ON THE CONVERGENCE OF MULTIPLE WALSH-FOURIER SERIES OF FUNCTIONS OF BOUNDED GENERALIZED VARIATION

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Abstract. The convergence of multiple Walsh-Fourier series of functions of bounded generalized variation is investigated. The sufficient and necessary conditions on the sequence $\Lambda = \{\lambda_n\}$ are found for the convergence of multiple Fourier series of functions of bounded partial Λ-variation.

1. Classes of Functions of Bounded Generalized Variation

In 1881 Jordan [13] introduced the class of functions of bounded variation and applied it to the theory of Fourier series. Hereafter this notion was generalized by many authors (quadratic variation, Φ-variation, Λ-variation ets., see [13, 22, 21, 14]). In two dimensional case the class BV of functions of bounded variation was introduced by Hardy [12].

Let $I := [0,1)$ and

$$J^k = \left(a^k, b^k\right) \subset I, \qquad k = 1, 2, \dots d.$$

Consider a measurable function $f(x)$ defined on R^d and 1-periodic with respect to each variable. For $d = 1$ we set

$$f\left(J^1\right) := f\left(b^1\right) - f\left(a^1\right).$$

If for a function of $d-1$ variables the expression $f\left(J^1 \times \cdots \times J^{d-1}\right)$ is already defined, then for a function of d variables the *mixed difference* is defined as follows:

$$f\left(J^1 \times \cdots \times J^d\right) := f\left(J^1 \times \cdots \times J^{d-1}, b^d\right) - f\left(J^1 \times \cdots \times J^{d-1}, a^d\right).$$

Let $E = \{J_k\}$ be a collection of nonoverlapping intervals from I ordered in arbitrary way and let Ω be the set of all such collections E.

For sequences of positive numbers $\Lambda^j = \{\lambda_n^j\}_{n=1}^{\infty}$, $j = 1, 2, \dots, d$, the $\left(\Lambda^1, \dots, \Lambda^d\right)$*-variation of f with respect to the index set*

$$D := \{1, 2, ..., d\}$$

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is defined as follows:

$$V_{\Lambda^1,\ldots,\Lambda^d}^{D}(f) := \sup_{\{J_{i_j}^j\}_{i_j=1}^{k_j}\in\Omega} \sum_{i_1,\ldots,i_d} \frac{\left|f\left(J_{i_1}^1\times\cdots\times J_{i_d}^d\right)\right|}{\lambda_{i_1}\cdots\lambda_{i_d}}.$$

For an index set $\alpha=\{j_1,...,j_p\}\subset D$ and any $x=(x_1,...,x_d)\in R^d$ we set $\widetilde{\alpha}:=D\setminus\alpha$ and denote by x_α the vector of R^p consisting of components $x_j, j\in\alpha$, i.e.

$$x_\alpha=\left(x_{j_1},...,x_{j_p}\right)\in R^p.$$

By $V_{\Lambda^{j_1},\ldots,\Lambda^{j_p}}^{\alpha}(f,x_{\widetilde{\alpha}})$ and $f\left(J_{i_{j_1}}^1\times\cdots\times J_{i_{j_p}}^p, x^{\widetilde{\alpha}}\right)$ we denote respectively the $\left(\Lambda^{j_1},...,\Lambda^{j_p}\right)$-variation and the mixed difference of f as a function of variables $x_{j_1},...,x_{j_p}$ over the p-dimensional cube I^p with fixed values $x_{\widetilde{\alpha}}$ of other variables. The $\left(\Lambda^{j_1},...,\Lambda^{j_p}\right)$ *−variation of f with respect to index set* α is defined as follows:

$$V_{\Lambda^{j_1},\ldots,\Lambda^{j_p}}^{\alpha}(f)=\sup_{x_{\widetilde{\alpha}}\in I^{d-p}} V_{\Lambda^{j_1},\ldots,\Lambda^{j_p}}^{\alpha}(f,x_{\widetilde{\alpha}}),$$

where $I^p:=[0,1)^p$.

Definition 1. *We say that the function f has Bounded total $\left(\Lambda^1,...,\Lambda^d\right)$-variation on I^d and write*

$$f\in BV_{\Lambda^1,\ldots,\Lambda^d}:=BV_{\Lambda^1,\ldots,\Lambda^d}(T^d),$$

if

$$V_{\Lambda^1,\ldots,\Lambda^d}(f):=\sum_{\alpha\subset D} V_{\Lambda^{j_1},\ldots,\Lambda^{j_p}}^{\alpha}(f)<\infty.$$

Definition 2. *We say that the function f is continuous in $\left(\Lambda^1,...,\Lambda^d\right)$-variation on I^d and write*

$$f\in CV_{\Lambda^1,\ldots,\Lambda^d}:=CV_{\Lambda^1,\ldots,\Lambda^d}(T^d),$$

if

$$\lim_{n\to\infty} V_{\Lambda^{j_1},\ldots,\Lambda^{j_{k-1}},\Lambda_n^{j_k},\Lambda^{j_{k+1}},\ldots,\Lambda^{j_p}}^{\alpha}(f)=0, \qquad k=1,2,\ldots,p$$

for any $\alpha\subset D$, $\alpha:=\{j_1,...,j_p\}$, where $\Lambda_n^{j_k}:=\left\{\lambda_s^{j_k}\right\}_{s=n}^{\infty}$.

