ON THE CONVERGENCE OF CESÀRO MEANS OF NEGATIVE ORDER OF DOUBLE TRIGONOMETRIC FOURIER SERIES OF FUNCTIONS OF BOUNDED PARTIAL GENERALIZED VARIATION

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Abstract. The convergence of Cesàro means of negative order of double trigonometric Fourier series of functions of bounded partial Λ-variation is investigated. The sufficient and neccessary conditions on the sequence $\Lambda = \{\lambda_n\}$ are found for the convergence of Cesàro means of Fourier series of functions of bounded partial Λ-variation.

1. Classes of Functions of Bounded Generalized Variation

In 1881 Jordan [9] introduced the class of functions of bounded variation and applied it to the theory of Fourier series. Hereinafter this notion was generalized by many authors (quadratic variation, Φ-variation, Λ-variation ets., see [2], [11], [14], [17]). In two dimensional case the class BV of functions of bounded variation was introduced by Hardy [8].

Let f be a real function of two variable of period 2π with respect to each variable. Given intervals $I = (a, b)$, $J = (c, d)$ and points x, y from $T := [0, 2\pi]$ we denote

$$f(I, y) := f(b, y) - f(a, y), \qquad f(x, J) = f(x, d) - f(x, c)$$

and

$$f(I, J) := f(a, c) - f(a, d) - f(b, c) + f(b, d).$$

Let $E = \{I_i\}$ be a collection of nonoverlapping intervals from T ordered in arbitrary way and let Ω be the set of all such collections E.

For the sequence of positive numbers $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ we denote

$$\Lambda V_1(f) = \sup_y \sup_{E\in\Omega} \sum_i \frac{|f(I_i, y)|}{\lambda_i} \quad (E = \{I_i\}),$$

$$\Lambda V_2(f) = \sup_x \sup_{F\in\Omega} \sum_j \frac{|f(x, J_j)|}{\lambda_j} \qquad (F = \{J_j\}),$$

$$\Lambda V_{1,2}(f) = \sup_{F,\, E\in\Omega} \sum_i \sum_j \frac{|f(I_i, J_j)|}{\lambda_i \lambda_j}.$$

[0]2000 Mathematics Subject Classification.42B08
Key words and phrases: Fourier series, Bounded Λ-variation, Cesàro means.

Definition 1. *We say that the function f has Bounded Λ-variation on $T^2 = [0, 2\pi]^2$ and write $f \in \Lambda BV$, if*

$$\Lambda V(f) := \Lambda V_1(f) + \Lambda V_2(f) + \Lambda V_{1,2}(f) < \infty.$$

We say that the function f has Bounded Partial Λ-variation and write $f \in P\Lambda BV$ if

$$P\Lambda V(f) := \Lambda V_1(f) + \Lambda V_2(f) < \infty.$$

If $\lambda_n \equiv 1$ (or if $0 < c < \lambda_n < C < \infty$, $n = 1, 2, \ldots$) the classes ΛBV and $P\Lambda BV$ coincide with the Hardy class BV and PBV respectively. Hence it is reasonable to assume that $\lambda_n \to \infty$ and since the intervals in $E = \{I_i\}$ are ordered arbitrarily, we will suppose, without loss of generality, that the sequence $\{\lambda_n\}$ is increasing. Thus,

$$1 < \lambda_1 \leq \lambda_2 \leq \ldots, \qquad \lim_{n\to\infty} \lambda_n = \infty. \tag{1}$$

In the case when $\lambda_n = n$, $n = 1, 2 \ldots$ we say *Harmonic Variation* instead of Λ-variation and write H instead of Λ (HBV, $PHBV$, $HV(f)$, ets).

The notion of Λ-variation was introduced by D. Waterman [14] in one dimensional case and A. Sahakian [13] in two dimensional case. The notion of bounded partial bounded variation (class PBV) was introduced in [6].

Definition 2 (Waterman [15])**.** *Let $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ and $\Lambda_k = \{\lambda_n\}_{n=k}^{\infty}$, $k = 1, 2, \ldots$. We say that the function f is continuous in Λ-variation and write $f \in C\Lambda BV$, if*

$$\lim_{k\to\infty} \Lambda_k V(f) = 0.$$

2. $(C; \alpha, \beta)$ $(-1 < \alpha, \beta < 0)$ Summability

Let $f \in L^1(T^2)$. The Fourier series of f with respect to the trigonometric system is the series

$$S[f, (x, y)] := \sum_{m,n=-\infty}^{+\infty} \widehat{f}(m, n) e^{imx} e^{iny},$$

where

$$\widehat{f}(m, n) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f(x, y) e^{-imx} e^{-iny} dx dy$$

are the Fourier coefficients of the function f. The rectangular partial sums are defined as follows:

$$S_{M,N}[f, (x, y)] := \sum_{m=-M}^{M} \sum_{n=-N}^{N} \widehat{f}(m, n) e^{imx} e^{iny},$$

The Cesàro $(C; \alpha, \beta)$, $\alpha, \beta > -1$, means of two-dimensional Fourier series are defined by

$$\sigma_{n,m}^{\alpha,\beta}(f; x, y) := \frac{1}{A_n^{\alpha}} \frac{1}{A_m^{\beta}} \sum_{i=0}^{n} \sum_{j=0}^{m} A_{n-i}^{\alpha-1} A_{m-j}^{\beta-1} S_{i,j}[f, (x, y)]$$

where

$$A_0^\alpha = 1,\ \ A_k^\alpha = \frac{(\alpha+1)\cdots(\alpha+k)}{k!}, \quad k = 1, 2, ...$$

It is well-known that (see [19], p. 157)

$$\sigma_{mn}^{(\alpha,\beta)} f(x,y) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x+t, y+s)\, K_m^\alpha(s)\, K_n^\beta(t)\, dsdt,$$

where the kernel K_n^α, $-1 < \alpha < 0$ satisfies the following conditions:

$$\left|K_n^{-\alpha}(u)\right| \le 2n, \quad u \in T, \tag{2}$$

$$K_n^\alpha(u) = \varphi_n^\alpha(u) + O\left(1/nt^2\right), \ \ 0 \le |u| \le \pi, \tag{3}$$

where

$$\varphi_n^\alpha(u) = \frac{\sin\left[(n+1/2+\alpha/2)\, u - \alpha\pi/2\right]}{A_n^\alpha \left[2\sin u/2\right]^{1+\alpha}}, \tag{4}$$

