a term of type $\mathcal{F}$ over X.

Definition 3.2. *Let $\mathcal{F}$ be a type. An identity of type F over X is an expression $t \approx t'$, where $t, t' \in T(X)$.*

Structures of equational logic are algebras with fuzzy operations and fuzzy equality. As usual, given an F-algebra $\mathcal{M}$ with fuzzy operations, a valutations (of variables from X) in $\mathcal{M}$ is a mapping $\nu : X \mapsto M$ assigning to each variable $x \in X$ some element $\nu(x) \in M$, a value $\|t\|_{\mathcal{M},\nu}$ of a term t, and a truth degree $\|t \approx t'\|_{\mathcal{M},\nu}$ of $t \approx t'$ are defined in straightforward way. The details follow.

Definition 3.3. *Let $\mathcal{M}$ be an $\mathbf{F}-$algebra of type $\langle \approx, F, \sigma \rangle$, $\nu : X \to M$ be a valuation, t be a term. A value $\|t\|_{\mathcal{M},\nu}$ of t in $\mathcal{M}$ under ν, i.e., $\|...\|_{\mathcal{M},\nu} : T(X) \to M$ is defined as follows:*

(i) if t is a variable x, then $\|t\|_{\mathcal{M},\nu} =^1 \nu(x)$;

(ii) if $t = f^{(n)}(x_1, \cdots, x_n)$, then $\|t\|_{\mathcal{M},\nu} =^\alpha a$ iff $f_M^{(n)}(\nu(x_1), \cdots, \nu(x_n), a) = \alpha$;

(iii) if $t = f^{(n)}(t_1, \cdots, t_n)$ and $\|t_i\|_{\mathcal{M},\nu} =^{\alpha_i} a_i$, then $\|t\|_{\mathcal{M},\nu} =^\alpha a$ iff $f_M^{(n)}(a_1, \cdots, a_n, a) = \alpha_{n+1}$ and $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_n \otimes \alpha_{n+1}$.

Note: condition (ii) says that if $f_M^{(n)}(\nu(x_1), \cdots, \nu(x_n), a) = \alpha$, then the element $\|t_i\|_{\mathcal{M},\nu} \in M$ is equal to $a \in M$ with the truth degree $\alpha \in L$.

Definition 3.4. *Let $\mathcal{M}$ be an $\mathbf{F}-$algebra of type $\langle \approx, F, \sigma \rangle$, $\nu : X \to M$ be a valuation, $t \approx t'$ be a formula of fuzzy equational logic (identity). A truth degree $\|t \approx t'\|_{\mathcal{M},\nu}$ of $t \approx t'$ in $\mathcal{M}$ under ν is defined by*

$$\|t \approx t'\|_{\mathcal{M},\nu} = \alpha \otimes \beta \to (a \approx^{\mathcal{M}} b),$$

where $\|t\|_{\mathcal{M},\nu} =^\alpha a$ and $\|t'\|_{\mathcal{M},\nu} =^\beta b$.

A truth degree $\|t \approx t'\|_{\mathcal{M}}$ of the identity $t \approx t'$ in $\mathcal{M}$ is defined by

$$\|t \approx t'\|_{\mathcal{M}} = \bigwedge_{\nu : X \to M} \|t \approx t'\|_{\mathcal{M},\nu}.$$

We say that an identity $\|t \approx t'\|$ is valid in M if $\|t \approx t'\|_{\mathcal{M}} = 1$, we denote by $\mathcal{M} \models t \approx t'$. This implies that for every $\nu : X \to M$ we have $\|t \approx t'\|_{\mathcal{M},\nu} = 1$, i.e., $\alpha \otimes \beta \leq (a \approx^{\mathcal{M}} b)$.

Example 3.5. [8] Let $t = f(x, y)$, $s = f(y, x)$, where f is a binary functional symbol. Also, let $\mathcal{M} = (M, F^M, \approx)$ be an $\mathbf{F}$-algebra, and f_M be an interpretation of the symbol f in $\mathcal{M}$, i.e.,

$$f_M : M \times M \times M \mapsto L.$$

If for a valuation $\nu : X \mapsto L$ we have $\nu(x) = a, \nu(y) = b$ and $f_M(a, b, m) = \alpha$, $f_M(b, a, n) = \beta$, then $\|t\|_{M,\nu} =^\alpha m$, $\|s\|_{M,\nu} =^\beta n$. Hence, we have

$\|t \approx s\|_{M,\nu} = (\alpha \otimes \beta) \mapsto (m \approx^M n) = 1$ iff $\alpha \otimes \beta \leq m \approx^M n$.

Thus, we say that an operation f_M is a commutative if for every $a, b, m, n \in M$ it satisfies the condition

$$f_M(a, b, m) \otimes f_M(b, a, n) \leq m \approx^M n.$$

Definition 3.6. *Let $Id(X)$ be the set of all identities of type $\mathcal{F} = (\approx, F, \sigma)$ over X. Any subset T of $Id(X)$ is called an equational theory of type $\mathcal{F}$ over X.*

An F-algebra $\mathcal{M}$ of type $\mathcal{F}$ is called a model of the set T, if

$$\mathcal{M} \models (t \approx s) \text{ for every } (t \approx s) \in T,$$

that is, if every identity $t \approx s$ of the set T is true in $\mathcal{M}$.

Theorem 3.7 (Compactness Theorem) *A fuzzy theory T has a model if and only if every finite subtheory of T has a model.*

REFERENCES

[1] R. Bělohl avek, V. Vychodil, "Fuzzy Equational Logic", *Springer, Berlin, Heidelberg*, 2005.

[2] A. Rosenfeld, "Fuzzy groups", *J. Math. Anal. Appl.* **35***(3)*, pp.512-517, 1971.

[3] R. Bělohl avek, V. Vychodil, "Algebras with fuzzy equalities", *Fuzzy Sets and Systems, 157*, pp. 161-201, 2006 .

[4] F. Klawonn, "Fuzzy points", *in: V. Novak, I. Perfilieva (Eds), Discovering World with Fuzzy Logic, Physica, Heidelberg*, pp.431-453, 2000.

[5] M. Demirci, "Fuzzy functions and their fundamental properties", *Fuzzy Sets and Systems* **106**, pp. 239-246, 1999.

[6] I. Bošnjak, R. Madarasz, G. Vojvodic, "Algebras of fuzzy sets", *Fuzzy Sets and Systems* **160**, pp.2979-2988, 2009.

[7] S. S. Davidov, J. Hatami, "Algebras with fuzzy operations", *Mathematical Problems of Computer Science, 38*, pp. 49-52, 2012.

[8] S. S. Davidov, J. Hatami, "F-groups", *14 Iranian Conference on Fuzzy System (www.icfs14.ir)*, pp. 138, 2014.