Definition 3. *We say that the function f has Bounded Partial $\left(\Lambda^1,...,\Lambda^d\right)$-variation and write*

$$f\in PBV_{\Lambda^1,\ldots,\Lambda^d}:=PBV_{\Lambda^1,\ldots,\Lambda^d}(T^d),$$

if

$$PV_{\Lambda^1,\ldots,\Lambda^d}(f):=\sum_{i=1}^{d} V_{\Lambda^i}^{\{i\}}(f)<\infty.$$

In the case $\Lambda^1 = \cdots = \Lambda^d = \Lambda$ we denote

$$BV_\Lambda := BV_{\Lambda^1,\dots,\Lambda^d}, \quad CV_\Lambda := CV_{\Lambda^1,\dots,\Lambda^d},$$

and

$$PBV_\Lambda := PBV_{\Lambda^1,\dots,\Lambda^d}.$$

If $\lambda_n \equiv 1$ (or if $0 < c < \lambda_n < C < \infty,\ n = 1, 2, \dots$) the classes BV_Λ and PBV_Λ coincide with the Hardy class BV and PBV respectively. Hence it is reasonable to assume that $\lambda_n \to \infty$, and since the intervals in the collection $E = \{J_i\}$ are ordered arbitrarily, we suppose, without loss of generality, that the sequence $\{\lambda_n\}$ is increasing. Thus,

$$1 < \lambda_1 \le \lambda_2 \le \dots, \qquad \lim_{n\to\infty} \lambda_n = \infty. \tag{1}$$

When $\lambda_n = n$ for all $n = 1, 2 \dots$ we say *Harmonic Variation* instead of Λ-variation and write H instead of Λ (BV_H, PBV_H, CV_H, ets).

Remark 1. *The notion of Λ-variation was introduced by Waterman* [21] *in one dimensional case, by Sahakian* [19] *in two dimensional case and by Sablin* [18] *in the case of higher dimensions. The notion of bounded partial variation (class PBV) was introduced by Goginava in* [7]*. These classes of functions of generalized bounded variation play an important role in the theory Fourier series.*

Observe, that the number of variations in Definition 1 of total variation is $2^d - 1$, while the number of variations in Definition 3 of partial variation is only d.

The statements of the following theorem are known.

Theorem A. 1) (Dragoshanski [5]) *If $d = 2$, then $BV_H = CV_H$.*

2) (Bakhvalov [1]) *For any $d \ge 2$,*

$$CV_H = \bigcup_\Gamma BV_\Gamma,$$

where the union is taken over all sequences $\Gamma = \{\gamma_n\}_{n=1}^\infty$ with $\gamma_n = o(n)$ as $n \to \infty$.

Theorem 1. *Let $\Lambda = \{\lambda_n\}_{n=1}^\infty$ and $d \ge 2$. If*

$$\frac{\lambda_n}{n} \downarrow 0 \quad \text{and} \quad \sum_{n=1}^\infty \frac{\lambda_n \log^{d-2} n}{n^2} < \infty, \tag{2}$$

then there exists a sequence $\Gamma = \{\gamma_n\}_{n=1}^\infty$ with

$$\gamma_n = o(n) \quad \text{as} \quad n \to \infty, \tag{3}$$

such that $PBV_\Lambda \subset BV_\Gamma$.

Proof of Theorem 1. Choosing the sequence $\{A_n\}_{n=1}^\infty$ such that

$$A_n \uparrow \infty, \qquad \frac{\lambda_n A_n}{n} \downarrow 0, \qquad \sum_{n=1}^\infty \frac{\lambda_n \log^{d-2} n A_n^d}{n^2} < \infty, \tag{4}$$

we set

$$\gamma_n = \frac{n}{A_n}, \qquad n = 1, 2, \ldots \tag{5}$$

We prove that there is a constant $C > 0$ such that

$$\sum_{i_1,\ldots,i_p} \frac{\left| f\left(J^1_{i_1} \times \cdots \times J^p_{i_p}, x^{\widetilde{\alpha}}\right)\right|}{\gamma_{i_1}\cdots\gamma_{i_p}} < C \cdot PV_\Lambda(f), \tag{6}$$

for any $f \in PBV_\Lambda$, $\{J^j_{i_j}\}_{i_j=1}^{k_j} \in \Omega$, $j = 1, 2, \ldots, d$, and $\alpha := \{i_1, ..., i_p\} \subset D$.