The coefficients A_n^α have following bounds:

$$c_1(\alpha) n^\alpha \le A_n^\alpha \le c_2(\alpha) n^\alpha. \tag{5}$$

Denote

$${}_1\Delta_i^m f(x,y) := f\left(x + \frac{2i\pi}{m},\ y\right) - f\left(x + \frac{(2i+1)\pi}{m}, y\right),$$

$${}_2\Delta_j^n f(x,y) := f\left(x,\ y + \frac{2j\pi}{n}\right) - f\left(x,\ y + \frac{(2j+1)\pi}{n}\right),$$

$$\Delta_{ij}^{mn} f(x,y) = f\left(x + \frac{2i\pi}{m},\ y + \frac{2j\pi}{n}\right) - f\left(x + \frac{(2i+1)\pi}{m},\ y + \frac{2j\pi}{n}\right)$$
$$- f\left(x + \frac{2i\pi}{m},\ y + \frac{(2j+1)\pi}{n}\right) + f\left(x + \frac{(2i+1)\pi}{m},\ y + \frac{(2j+1)\pi}{n}\right).$$

3. Formulation of Problems

Let $C(T^2)$ be the space of 2π-periodic with respect to each variable continuous functions with the norm

$$\|f\|_C := \sup_{x,y \in T^2} |f(x,y)|.$$

For the function $f(x,y)$ we denote by $f(x \pm 0, y \pm 0)$ the open coordinate quadrant limits (if exist) at the point (x,y) and set

$$\sum f(x \pm 0, y \pm 0) \tag{6}$$
$$= \left\{ f(x+0, y+0) + f(x+0, y-0) + f(x-0, y+0) + f(x-0, y-0) \right\}.$$

The well known Dirichlet-Jordan theorem (see [19]) states that the Fourier series of a function $f(x)$, $x \in T$ of bounded variation converges at every point x to the value $\left[f(x+0) + f(x-0)\right]/2$. If f is in addition continuous on T the Fourier series converges uniformly on T. This result was generalized by Waterman [14].

Theorem W1 (Waterman [14])**.** *If $f \in HBV$, then $S[f]$ converges at every point x to the value $[f(x+0)+f(x-0)]/2$. If f is in addition continuous on T, then $S[f]$ converges uniformly on T.*

Hardy [8] generalized the Dirichlet-Jordan theorem to the double Fourier series. He proved that if function $f(x,y)$ has bounded variation in the sense of Hardy ($f \in BV$), then $S[f]$ converges at any point (x,y) to the value $\frac{1}{4}\sum f(x \pm 0, y \pm 0)$. If f is in addition continuous on T^2 then $S[f]$ converges uniformly on T^2.

Theorem S (Sahakian [13])**.** *The Fourier series of a function $f(x,y) \in HBV$ converges to $\frac{1}{4}\sum f(x \pm 0, y \pm 0)$ at any point (x,y), where the quadrant limits (6) exist. The convergence is uniformly on any compact K, where the function f is continuous.*

Analogs of Theorem S for higher dimensions can be found in [12] and [1]. Convergence of spherical and other partial sums of double Fourier series of functions of bounded Λ-variation was investigated in details by Dyachenko (see [3], [4], [5] and references therein).

The first author [6] has proved that in Hardy's theorem there is no need to require the boundedness of mixed variation. In particular, the following is true

Theorem G1 (Goginava [6])**.** *Let $f \in C\left(T^2\right) \bigcap PBV$. Then $S[f]$ converges uniformly on T^2.*

For one-dimensional Fourier series Waterman [15] proved the following

Theorem W2 (Waterman [15])**.** *Let $0 < \alpha < 1$. The Fourier series of a function $f \in \{n^{1-\alpha}\}BV$ is everywhere $(C,-\alpha)$ bounded and is uniformly $(C,-\alpha)$ bounded on each closed interval of continuity of f.*

If $f \in C\{n^{1-\alpha}\}BV$, then $S[f]$ is everywhere $(C,-\alpha)$ summable to the value $[f(x+0)+f(x-0)]/2$ and the summability is uniform on each closed interval of continuity.

Later Sablin proved in [12], that for $0 < \alpha < 1$ the classes $\{n^{1-\alpha}\}BV$ and $C\{n^{1-\alpha}\}BV$ coincide.

Zhizhiashvili [18] has inverstigated the convergence of Cesàro means of double trigonometric Fourier series. In particular, the following theorem was proved.

Theorem Zh (Zhizhiashvili [18])**.** *If $f \in BV$, then the double Fourier series of f is $(C;-\alpha,-\beta)$ summable to $\frac{1}{4}\sum f(x \pm 0, y \pm 0)$ in any point (x,y). The convergence is uniformly on any compact K, where the function f is continuous.*

For functions of partial bounded variation the problem was solved by the first author in [7].

Theorem G2 (Goginava [7]). *Let $f \in C\left(T^2\right) \cap PBV$ and $\alpha+\beta<1,\ \alpha,\beta>0$. Then the double trigonometric Fourier series of the function f is uniformly $(C;-\alpha,-\beta)$ summable in the sense of Pringsheim.*

Theorem G3 (Goginava [7]). *Let $\alpha+\beta\geq 1,\ \alpha,\beta>0$. Then there exists a continuous function $f_0 \in PBV$ such that the Cesàro $(C;-\alpha,-\beta)$ means $\sigma_{n,m}^{-\alpha,-\beta}(f_0;0,0)$ of the double trigonometric Fourier series of f_0 diverge over cubes.*

In this paper we consider the following problem. *Let $\alpha,\beta\in(0,1)$, $\alpha+\beta<1$. Under what conditions on the sequence $\Lambda=\{\lambda_n\}$ the double Fourier series of the function $f\in P\Lambda BV$ is $(C;-\alpha,-\beta)$ summable.*

The solution is given in Theorems 1 and 2 bellow.

4. Main Results

Theorem 1. *Let $\alpha,\beta\in(0,1)$, $\alpha+\beta<1$ and the sequence $\Lambda=\{\lambda_k\}$ satisfies the conditions:*

$$\frac{\lambda_k}{k^{1-(\alpha+\beta)}}\downarrow 0,\qquad \sum_{k=1}^{\infty}\frac{\lambda_k}{k^{2-(\alpha+\beta)}}<\infty.$$

Then the double Fourier series of the function $f\in P\Lambda BV$ is $(C;-\alpha,-\beta)$ summable to $\frac{1}{4}\sum f(x\pm 0,y\pm 0)$ at any point (x,y), where the quadrant limits (6) exist. The convergence is uniform on any compact K, where the function f is continuous.

