GEOMETRIC PROBABILITY CALCULATION FOR A TRIANGLE

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Let $P(L(\omega) \subset D)$ is the probability that a random segment of length $l$ in $\mathbb{R}^n$ having a common point with body $D$ entirely lies in $D$. In the paper, using a relationship between $P(L(\omega) \subset D)$ and covariogram of $D$, the explicit form of $P(L(\alpha) \subset D)$ for arbitrary triangle on the plane is obtained.

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**Introduction.** Let $\mathbb{R}^n$ ($n \geq 2$) be the $n$-dimensional Euclidean space, $D \subset \mathbb{R}^n$ be a bounded convex body with inner points, and $V_n$ be the $n$-dimensional Lebesgue measure in $\mathbb{R}^n$.

Consider the set of the segments of a constant length that are contained in $D$. The measure evaluation problem of such segment sets no simple solution and depends on the shape of $D$. It is known the explicit form for the kinematic measures of the disk, the rectangle, if the length of the segment is less than the smaller side of the rectangle (see [1, 2]), the equilateral triangle, the rectangle and the regular pentagon (for an arbitrary length of the segment) [3].

**Definition 1.** (see [2]). The function

$$C(D, h) = V_n(D \cap (D + h)), \quad h \in \mathbb{R}^n,$$

is called the covariogram of the body $D$. Here $D + h = \{x + h, x \in D\}$.

Let $S^{n-1}$ denote the $(n - 1)$-dimensional unit sphere in $\mathbb{R}^n$ centered at the origin. We consider a random line, which is parallel to $u \in S^{n-1}$ and intersects $D$, that is, an element from the set:

$$\Omega_1(u) = \{\text{lines, which are parallel to } u \text{ and intersect } D\}.$$

Let $\Pi_{u_\perp}D$ be the orthogonal projection of $D$ onto the hyperplane $u_\perp$ (here $u_\perp$ stands for the hyperplane with normal $u$ and passing through the origin).

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A random line, which is parallel to \( \mathbf{u} \) and intersects \( \mathbf{D} \), has an intersection point (denoted by \( x \)) with \( \Pi_{\mathbf{r}_u} \mathbf{D} \). We can identify the points of \( \Pi_{\mathbf{r}_u} \mathbf{D} \) and the lines, which intersect \( \mathbf{D} \) and are parallel to \( \mathbf{u} \), meaning that we can identify the sets \( \Omega_1(\mathbf{u}) \) and \( \Pi_{\mathbf{r}_u} \mathbf{D} \). Assuming that the intersection point \( x \) is uniformly distributed over the convex body \( \Pi_{\mathbf{r}_u} \mathbf{D} \), we can define the following distribution function.

**Definition** 2. The function

\[
F(\mathbf{u},t) = \frac{V_{n-1}\{x \in \Pi_{\mathbf{r}_u} \mathbf{D} : V_1(\mathbf{g}(\mathbf{u},x) \cap \mathbf{D}) < t\}}{b_D(\mathbf{u})}
\]

is called orientation-dependent chord length distribution function of \( \mathbf{D} \) in direction \( \mathbf{u} \) at a point \( t \in \mathbb{R}^1 \), where \( \mathbf{g}(\mathbf{u},x) \) is the line, which is parallel to \( \mathbf{u} \) and intersects \( \Pi_{\mathbf{r}_u} \mathbf{D} \) at the point \( x \) and \( b_D(\mathbf{u}) = V_{n-1}(\Pi_{\mathbf{r}_u} \mathbf{D}) \).

Observe that each vector \( \mathbf{h} \in \mathbb{R}^n \) can be represented in the form \( \mathbf{h} = (\mathbf{u},t) \), where \( \mathbf{u} \) is the direction of \( \mathbf{h} \), and \( t \) is the length of \( \mathbf{h} \).