To prove (6) observe, that

$$\sum_{i_1,\ldots,\, i_p} \frac{\left| f\left(J^1_{i_1} \times \cdots \times J^p_{i_p}, x^{\widetilde{\alpha}}\right)\right|}{\gamma_{i_1}\cdots\gamma_{i_p}} \tag{7}$$

$$= \sum_{\sigma} \sum_{i_{\sigma(1)} \le \cdots \le i_{\sigma(p)}} \frac{\left| f\left(J^1_{i_1} \times \cdots \times J^p_{i_p}, x^{\widetilde{\alpha}}\right)\right|}{\gamma_{i_1}\cdots\gamma_{i_p}} < \infty,$$

where the sum is taken over all rearrangements $\sigma = \{\sigma(k)\}_{k=1}^p$ of the set $\{1, 2, \ldots, p\}$.

Denoting $M = PV_\Lambda(f)$ and using (5), (4) and (2) we obtain:

$$\sum_{i_1 \le i_2 \le \cdots \le i_p} \frac{\left| f\left(J^1_{i_1} \times \cdots \times J^p_{i_p}, x^{\widetilde{\alpha}}\right)\right|}{\gamma_{i_1}\cdots\gamma_{i_p}}$$

$$= \sum_{i_1 \le i_2 \le \cdots \le i_{p-1}} \frac{A_{i_1}\cdots A_{i_{p-1}}}{i_1 \cdots i_{p-1}} \sum_{i_p \ge i_{p-1}} \frac{\left| f\left(J^1_{i_1} \times \cdots \times J^p_{i_p}, x^{\widetilde{\alpha}}\right)\right|}{\lambda_{i_p}} \cdot \frac{\lambda_{i_p} A_{i_p}}{i_p}$$

$$\le M \sum_{i_1 \le i_2 \le \cdots \le i_{p-1}} \frac{A^p_{i_{p-1}} \lambda_{i_{p-1}}}{i^2_{p-1}} \cdot \frac{1}{i_1 \cdots i_{p-2}}$$

$$= M \sum_{i_{i_{p-1}}=1}^{\infty} \frac{A^p_{i_{p-1}} \lambda_{i_{p-1}}}{i^2_{p-1}} \sum_{i_{p-2}=1}^{i_{p-1}} \frac{1}{i_{p-2}} \sum_{i_{p-3}=1}^{i_{p-2}} \frac{1}{i_{p-3}} \cdots \sum_{i_1=1}^{i_2} \frac{1}{i_1}$$

$$\le M \sum_{i_{p-1}=1}^{\infty} \frac{A^p_{i_{p-1}} \lambda_{i_{p-1}}}{i^2_{p-1}} \left(\sum_{i=1}^{i_{p-1}} \frac{1}{i} \right)^{p-2} \le C \cdot M \sum_{n=1}^{\infty} \frac{A^p_n \lambda_n \log^{d-2} n}{n^2} < \infty.$$

Similarly we can prove that all other summands in the right hind side of (7) are finite. Theorem 1 is proved. □

2. Walsh function

We denote the set of all non-negative integers by $\mathbf{N}$, the set of all integers by $\mathbf{Z}$ and the set of dyadic rational numbers in the unit interval $I := [0,1)$ by $\mathbf{Q}$. Each element of $\mathbf{Q}$ has the form $\frac{p}{2^n}$ for some $p,n \in \mathbf{N}$, $0 \le p \le 2^n$.

By a dyadic interval in I we mean an interval $I_N^l := [l2^{-N}, (l+1)\,2^{-N})$ for some $l \in \mathbf{N}$, $0 \le l < 2^N$. Given $N \in \mathbf{N}$ and $x \in I$, we denote by $I_N(x)$ the dyadic interval of lenght 2^{-N} that contains x. We denote $I_N := [0, 2^{-N})$.

Let $r_0(x)$ be the function defined on the real line by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0,1/2) \\ -1, & \text{if } x \in [1/2,1) \end{cases}, \qquad r_0(x+1) = r_0(x), \qquad x \in \mathbf{R}.$$

The Rademacher system is defined by

$$r_n(x) = r_0(2^n x), \quad x \in I, \quad n = 0,1,\ldots.$$

Let $w_0, w_1, \ldots$ represent the Walsh functions, i.e. $w_0(x) \equiv 1$ and if $n = 2^{n_1} + \cdots + 2^{n_s}$ is a positive integer with $n_1 > n_2 > \cdots > n_s$ then

$$w_n(x) = r_{n_1}(x) \cdots r_{n_s}(x), \qquad x \in I.$$

The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that (see [20, 11]):

$$D_n(t) = w_n(t) \sum_{j=0}^{\infty} \delta_j w_{2^j}(t)\, D_{2^j}(t), \tag{8}$$

where $n = \sum_{j=0}^{\infty} \delta_j 2^j$, $\delta_j = 0$ or 1.