Theorem 2. *Let $\alpha,\beta\in(0,1)$, $\alpha+\beta<1$ and the sequence $\Lambda=\{\lambda_k\}$ satisfies the conditions:*

$$\frac{\lambda_k}{k^{1-(\alpha+\beta)}}\downarrow 0,\qquad \sum_{k=1}^{\infty}\frac{\lambda_k}{k^{2-(\alpha+\beta)}}=\infty.$$

Then there exists a continuous function $f\in P\Lambda BV$ for which $(C;-\alpha,-\beta)$ means of two-dimensional Fourier series diverges over cubes at $(0,0)$.

Corollary 1. *Let $\alpha,\beta\in(0,1)$, $\alpha+\beta<1$.*
a)If $f\in P\left\{\frac{n^{1-(\alpha+\beta)}}{\log^{1+\varepsilon}(n+1)}\right\}BV$ for some $\varepsilon>0$, then the double Fourier series of the function f is $(C;-\alpha,-\beta)$ summable to $\frac{1}{4}\sum f(x\pm 0,y\pm 0)$ in any point (x,y), where the quadrant limits (6) exist. The convergence is uniform on any compact K, where the function f is continuous.

b) There exists a continuous function $f\in P\left\{\frac{n^{1-(\alpha+\beta)}}{\log(n+1)}\right\}BV$ such that $(C;-\alpha,-\beta)$ means of two-dimensional Fourier series of f diverges over cubes at $(0,0)$.

Corollary 2. *Let $\alpha,\beta\in(0,1)$, $\alpha+\beta<1$ and $f\in PBV$. Then the double Fourier series of the function f is $(C;-\alpha,-\beta)$ summable to $\frac{1}{4}\sum f(x\pm 0,y\pm 0)$ in any point (x,y), where the quadratic limits (6) exist. The convergence is uniform on any compact K, where the function f is continuous.*

5. Proofs

Proof of Theorem 1. It is easy to show that

$$\sigma_{mn}^{(-\alpha,-\beta)} f(x,y) - \frac{1}{4}\sum f(x \pm 0, y \pm 0)$$

$$= \frac{1}{\pi^2}\sum_{i=1}^{4}\int_0^{\pi}\int_0^{\pi}\varphi_i(x,y,s,t)\,K_m^{-\alpha}(s)\,K_n^{-\beta}(t)\,dsdt$$

$$= \sum_{i=1}^{4} I_{mn}^{(k)}(x,y).$$

where

$$\varphi_1(x,y,s,t) := f(x+s,y+t) - f(x+0,y+0),$$
$$\varphi_2(x,y,s,t) := f(x-s,y+t) - f(x-0,y+0),$$
$$\varphi_3(x,y,s,t) := f(x+s,y-t) - f(x+0),y-0),$$
$$\varphi_4(x,y,s,t) := f(x-s,y-t) - f(x-0,y-0).$$

For $I_{mn}^{(1)}(x,y)$ we can write

$$\pi^2 I_{mn}^{(1)}(x,y) \tag{7}$$

$$= \left(\int_0^{\pi/m}\int_0^{\pi/n} + \int_0^{\pi/m}\int_{\pi/n}^{\pi} + \int_{\pi/m}^{\pi}\int_0^{\pi/n} + \int_{\pi/m}^{\pi}\int_{\pi/n}^{\pi}\right)\left(\varphi_1(x,y,s,t)\,K_m^{-\alpha}(s)\,K_n^{-\beta}(t)\,dsdt\right)$$

$$= \sum_{k=1}^{4} I_{mn}^{(1k)}(x,y).$$

From (2) we have

$$\left|I_{mn}^{(11)}(x,y)\right| \le c(\alpha,\beta)\,mn\int_0^{\pi/m}\int_0^{\pi/n}|\varphi_1(x,y,s,t)|\,dsdt \tag{8}$$

$$\le c(\alpha,\beta)\sup_{0<s<\pi/m,0<t<\pi/n}|\varphi_1(x,y,s,t)| = o(1) \quad \text{as } m,n\to\infty.$$

Using (3), we obtain

$$I_{mn}^{(12)}(x,y) = \int_0^{\pi/m}\int_{\pi/n}^{\pi}\varphi_1(x,y,s,t)\,K_m^{-\alpha}(s)\,\varphi_n^{-\beta}(t)\,dsdt \tag{9}$$

$$+\int_0^{\pi/m}\int_{\pi/n}^{\pi}\varphi_1(x,y,s,t)\,K_m^{-\alpha}(s)\,O\left(\frac{1}{nt^2}\right)(t)\,dsdt$$

$$= I_{mn}^{(121)}(x,y) + I_{mn}^{(122)}(x,y).$$

We can write

$$
\begin{aligned}
(10)\qquad & \left|I_{mn}^{(122)}(x,y)\right| \\
\leq\ & \int_0^{\pi/m}\int_{\pi/n}^{\pi/\sqrt{n}} \left|\varphi_1(x,y,s,t)\right| \left|K_m^{-\alpha}(s)\right| O\left(\frac{1}{nt^2}\right) dsdt \\
&+ \int_0^{\pi/m}\int_{\pi/\sqrt{n}}^{\pi} \left|\varphi_1(x,y,s,t)\right| \left|K_m^{-\alpha}(s)\right| O\left(\frac{1}{nt^2}\right) dsdt \\
\leq\ & c(\alpha,\beta,f)\left\{\sup_{0<s<\pi/m,0<t<\pi/\sqrt{n}} \left|\varphi_1(x,y,s,t)\right| + \int_{\pi/\sqrt{n}}^{\pi} \frac{dt}{nt^2}\right\} \\
=\ & o(1) \quad \text{as } n,m\to\infty.
\end{aligned}
$$

In order to estimate $I_{mn}^{(121)}(x,y)$ it is enough to estimate the following expression

$$
J_{mn}(x,y) := n^{\beta}\int_0^{\pi/m}\int_{\pi/n}^{\pi} \varphi_1(x,y,s,t) K_m^{-\alpha}(s) w_{\beta}(t)\sin ntdsdt,
$$

where

$$
w_{\beta}(t) = \frac{\cos\frac{1-\beta}{2}t}{(\sin t/2)^{1-\beta}}.
$$

We have

$$
J_{mn}(x,y) = n^{\beta}\sum_{i=1}^{n-1}\int_0^{\pi/m} K_m^{-\alpha}(s)\left(\int_{i\pi/n}^{(i+1)\pi/n} \varphi_1(x,y,s,t) w_{\beta}(t)\sin ntdt\right) ds
$$

$$
\begin{aligned}
= n^{\beta}\sum_{i=1}^{(n-1)/2}\int_0^{\pi/m} K_m^{-\alpha}(s)\Bigg(\int_0^{\pi/n}\left[\varphi_1\left(x,y,s,t+\frac{2i\pi}{n}\right)-\varphi_1\left(x,y,s,t+\frac{(2i+1)\pi}{n}\right)\right] \\
w_{\beta}\left(t+\frac{2i\pi}{n}\right)\sin ntdt\Bigg) ds \\
+n^{\beta}\sum_{i=1}^{(n-1)/2}\int_0^{\pi/m} K_m^{-\alpha}(s)\Bigg(\int_0^{\pi/n}\varphi_1\left(x,y,s,t+\frac{(2i+1)\pi}{n}\right) \\
\left[w_{\beta}\left(t+\frac{2i\pi}{n}\right)-w_{\beta}\left(t+\frac{(2i+1)\pi}{n}\right)\right]\sin ntdt\Bigg) ds \\
= J_{mn}^{(1)}(x,y) + J_{mn}^{(2)}(x,y).
\end{aligned}
$$