Let \( L(\omega) \) be a random segment of length \( l > 0 \), which is parallel to a given fixed direction \( \mathbf{u} \in S^{n-1} \) and intersects \( \mathbf{D} \). Consider the random variable \( |L(\omega)| := V_1(L(\omega) \cap \mathbf{D}) \), where \( L(\omega) \in \Omega_2(\mathbf{u}) \), and the set \( \Omega_2(\mathbf{u}) \) is defined as follows:

\[
\Omega_2(\mathbf{u}) = \{ \text{segments of lengths } l, \text{ which are parallel to } \mathbf{u} \text{ and intersect } \mathbf{D} \}.
\]

Observe that each random segment \( L(\omega) \) lying on a line \( \mathbf{g}(\mathbf{u},x) \) can be specified by the coordinates \( (\mathbf{g}(\mathbf{u},x),y) \), where \( y \) is the one-dimensional coordinate of the center of \( L(\omega) \) on the line \( \mathbf{g}(\mathbf{u},x) \). As the origin on the line \( \mathbf{g}(\mathbf{u},x) \) we take one of the intersection points of the line \( \mathbf{g}(\mathbf{u},x) \) with the boundary of domain \( \mathbf{D} \). Using the above notation, we can identify \( \Omega_2(\mathbf{u}) \) with the following set:

\[
\Omega_2(\mathbf{u}) = \left\{ (x,y) : x \in \Pi_{\mathbf{r}_u} \mathbf{D}, \quad y \in \left[ -\frac{l}{2}, \chi(u,x) + \frac{l}{2} \right] \right\},
\]

where \( \chi(u,x) = V_1(\mathbf{g}(\mathbf{u},x) \cap \mathbf{D}) \). Note that the set \( \Omega_2(\mathbf{u}) \) does not depend on the choice of the origin of the line \( \mathbf{g}(\mathbf{u},x) \), and the choice of the positive direction follows from the explicit form of the range of \( y \). Further, we set

\[
B^{\mathbf{u}}_D = \left\{ (x,y) \in \Omega_2(\mathbf{u}) : |L(\mathbf{x},y)| < t \right\}, \quad t \in \mathbb{R}^1,
\]

and observe that the sets \( \Omega_2(\mathbf{u}) \) and \( B^{\mathbf{u}}_D \) are measurable subsets of \( \mathbb{R}^n \).

**Definition** 3. The function

\[
F_{L}(\mathbf{u},t) = \frac{V_n(B^{\mathbf{u}}_D)}{V_n(\Omega_2(\mathbf{u}))} = \frac{1}{V_n(\Omega_2(\mathbf{u}))} \int_{B^{\mathbf{u}}_D} dx dy
\]

is called orientation-dependent distribution function of the length of a random segment \( L \) in direction \( \mathbf{u} \in S^{n-1} \).

Let \( G_n \) be the space of all lines \( g \in \mathbb{R}^n \). A line \( g \in G_n \) can be specified by its direction \( \mathbf{u} \in S^{n-1} \) and its intersection point \( x \) in the hyperplane \( \mathbf{u} \perp \). The density \( d\mathbf{u} \) is the volume element of the unit sphere \( S^{n-1} \), and \( dx \) is the volume element on \( \mathbf{u} \perp \) at \( x \). Let \( \mu(\cdot) \) be a locally finite measure on \( G_n \), invariant under the group of Euclidian motions. It is well known that the element of \( \mu(\cdot) \) up to a constant factor has the following form (see [1]):

\[
\mu(\mathbf{d}g) = dg = d\mathbf{u} dx.
\]
Denote by \( O_{n-1} = \sigma_{n-1} \left(S^{n-1}\right) \) the surface area of the unit sphere in \( \mathbb{R}^n \). For each bounded convex body \( D \), we denote the set of lines that intersect \( D \) by

\[ [D] = \{g \in G_n, g \cap D \neq \emptyset\}. \]

We have (see [1])

\[ \mu([D]) = \frac{O_{n-2}V_{n-1}(\partial D)}{2(n-1)}. \]

A random line in \( [D] \) is the one with distribution proportional to the restriction of \( \mu \) to \( [D] \). Therefore, for any \( t \in \mathbb{R}^1 \) we have

\[ F(t) = \frac{\mu(\{g \in [D], V_1(g \cap D) < t\})}{\mu([D])}, \]

which is called the chord length distribution function of \( D \). Let \( L \) be a random segment of length \( l \) in \( \mathbb{R}^n \) and let \( K(\cdot) \) be the kinematic measure of \( L \) [1]. If \( g \in G_n \) is the line containing \( L \) and \( y \) is the one-dimensional coordinate of the center of \( L \) on the line \( g \), then the element of the kinematic measure up to a constant factor is given by

\[ dK = dg dy dK_1, \]

where \( dy \) is the one-dimensional Lebesgue measure on \( g \) and \( dK_1 \) is a motion element in \( \mathbb{R}^n \) that leaves \( g \) unchanged (see [1, 4–7]).