Using the following inequality:

$$\left|w_\beta\left(t+\frac{2i\pi}{n}\right)-w_\beta\left(t+\frac{(2i+1)\pi}{n}\right)\right|\leq\frac{c(\beta)n^{1-\beta}}{i^{2-\beta}}, \tag{11}$$

for $J_{mn}^{(2)}(x,y)$ we can write

$$\left|J_{mn}^{(2)}(x,y)\right| \tag{12}$$

$$\leq c(\beta)mn\sum_{i=1}^{(n-1)/2}\frac{1}{i^{2-\beta}}\int_0^{\pi/m}\left(\int_0^{\pi/n}\left|\varphi_1\left(x,y,s,t+\frac{(2i+1)\pi}{n}\right)\right|dt\right)ds$$

$$\leq c(\beta)nm\sum_{i\leq\sqrt{n}}\frac{1}{i^{2-\beta}}\int_0^{\pi/m}\left(\int_0^{\pi/n}\left|\varphi_1\left(x,y,s,t+\frac{(2i+1)\pi}{n}\right)\right|dt\right)ds$$

$$+c(\beta)nm\sum_{\sqrt{n}<i\leq(n-1)/2}\frac{1}{i^{2-\beta}}\int_0^{\pi/m}\left(\int_0^{\pi/n}\left|\varphi_1\left(x,y,s,t+\frac{(2i+1)\pi}{n}\right)\right|dt\right)ds$$

$$\leq c(\beta)\sup_{0<s<\pi/n,0<s<4\pi/\sqrt{n}}|\varphi_1(x,y,s,t)|+c(\beta,f)\left(\frac{1}{\sqrt{n}}\right)^{1-\beta}=o(1),$$

as $n,m\to\infty$.

To estimate $J_{mn}^{(1)}(x,y)$, we denote

$$\mu(n,m):=\left[\min\left\{\frac{1}{2}\ln n-1,\ (s(n,m))^{-1}\right\}\right], \tag{13}$$

where $[a]$ is the integer part of a and

$$s(n,m):=\sup_{0<s<\pi/m,\ 0<t<\pi\ln n/n}|\varphi_1(x,y,s,t)|. \tag{14}$$

Then we have

$$\left|J_{mn}^{(1)}(x,y)\right|\leq c(\beta)nm\int_0^{\pi/m}\left(\int_0^{\pi/n}\sum_{i=1}^{\mu(n,m)}\frac{1}{i^{1-\beta}}\left|\varphi_1\left(x,y,s,t+\frac{2i\pi}{n}\right)\right.\right. \tag{15}$$

$$\left.\left.-\varphi_1\left(x,y,s,t+\frac{(2i+1)\pi}{n}\right)\right|dtds\right)$$

$$+c(\beta)nm\int_0^{\pi/m}\left(\int_0^{\pi/n}\sum_{i=\mu(n,m)}^{(n-1)/2}\frac{1}{i^{1-\beta}}\left|\varphi_1\left(x,y,s,t+\frac{2i\pi}{n}\right)\right.\right.$$

$$\left.\left.-\varphi_1\left(x,y,s,t+\frac{(2i+1)\pi}{n}\right)\right|dt\right)ds$$

$$\leq c(\beta)\sup_{0<s<\pi/m,\ 0<s<(2\mu(n,m)+1)\pi/n}|\varphi_1(x,y,s,t)|(\mu(n,m))^\beta$$

$$+c(\beta)\,nm\int\limits_0^{\pi/m}\left(\int\limits_0^{\pi/n}\sum_{i=\mu(n,m)}^{(n-1)/2}\frac{1}{\lambda_i}\left|\varphi_1\left(x,y,s,t+\frac{2i\pi}{n}\right)\right.\right.$$

$$\left.\left.-\varphi_1\left(x,y,s,t+\frac{(2i+1)\pi}{n}\right)\right|\frac{\lambda_i}{i^{1-\beta}}dt\right)ds$$

$$\leq c(\beta)\sup_{0<s<\pi/m,\ 0<t<\pi\ln n/n}|\varphi_1(x,y,s,t)|\,(\mu(n,m))^{\beta}$$

$$+c(\beta)\,nm\frac{\lambda_{\mu(n,m)}}{(\mu(n,m))^{1-\beta}}\int\limits_0^{\pi/m}\left(\int\limits_0^{\pi/n}\sum_{i=\mu(n,m)}^{(n-1)/2}\frac{1}{\lambda_i}\left|\varphi_1\left(x,y,s,t+\frac{2i\pi}{n}\right)\right.\right.$$

$$\left.\left.-\varphi_1\left(x,y,s,t+\frac{(2i+1)\pi}{n}\right)\right|dt\right)ds$$

$$\leq c(\beta)\,s(n,m)\,(\mu(n,m))^{\beta}+c(\beta)\frac{\lambda_{\mu(n,m)}}{(\mu(n,m))^{1-\beta}}V_2\Lambda(f)=o(1)\quad\text{as } n,m\to\infty.$$

Combining (9), (10), (12) and (15) we conclude that

$$I_{mn}^{(12)}(x,y)\to 0\ \text{ as }\ m,n\to\infty. \tag{16}$$

Analogously, we can prove that

$$I_{mn}^{(13)}(x,y)\to 0\ \text{ as }\ m,n\to\infty. \tag{17}$$

In order to estimate $I_{mn}^{(14)}(x,y)$ it is enough to estimate the following expression

$$L_{mn}(x,y):=m^{\alpha}n^{\beta}\int\limits_{\pi/m}^{\pi}\int\limits_{\pi/n}^{\pi}\varphi_1(x,y,s,t)\,w_{\alpha}(s)\,w_{\beta}(t)\sin ms\sin nt\,dt\,ds.$$