Note that in the case, where the segment is orientated, the constant factor is equal to 1, while for the unoriented segment it is equal to 1/2. In this paper we consider only the case of unoriented segments. The length \( |L| \) of a random segment \( L \), provided that it hits the body \( D \), has the following distribution function:

\[ F_{|L|}(t) = \frac{K(L : L \cap D \neq \emptyset, V_1(L \cap D) < t)}{K(L : L \cap D \neq \emptyset)}, \quad t \in \mathbb{R}^1. \]

Denote by \( P(L(\omega) \subset D) \) probability, that random segment of length \( l \) in \( \mathbb{R}^n \) having a common point with body \( D \) entirely lying in body \( D \) (in this case the direction of the segment \( L(\omega) \) is arbitrary).

**Proposition** (see [7]). Probability \( P(L(\omega) \subset D) \) in terms of chord length distribution function \( F(t) \) has the following form:

\[ P(L(\omega) \subset D) = \frac{O_{n-2}V_{n-1}(\partial D) \int_0^l F(z)dz - l}{(n-1)O_{n-1}V_n(D) + lO_{n-2}V_{n-1}(\partial D)}. \]

**Case of a Triangle.** For any body \( D \) of the \( \mathbb{R}^n \) we have (see [7])

\[ P(L(\omega) \subset D) = \frac{1}{O_{n-1}} \int_{S^{n-1}} \frac{C(D, u, l)}{V_n(D) + l \cdot b_D(u)} du, \]

while the kinematic measure of the segments entire lying in \( D \) is calculated by the following formula:

\[ K(L(\omega) \subset D) = \int_{S^{n-1}} C(D, u, l) du. \]

For any planar bounded convex domain we have

\[ P(L(\omega) \subset D) = \frac{1}{\pi S(D) + l|\partial D|} \int_0^\pi C(D, u, l) du. \quad (1) \]
Denote by $\Delta$ a triangle in the plane. The main result of the present paper is the following statement.

**Theorem.** Probability $P(L(\omega) \subset \Delta)$ for arbitrary triangle has the explicit forms (2)–(8) depending on the value of $l$.

**Proof.** Without loss of the generality we assume, that $AB \equiv a$ is the longest side of $\triangle ABC$, $\angle CAB \equiv \alpha$ is the smallest angle, and $\angle ABC \equiv \beta$. Thus, we have $BC = \frac{a \sin \alpha}{\sin(\alpha + \beta)}$, $CA = \frac{a \sin \beta}{\sin(\alpha + \beta)}$, $\angle BCA = \pi - (\alpha + \beta)$. Since $AB$ is the longest side, then $\angle BCA$ is the biggest angle. Therefore $\alpha \leq \beta \leq \pi - (\alpha + \beta)$.

Covariogram of a triangle $\Delta$ with side $a$ has the form (see [3]):

$$C(\Delta, u, l) = \begin{cases} 
\frac{(a \sin \beta - t \sin(u + \beta))^2 \sin \alpha}{2 \sin \beta \sin(\alpha + \beta)}, & u \in [0, \alpha], t \in [0, \frac{a \sin \beta}{\sin(u + \beta)}], \\
\frac{(a \sin \alpha - t \sin(u - \alpha))^2 \sin \beta}{2 \sin \alpha \sin(\alpha + \beta)}, & u \in [\alpha, \pi - \alpha], t \in [0, \frac{a \sin \alpha}{\sin(u - \alpha)}], \\
\frac{(a \sin \alpha + t \sin(u + \beta))^2 \sin \alpha}{2 \sin \alpha \sin(\alpha + \beta)}, & u \in [0, \frac{a \sin \alpha}{\sin(u + \beta)}], t \in [0, \frac{a \sin \alpha \sin \beta}{\sin(\alpha + \beta) \sin u}], \\
\frac{(a \sin \alpha + t \sin(u - \alpha))^2 \sin \beta}{2 \sin \alpha \sin(\alpha + \beta)}, & u \in [\pi + \alpha, 2\pi - \alpha], t \in [0, \frac{a \sin \alpha}{\sin(u - \alpha)}].
\end{cases}$$

Let consider the following cases

a) $0 \leq l \leq \frac{a \sin \alpha \sin \beta}{\sin(\alpha + \beta)}$.