We have

$$L_{mn}(x,y)=m^{\alpha}n^{\beta}\sum_{i=1}^{(m-1)/2}\sum_{j=1}^{(n-1)/2}\int\limits_0^{\pi/m}\int\limits_0^{\pi/n}\varphi_1\left(x,y,s+\frac{2i\pi}{m},t+\frac{2j\pi}{n}\right) \tag{18}$$

$$\times w_{\alpha}\left(s+\frac{2i\pi}{m}\right)w_{\beta}\left(t+\frac{2j\pi}{n}\right)\sin ms\sin nt\,dt\,ds$$

$$-m^{\alpha}n^{\beta}\sum_{i=1}^{(m-1)/2}\sum_{j=1}^{(n-1)/2}\int\limits_0^{\pi/m}\int\limits_0^{\pi/n}\varphi_1\left(x,y,s+\frac{(2i+1)\pi}{m},t+\frac{2j\pi}{n}\right)$$

$$\times w_{\alpha}\left(s+\frac{(2i+1)\pi}{m}\right)w_{\beta}\left(t+\frac{2j\pi}{n}\right)\sin ms\sin nt\,dt\,ds$$

$$-m^{\alpha}n^{\beta}\sum_{i=1}^{(m-1)/2}\sum_{j=1}^{(n-1)/2}\int\limits_0^{\pi/m}\int\limits_0^{\pi/n}\varphi_1\left(x,y,s+\frac{2i\pi}{m},t+\frac{(2j+1)\pi}{n}\right)$$

$$\times w_\alpha\left(s+\frac{2i\pi}{m}\right)w_\beta\left(t+\frac{(2j+1)\pi}{n}\right)\sin ms\sin nt dtds$$

$$+m^\alpha n^\beta\sum_{i=1}^{(m-1)/2}\sum_{j=1}^{(n-1)/2}\int_0^{\pi/m}\int_0^{\pi/n}\varphi_1\left(x,y,s+\frac{(2i+1)\pi}{m},t+\frac{(2j+)1\pi}{n}\right)$$

$$\times w_\alpha\left(s+\frac{(2i+1)\pi}{m}\right)w_\beta\left(t+\frac{(2j+)1\pi}{n}\right)\sin ms\sin nt dtds$$

$$=m^\alpha n^\beta\sum_{i=1}^{(m-1)/2}\sum_{j=1}^{(n-1)/2}\int_0^{\pi/m}\int_0^{\pi/n}\left[\varphi_1\left(x,y,s+\frac{2i\pi}{m},t+\frac{2j\pi}{n}\right)\right.$$

$$-\varphi_1\left(x,y,s+\frac{(2i+1)\pi}{m},t+\frac{2j\pi}{n}\right)-\varphi_1\left(x,y,s+\frac{2i\pi}{m},t+\frac{(2j+1)\pi}{n}\right)$$

$$\left.+\varphi_1\left(x,y,s+\frac{(2i+1)\pi}{m},t+\frac{(2j+1)\pi}{n}\right)\right]$$

$$\times w_\alpha\left(s+\frac{2i\pi}{m}\right)w_\beta\left(t+\frac{2j\pi}{n}\right)\sin ms\sin nt dtds$$

$$+m^\alpha n^\beta\sum_{i=1}^{(m-1)/2}\sum_{j=1}^{(n-1)/2}\int_0^{\pi/m}\int_0^{\pi/n}\left[\varphi_1\left(x,y,s+\frac{(2i+1)\pi}{m},t+\frac{2j\pi}{n}\right)\right.$$

$$\left.-\varphi_1\left(x,y,s+\frac{(2i+1)\pi}{m},t+\frac{(2j+1)\pi}{n}\right)\right]$$

$$\times\left[w_\alpha\left(s+\frac{2i\pi}{m}\right)-w_\alpha\left(s+\frac{(2i+1)\pi}{m}\right)\right]w_\beta\left(t+\frac{2j\pi}{n}\right)\sin ms\sin nt dtds$$

$$+m^\alpha n^\beta\sum_{i=1}^{(m-1)/2}\sum_{j=1}^{(n-1)/2}\int_0^{\pi/m}\int_0^{\pi/n}\left[\varphi_1\left(x,y,s+\frac{2i\pi}{m},t+\frac{(2j+1)\pi}{n}\right)\right.$$

$$\left.-\varphi_1\left(x,y,s+\frac{(2i+1)\pi}{m},t+\frac{(2j+)1\pi}{n}\right)\right]$$

$$\times\left[w_\beta\left(s+\frac{2j\pi}{n}\right)-w_\beta\left(s+\frac{(2j+1)\pi}{n}\right)\right]w_\alpha\left(s+\frac{2i\pi}{m}\right)\sin ms\sin nt dtds$$

$$+m^\alpha n^\beta\sum_{i=1}^{(m-1)/2}\sum_{j=1}^{(n-1)/2}\int_0^{\pi/m}\int_0^{\pi/n}\varphi_1\left(x,y,s+\frac{(2i+1)\pi}{m},t+\frac{(2j+1)\pi}{n}\right)$$

$$\times\left[w_\beta\left(s+\frac{2j\pi}{n}\right)-w_\beta\left(s+\frac{(2j+1)\pi}{n}\right)\right]$$

$$\left[w_\alpha\left(s+\frac{2i\pi}{m}\right)-w_\alpha\left(s+\frac{(2i+1)\pi}{m}\right)\right]\sin ms\sin nt dtds$$

$$= \sum_{k=1}^{4} L_{mn}^{(k)}(x,y).$$

By (11) we obtain

$$\left|L_{mn}^{(4)}(x,y)\right| \leq c(\alpha,\beta)\, mn \sum_{i=1}^{\left[\sqrt{m}\right]} \frac{1}{i^{2-\alpha}} \sum_{j=1}^{\left[\sqrt{n}\right]} \frac{1}{j^{2-\beta}} \tag{19}$$

$$\int_0^{\pi/m}\int_0^{\pi/n} \left|\varphi_1\left(x,y,s+\frac{(2i+1)\pi}{m},t+\frac{(2j+1)\pi}{n}\right)\right|$$

$$+c(\alpha,\beta,f)\sum_{i=1}^{\infty}\frac{1}{i^{2-\alpha}}\sum_{j=\left[\sqrt{n}\right]}^{\infty}\frac{1}{j^{2-\beta}}$$

$$+c(\alpha,\beta,f)\, mn\sum_{i=\left[\sqrt{m}\right]}^{\infty}\frac{1}{i^{2-\alpha}}\sum_{j=1}^{\infty}\frac{1}{j^{2-\beta}}$$

$$\leq c(\alpha,\beta)\sup_{0<s<4\pi/\sqrt{m},0<t<4\pi/\sqrt{n}}\left|\varphi_1(x,y,s,t)\right| + o(1)$$

$$= o(1) \quad \text{as } n,m\to\infty.$$

Let

$$\tau(n,m) := \left[\min\left\{\frac{1}{2}\ln n - 1, \frac{1}{2}\ln m - 1, (l(n,m))^{-1}\right\}\right], \tag{20}$$

where

$$l(n,m) := \sup_{0<s<\pi\ln m/m,\ 0<t<\pi\ln n/n}\left|\varphi_1(x,y,s,t)\right| \tag{21}$$

Then we can write

$$\left|L_{mn}^{(3)}(x,y)\right| \tag{22}$$

$$\leq c(\alpha,\beta)\, mn\sum_{i=1}^{\tau(n,m)}\frac{1}{i^{1-\alpha}}\sum_{j=1}^{\tau(n,m)}\frac{1}{j^{2-\beta}}\int_0^{\pi/m}\int_0^{\pi/n}\left|\varphi_1\left(x,y,s+\frac{2i\pi}{m},t+\frac{(2j+1)\pi}{n}\right)\right.$$

$$\left.-\varphi_1\left(x,y,s+\frac{(2i+1)\pi}{m},t+\frac{(2j+1)\pi}{n}\right)\right| dtds$$

$$+c(\alpha,\beta)\, mn\sum_{i=\tau(n,m)}^{(m-1)/2}\frac{1}{\lambda_i}\frac{\lambda_i}{i^{1-\alpha}}\sum_{j=1}^{\tau(n,m)}\frac{1}{j^{2-\beta}}\int_0^{\pi/m}\int_0^{\pi/n}\left|\varphi_1\left(x,y,s+\frac{2i\pi}{m},t+\frac{(2j+1)\pi}{n}\right)\right.$$

$$\left.-\varphi_1\left(x,y,s+\frac{(2i+1)\pi}{m},t+\frac{(2j+1)\pi}{n}\right)\right| dtds$$

$$+c(\alpha,\beta)\,mn\sum_{i=1}^{(m-1)/2}\frac{1}{\lambda_i}\frac{\lambda_i}{i^{1-\alpha}}\sum_{j=\tau(n,m)}^{(n-1)/2}\frac{1}{j^{2-\beta}}\int_0^{\pi/m}\int_0^{\pi/n}\left|\varphi_1\left(x,y,s+\frac{2i\pi}{m},t+\frac{(2j+1)\pi}{n}\right)\right.$$

$$\left.-\varphi_1\left(x,y,s+\frac{(2i+1)\pi}{m},t+\frac{(2j+1)\pi}{n}\right)\right|dtds$$

$$\leq c(\alpha,\beta)\,l(n,m)\,(\tau(n,m))^{\alpha+\beta}$$

$$+c(\alpha,\beta)\frac{\lambda_{\tau(n,m)}}{(\tau(n,m))^{1-\alpha}}V_1\Lambda(f)+c(\alpha,\beta)\frac{1}{(\tau(n,m))^{1-\beta}}V_1\Lambda(f)$$

$$=o(1)\quad\text{as } n,m\to\infty.$$

Analogously, we can prove that

$$\left|L_{mn}^{(2)}(x,y)\right|=o(1)\quad\text{as } n,m\to\infty.$$

For $L_{mn}^{(1)}(x,y)$ we can write

$$\left|L_{mn}^{(1)}(x,y)\right| \tag{23}$$

$$\leq c(\alpha,\beta)\,mn\left\{\sum_{i=1}^{\tau(n,m)}\frac{1}{i^{1-\alpha}}\sum_{j=1}^{\tau(n,m)}\frac{1}{j^{1-\beta}}+\sum_{i=\tau(n,m)}^{(m-1)/2}\frac{1}{i^{1-\alpha}}\sum_{j=1}^{\tau(n,m)}\frac{1}{j^{1-\beta}}\right.$$

$$\left.+\sum_{i=1}^{\tau(n,m)}\frac{1}{i^{1-\alpha}}\sum_{j=\tau(n,m)}^{(n-1)/2}\frac{1}{j^{1-\beta}}+\sum_{i=\tau(n,m)}^{(m-1)/2}\frac{1}{i^{1-\alpha}}\sum_{j=\tau(n,m)}^{(n-1)/2}\frac{1}{j^{1-\beta}}\right\}$$

$$\left(\int_0^{\pi/m}\int_0^{\pi/n}\left|\Delta_{ij}^{mn}f(x+s,y+t)\right|dsdt\right)=\sum_{k=1}^{4}L_{mn}^{(1k)}(x,y).$$

From (20) and (21) we obtain that

$$\left|L_{mn}^{(11)}(x,y)\right|\leq c(\alpha,\beta)\left(l(n,m)^{1-\alpha-\beta}\right)\to 0\quad\text{as } m,n\to\infty. \tag{24}$$

Next, we have

$$\left|L_{mn}^{(13)}(x,y)\right|\leq c(\alpha,\beta)\,mn\left\{\sum_{i=1}^{\tau(n,m)}\frac{1}{i^{1-\alpha}}\sum_{j=\tau(n,m)}^{(n-1)/2}\frac{1}{j^{1-\beta}}\right. \tag{25}$$

$$\left.\left(\int_0^{\pi/m}\int_0^{\pi/n}\left|\Delta_{ij}^{mn}f(x+s,y+t)\right|dsdt\right)\right\}$$

$$\leq c(\alpha,\beta)\,n\left\{\sum_{i=1}^{\tau(n,m)}\frac{1}{i^{1-\alpha}}\left(\int_0^{\pi/n}\sup_x\sum_{j=\tau(n,m)}^{(n-1)/2}\frac{\lambda_j}{j^{1-\beta}}\frac{1}{\lambda_j}\left|{}_2\Delta_j^n f(x,y+t)\right|dt\right)\right\}$$

$$\leq c(\alpha,\beta)\frac{\tau(n,m)}{(\tau(n,m))^{1-\beta-\alpha}}V_2\Lambda(f)\to 0,\quad \text{as } n,m\to\infty.$$