Using (1), we get

$$P(L(\omega) \subset \Delta) = \frac{1}{\pi S(\Delta) + l |\partial \Delta|} \int_0^\pi C(\Delta, u, l) \, du =$$

$$= \frac{2 \sin(\alpha + \beta)}{\pi a^2 \sin \alpha \sin \beta + 2al(\sin \alpha + \sin \beta + \sin(\alpha + \beta))} \times$$

$$\times \left( \int_0^\alpha \frac{(a \sin \beta - l \sin(u + \beta))^2 \sin \alpha}{2 \sin \beta \sin(\alpha + \beta)} \, du + \right.$$}

$$+ \int_\frac{\pi - \beta}{2} \frac{(a \sin \alpha \sin \beta - l \sin u \sin(\alpha + \beta))^2}{2 \sin \alpha \sin \beta \sin(\alpha + \beta)} \, du + \left. \int_{\frac{\pi - \beta}{2}}^\alpha \frac{(a \sin \alpha - l \sin(u - \alpha))^2 \sin \beta}{2 \sin \alpha \sin(\alpha + \beta)} \, du \right).$$
We set
\[ f_1(x, y) = \frac{\sin \alpha}{\sin \beta} \int_x^y (a \sin \beta - l \sin(u + \beta))^2 du = a^2 \sin \alpha \sin \beta (y - x) - \]
\[ -4al \sin \alpha \sin \left( \frac{y + x}{2} + \beta \right) \sin \left( \frac{y - x}{2} \right) + \frac{l^2 \sin \alpha}{2 \sin \beta} ((y - x) - \sin(y - x) \cos(y + x + 2\beta)), \]
\[ f_2(x, y) = \frac{1}{\sin \alpha \sin \beta} \int_x^y (a \sin \alpha \sin \beta - l \sin u \sin(\alpha + \beta))^2 du = a^2 \sin \alpha \sin \beta (y - x) - \]
\[ 4al \sin(\alpha + \beta) \sin \left( \frac{y + x}{2} \right) \sin \left( \frac{y - x}{2} \right) + \frac{l^2 \sin^2(\alpha + \beta)}{2 \sin \alpha \sin \beta} ((y - x) - \sin(y - x) \cos(y + x)), \]
\[ f_3(x, y) = \frac{\sin \beta}{\sin \alpha} \int_x^y (a \sin \alpha - l \sin(u - \alpha))^2 du = a^2 \sin \alpha \sin \beta (y - x) - \]
\[ -4al \sin \beta \sin \left( \frac{y + x - \alpha}{2} \right) \sin \left( \frac{y - x}{2} \right) + \frac{l^2 \sin \beta}{2 \sin \alpha} ((y - x) - \sin(y - x) \cos(y + x - 2\alpha)). \]

Hence, for \( 0 \leq l \leq \frac{a \sin \alpha \sin \beta}{\sin(\alpha + \beta)} \) we get
\[
P(L(\omega) \subset \Delta) = \frac{f_1(0, \alpha) + f_2(\alpha, \pi - \beta) + f_3(\pi - \beta, \pi)}{\pi a^2 \sin \alpha \sin \beta + 2al(\sin \alpha + \sin \beta + \sin(\alpha + \beta))}. \tag{2} \]

b) \( \frac{a \sin \alpha \sin \beta}{\sin(\alpha + \beta)} \leq l \leq a \sin \alpha. \) We have
\[
P(L(\omega) \subset \Delta) = \frac{f_1(0, \alpha) + f_2(\alpha, \alpha + \varphi_1) + f_2(\pi - \beta - \varphi_1, \pi - \beta) + f_3(\pi - \beta, \pi)}{\pi a^2 \sin \alpha \sin \beta + 2al(\sin \alpha + \sin \beta + \sin(\alpha + \beta))}, \tag{3} \]
where \( \varphi_1 = \arcsin \frac{a \sin \alpha \sin \beta}{l \sin(\alpha + \beta)} - \alpha, \) \( \varphi_1 = \arcsin \frac{a \sin \alpha \sin \beta}{l \sin(\alpha + \beta)} - \beta. \)