Analogously, we can prove that

$$\left|L_{mn}^{(12)}(x,y)\right|\to 0,\quad \text{as } n,m\to\infty. \tag{26}$$

From the condition of the Theorem 1 we can write

$$\begin{aligned}
&\left|L_{mn}^{(14)}(x,y)\right| \\
&\leq c(\alpha,\beta)\,nm\sum_{i=\tau(n,m)}^{(m-1)/2}\frac{1}{i^{1-\alpha}}\sum_{j=\tau(n,m)}^{(n-1)/2}\frac{1}{j^{1-\beta}} \\
&\left(\int\limits_0^{\pi/m}\int\limits_0^{\pi/n}\left|\Delta_{ij}^{mn}f(x+s,y+t)\right|dsdt\right) \\
&\leq c(\alpha,\beta)\,nm\left\{\sum_{i=\tau(n,m)}^{(m-1)/2}\frac{1}{i^{1-\alpha}}\sum_{j=i}^{(n-1)/2}\frac{\lambda_j}{j^{1-\beta}}\frac{1}{\lambda_j}\right. \\
&\left.+\sum_{j=\tau(n,m)}^{(m-1)/2}\frac{1}{j^{1-\beta}}\sum_{i=j}^{(n-1)/2}\frac{1}{\lambda_i}\frac{\lambda_i}{i^{1-\alpha}}\right\} \\
&\left(\int\limits_0^{\pi/m}\int\limits_0^{\pi/n}\left|\Delta_{ij}^{mn}f(x+s,y+t)\right|dsdt\right) \\
&\leq c(\alpha,\beta)\,n\sum_{i=\tau(n,m)}^{(m-1)/2}\frac{\lambda_i}{i^{2-(\alpha+\beta)}}\left(\int\limits_0^{\pi/n}\sup_x\sum_{j=i}^{(n-1)/2}\frac{1}{\lambda_j}\left|{}_2\Delta_j^n f(x,y+t)\right|dt\right) \\
&+c(\alpha,\beta)\,m\sum_{j=\tau(n,m)}^{(m-1)/2}\frac{\lambda_j}{i^{2-(\alpha+\beta)}}\left(\int\limits_0^{\pi/m}\sup_y\sum_{i=j}^{(m-1)/2}\frac{1}{\lambda_j}\left|{}_1\Delta_i^m f(x+s,y)\right|ds\right) \\
&\leq c(\alpha,\beta)\left(V_1\Lambda(f)+V_2\Lambda(f)\right)\sum_{j=\tau(n,m)}^{\infty}\frac{\lambda_j}{j^{2-(\alpha+\beta)}}\to 0,
\end{aligned} \tag{27}$$

as $n,m\to\infty$.

Combining (23)-(27) we conclude that

$$L_{mn}^{(1)}(x,y)\to 0\quad \text{as } m,n\to\infty. \tag{28}$$

From (18), (19), (22) and (28) we obtain

$$L_{mn}(x,y)\to 0\quad \text{as } m,n\to\infty. \tag{29}$$

Finally, combining (8), (16), (17) and (29) we get

$$I_{mn}^{(1)}(x,y)\to 0\quad as\ \ m,n\to\infty.$$

Analogously, we can prove that

$$I_{mn}^{(k)}(x,y)\to 0 \quad as\ \ m,n\to\infty,\ \ k=2,3,4.$$

To complete the proof of Theorem 1, note that if f is continuous on some compact K, then the relations

$$\lim_{s,t\to 0}\varphi_i(x,y,s,t)=0,\quad i=1,2,3,4,$$

hold uniformly on $(x,y)\in K$ and all estimates in the proof also hold uniformly on $(x,y)\in K$. Hence the $(C;-\alpha,\beta)$ means $\sigma_{n,m}^{\alpha,\beta}(f;x,y)$ will converge to$f$ uniformly on K. □

Proof of Theorem 2. It is not hard to see, that for any sequence $\Lambda=\{\lambda_n\}$ satisfying (1) the class $C\left(T^2\right)\bigcap P\Lambda BV$ is a Banach space with the norm

$$\|f\|_{P\Lambda BV}:=\|f\|_C+P\Lambda V(f).$$

Denote

$$A_{i,j}:=\left[\frac{\pi i-\alpha\pi/2}{N+1/2-\alpha/2},\frac{\pi(i+1)-\alpha\pi/2}{N+1/2-\alpha/2}\right)\times\left[\frac{\pi j-\beta\pi/2}{N+1/2-\beta/2},\frac{\pi(j+1)-\beta\pi/2}{N+1/2-\beta/2}\right)$$

and

$$W:=\left\{(i,j):j<i<2j,1<j<\frac{N-1}{2}\right\}.$$

Let

$$\begin{aligned} f_N(x,y)\ \ :\ &=\sum_{(i,j)\in W}t_j\mathbf{1}_{A_{i,j}}(x,y)\sin\left[(N+1/2-\alpha/2)x+\alpha\pi/2\right]\\ &\times\sin\left[(N+1/2-\beta/2)y+\beta\pi/2\right],\end{aligned}$$

where

$$t_j:=\left(\sum_{i=1}^{j}\frac{1}{\lambda_i}\right)^{-1}.$$

First, we prove that $f\in P\Lambda BV$. Indeed, let

$$y\in\left[\frac{\pi j-\beta\pi/2}{N+1/2-\beta/2},\frac{\pi(j+1)-\beta\pi/2}{N+1/2-\beta/2}\right).$$

Then it is evident that

$$\sum_i\frac{|f(I_i,y)|}{\lambda_i}\le c\left(\sum_{i=j}^{2j-1}\frac{1}{\lambda_{2j-i}}\right)t_j\le c<\infty,$$

consequently,

$$V_1\Lambda(f)<\infty. \tag{30}$$

Let

$$x\in\left[\frac{\pi i-\alpha\pi/2}{N+1/2-\alpha/2},\frac{\pi(i+1)-\alpha\pi/2}{N+1/2-\alpha/2}\right)$$

then from construction of the function f we have

$$\sum_j \frac{|f(x,J_j)|}{\lambda_j} \le c \sum_{j=[i/2]}^{i} \frac{t_j}{\lambda_{j-[i/2]+1}} \le ct_{[i/2]} \left(\sum_{j=1}^{i-[i/2]+1} \frac{1}{\lambda_j} \right) \le c < \infty.$$

Hence

$$V_2\Lambda(f) < \infty. \tag{31}$$

Combining (30) and (31) and we conclude that $f \in P\Lambda BV$.

From (2)-(5) we can write

$$\begin{aligned} &\pi^2 \sigma_{N,N}^{(-\alpha,-\beta)} f_N(0,0) \qquad (32)\\ &= \int_{T^2} f_N(x,y) K_N^{-\alpha}(x) K_N^{-\beta}(y)\, dxdy \\ &= \sum_{(i,j)\in W} t_j \int_{A_{i,j}} \sin[(N+1/2-\alpha/2)x+\alpha\pi/2] \sin[(N+1/2-\beta/2)y+\beta\pi/2] \\ &\quad \times O\left(\frac{1}{Nx^2}\right) O\left(\frac{1}{Ny^2}\right) dxdy \\ &+ \sum_{(i,j)\in W} t_j \int_{A_{i,j}} \sin[(N+1/2-\alpha/2)x+\alpha\pi/2] \frac{\sin^2[(N+1/2-\beta/2)y+\beta\pi/2]}{A_N^{-\beta}(2\sin y/2)^{1-\beta}} \\ &\quad \times O\left(\frac{1}{Nx^2}\right) dxdy \\ &+ \sum_{(i,j)\in W} t_j \int_{A_{i,j}} \frac{\sin^2[(N+1/2-\alpha/2)x+\alpha\pi/2]}{A_N^{-\alpha}(2\sin x/2)^{1-\alpha}} \sin[(N+1/2-\beta/2)y+\beta\pi/2] \\ &\quad \times O\left(\frac{1}{Ny^2}\right) dxdy \\ &+ \sum_{(i,j)\in W} t_j \int_{A_{i,j}} \frac{\sin^2[(N+1/2-\alpha/2)x+\alpha\pi/2]}{A_N^{-\alpha}(2\sin x/2)^{1-\alpha}} \frac{\sin^2[(N+1/2-\beta/2)y+\beta\pi/2]}{A_N^{-\beta}(2\sin y/2)^{1-\beta}} dxdy \\ &= \sum_{k=1}^{4} F_N^{(k)}(x,y) \end{aligned}$$

It is easy to show that

$$\begin{aligned} &\left| F_N^{(1)}(x,y) \right| \le c \sum_{(i,j)\in W} \frac{t_j}{ij} \qquad (33)\\ &= c \sum_{j=1}^{[(N-1)/2]} \frac{t_j}{j} \sum_{i=j+1}^{2j-1} \frac{1}{i} \le c \sum_{j=1}^{[(N-1)/2]} \frac{t_j}{j}, \end{aligned}$$

$$\left|F_N^{(2)}(x,y)\right| \leq c(\alpha,\beta)\sum_{(i,j)\in W}\frac{t_j}{ij^{1-\beta}} \tag{34}$$

$$= c(\alpha,\beta)\sum_{j=1}^{[(N-1)/2]}\frac{t_j}{j^{1-\beta}}\sum_{i=j+1}^{2j-1}\frac{1}{i}$$

$$\leq c(\alpha,\beta)\sum_{j=1}^{[(N-1)/2]}\frac{t_j}{j^{1-\beta}},$$

$$\left|F_N^{(3)}(x,y)\right| \leq c(\alpha,\beta)\sum_{j=1}^{[(N-1)/2]}\frac{t_j}{j}\sum_{i=j+1}^{2j-1}\frac{1}{i^{1-\alpha}} \tag{35}$$

$$\leq c(\alpha,\beta)\sum_{j=1}^{[(N-1)/2]}\frac{t_j}{j^{1-\alpha}}.$$

From the construction of the function f_N we can write

$$\left|F_N^{(4)}(x,y)\right| \tag{36}$$

$$= \frac{1}{(N+1/2-\alpha/2)(N+1/2-\beta/2)}\sum_{(i,j)\in W}t_j$$

$$\int_{\pi i}^{\pi(i+1)}\int_{\pi j}^{\pi(j+1)}\frac{\sin^2 u}{A_N^{-\alpha}\left(2\sin\frac{u-\alpha\pi/2}{2(N+1/2-\alpha/2)}\right)^{1-\alpha}}\frac{\sin^2 v}{A_N^{-\alpha}\left(2\sin\frac{v-\beta\pi/2}{2(N+1/2-\beta/2)}\right)^{1-\beta}}dudv$$

$$\geq \frac{c(\alpha,\beta)N^{\alpha+\beta}}{N^2}\sum_{(i,j)\in W}t_j\frac{N^{2-(\alpha+\beta)}}{i^{1-\alpha}j^{1-\beta}}\int_{\pi i}^{\pi(i+1)}\sin^2 udu\int_{\pi j}^{\pi(j+1)}\sin^2 vdv$$

$$\geq c(\alpha,\beta)\sum_{(i,j)\in W}\frac{t_j}{j^{1-\beta}}\frac{1}{i^{1-\alpha}} \geq c(\alpha,\beta)\sum_{j=1}^{[(N-1)/2]}\frac{t_j}{j^{1-\beta}}\sum_{i=j+1}^{2j-1}\frac{1}{i^{1-\alpha}}$$

$$\geq c(\alpha,\beta)\sum_{j=1}^{[(N-1)/2]}\frac{t_j}{j^{1-(\beta+\alpha)}}.$$

Combining (32)-(36) we conclude that

$$\pi^2\left|\sigma_{N,N}^{(-\alpha,-\beta)}f_N(0,0)\right| \geq c(\alpha,\beta)\sum_{j=j_0}^{[(N-1)/2]}\frac{t_j}{j^{1-(\beta+\alpha)}}. \tag{37}$$

Let $\lambda_j = j^{1-(\alpha+\beta)}\gamma_j,\ \ \gamma_j \geq \gamma_{j+1},\ j=1,2,....$ Then we can write

$$\frac{1}{t_j} = \sum_{i=1}^{j}\frac{1}{\lambda_i} = \sum_{i=1}^{j}\frac{1}{i^{1-(\alpha+\beta)}\gamma_i} \leq c(\alpha,\beta)\frac{j^{\alpha+\beta}}{\gamma_j}.$$

Consequently,

$$t_j j^{\alpha+\beta} \geq c(\alpha,\beta)\gamma_j. \tag{38}$$

Combining (37) and (38) we obtain

$$\pi^2 \left|\sigma_{N,N}^{(-\alpha,-\beta)} f_N(0,0)\right| \geq c(\alpha,\beta) \sum_{j=j_0}^{[(N-1)/2]} \frac{\gamma_j}{j} \tag{39}$$

$$= c(\alpha,\beta) \sum_{j=j_0}^{[(N-1)/2]} \frac{t_j}{j^{2-(\beta+\alpha)}} \to \infty \text{ as } N \to \infty.$$

Applying the Banach-Steinhaus Theorem, from (39) we obtain that there exists a continuous function $f \in P\Lambda BV$ such that

$$\sup_N \left|\sigma_{N,N}^{(-\alpha,-\beta)} f_N(0,0)\right| = +\infty.$$

□

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