c) \( a \sin \alpha \leq l \leq \min \left\{ \frac{a \sin \alpha}{\sin(\alpha + \beta)}, \frac{a \sin \beta}{\sin(\alpha + \beta)} \right\}, \) for which we have
\[
P(L(\omega) \subset \Delta) = \frac{1}{\pi a^2 \sin \alpha \sin \beta + 2al(\sin \alpha + \sin \beta + \sin(\alpha + \beta))} \times \]
\[ (f_1(0, \alpha) + f_2(\alpha, \alpha + \varphi_1) + f_2(\pi - \beta - \varphi_1, \pi - \beta) + f_3(\pi - \beta, \pi)) \]
\[ + f_2(\pi - \beta - \varphi_1, \pi - \beta) + f_3(\pi - \beta, \pi + \varphi_2) + f_3(\pi - \varphi_2, \pi)), \]
where \( \varphi_2 = \alpha + \beta - \pi + \arcsin \frac{a \sin \alpha}{l}, \varphi_2 = \arcsin \frac{a \sin \alpha}{l} - \alpha : \]

c1) if \( \sin \beta \leq \frac{a \sin \alpha}{\sin(\alpha + \beta)} \), we consider \( a \sin \beta \leq l \leq \frac{a \sin \alpha}{\sin(\alpha + \beta)}, \) so
\[ P(L(\omega) \subset \Delta) = \frac{1}{\pi a^2 \sin \alpha \sin \beta + 2al(\sin \alpha + \sin \beta + \sin(\alpha + \beta))} \times \]
\[ \times (f_1(0, \phi_3) + f_1(\alpha - \phi_3, \alpha) + f_2(\alpha, \alpha + \varphi_1) + f_2(\pi - \beta - \phi_1, \pi - \beta) + \]
\[ + f_3(\pi - \beta, \pi - \beta + \varphi_2) + f_3(\pi - \phi_2, \pi)), \]
\[ \text{where } \phi_3 = \arcsin \frac{a \sin \beta}{l} - \beta, \phi_3 = \alpha + \beta - \pi + \arcsin \frac{a \sin \beta}{l}; \]
\[ \text{c2) if } \frac{\sin \alpha}{\sin(\alpha + \beta)} \leq \sin \beta, \text{ we consider } \frac{a \sin \alpha}{\sin(\alpha + \beta)} \leq l \leq a \sin \beta, \text{ then} \]
\[ P(L(\omega) \subset \Delta) = \frac{f_1(0, \alpha) + f_2(\alpha, \alpha + \varphi_1) + f_3(\pi - \phi_2, \pi)}{\pi a^2 \sin \alpha \sin \beta + 2al(\sin \alpha + \sin \beta + \sin(\alpha + \beta))}. \]
\[ \text{d) max } \left\{ \frac{a \sin \alpha}{\sin(\alpha + \beta)}, \frac{a \sin \beta}{l} \right\} \leq l \leq \frac{a \sin \beta}{\sin(\alpha + \beta)}, \]
\[ P(L(\omega) \subset \Delta) = \frac{f_1(0, \varphi_3) + f_1(\alpha - \phi_3, \alpha) + f_2(\alpha, \alpha + \varphi_1) + f_3(\pi - \phi_2, \pi)}{\pi a^2 \sin \alpha \sin \beta + 2al(\sin \alpha + \sin \beta + \sin(\alpha + \beta))}. \]
\[ \text{e) } \frac{a \sin \beta}{\sin(\alpha + \beta)} \leq l \leq a \text{ we have} \]
\[ P(L(\omega) \subset \Delta) = \frac{f_1(0, \varphi_3) + f_3(\pi - \phi_2, \pi)}{\pi a^2 \sin \alpha \sin \beta + 2al(\sin \alpha + \sin \beta + \sin(\alpha + \beta))}. \]

Obviously, if \( l > a \), the probability \( P(L(\omega) \subset \Delta) \) is zero. \( \square \)

Particularly, for regular triangle with a side \( a \) and \( \alpha = \beta = 60^\circ \), among all 5 subcases a)–e) there are only two cases, namely
\[ 0 \leq l \leq \sin \alpha \quad \text{and} \quad \sin \alpha \leq l \leq a \]
and result of Theorem coincides with the result of [7] (Eqs. (4.3), (4.4)) for a regular triangle.

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