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in [0, 2^{-n}), \\ 0, & \text{if } x \in [2^{-n}, 1) \end{cases} \tag{9}$$

$$|D_n(x)| \le \min\left(n, \frac{1}{x}\right), \quad x \in (0,1), \tag{10}$$

$$|D_{m_A}(x)| \ge \frac{1}{4x}, \quad 2^{-2A-1} \le x < 1, \tag{11}$$

where

$$m_A := 2^{2A-2} + 2^{2A-4} + \cdots + 2^2 + 2^0. \tag{12}$$

Given $x \in I$, the expansion

$$x = \sum_{k=0}^{\infty} x_k 2^{-(k+1)}, \tag{13}$$

where $x_k = 0$ or 1, is called the dyadic expansion of x. If $x \in I \backslash \mathbf{Q}$, then (13) is uniquely determined. For the dyadic expansion of $x \in \mathbf{Q}$ we choose the one with $\lim_{k\to\infty} x_k = 0$.

The dyadic sum of $x, y \in I$ in terms of the dyadic expansion of x and y is defined by

$$x \dotplus y = \sum_{k=0}^{\infty} |x_k - y_k| \, 2^{-(k+1)}.$$

We consider the multiple Walsh system

$$w_{n_1}(x_1) \times \cdots \times w_{n_d}(x_d), \qquad n_i \in \mathbf{N}, \quad i = 1, 2, ..., d$$

on the d-dimensional unit cube $I^d = [0,1) \times \cdots \times [0,1)$.

If $f \in L^1(I^d)$, then

$$\hat{f}(n_1, ..., n_d) = \int_{I^d} f(x_1, ..., x_d) \, w_{n_1}(x_1) \cdots w_{n_d}(x_d) \, dx_1 \cdots dx_d$$

is the $(n_1, ..., n_d)$-th Walsh-Fourier coefficient of f.

The rectangular partial sums of d-dimensional Fourier series with respect to the Walsh system are defined by

$$S_{m_1,...,m_d} f(x_1, ..., x_d) = \sum_{n_1=0}^{m_1-1} \cdots \sum_{n_d=0}^{m_d-1} \hat{f}(n_1, ..., n_d) \, w_{n_1}(x_1) \cdots w_{n_d}(x_d).$$

Denoting

$$h_{\{i\}} := (0, ..., 0, h_i, 0, ..., 0) \in \mathbf{R}^d$$

and

$$\Theta\left(f, x, h_{\{i\}}\right) := f\left(x + h_{\{i\}}\right) - f(x), \qquad x \in \mathbf{R}^d,$$

the symbols $\Theta\left(f, x, h_{\{\alpha_1,...,\alpha_p\}}\right)$ will stand for the expression which can be obtained by consecutive applying of Θ to the arguments with indices $\{\alpha_1, ..., \alpha_p\}$.

We denote by $C(I^d)$ the space of continuous, 1-periodic with respect to each variable functions defined on $\mathbf{R}^d$ with the norm

$$\|f\|_C := \sup_{x \in I^d} |f(x)|.$$

For $f \in C(I^d)$ the expressions

$$\omega_{\alpha_1,...,\alpha_p}(\delta_{\alpha_1}, ..., \delta_{\alpha_1}; f)_C := \sup_{|h_{\alpha_i}| \le \delta_{\alpha_i}, \ i=1,...,p} \left\|\Theta\left(f, \cdot, h_{\{\alpha_1,...,\alpha_p\}}\right)\right\|_C$$

are called modulus of continuity of function f.

3. Convergence of d-dimensional Walsh-Fourier series

In this paper we consider convergence of **only rectangular partial sums** (convergence in the sense of Pringsheim) of d-dimensional Walsh-Fourier series.

We say that $f(x_1, ..., x_d)$ is continuous at $(x_1, ..., x_d)$ if

$$\lim_{h_i \to 0+, \ i=1,...,d} f(x_1 \dotplus h_1, ..., x_d \dotplus h_d) = f(x_1, ..., x_d). \tag{14}$$

The well known Dirichlet-Jordan theorem (see [23]) states that the Fourier series of a function $f(x)$, $x \in T$ of bounded variation converges at every point x to the value $\left[f(x+0)+f(x-0)\right]/2$.

Hardy [12] generalized the Dirichlet-Jordan theorem to the double Fourier series. He proved that if function $f(x,y)$ has bounded variation in the sense of Hardy ($f \in BV$), then $S[f]$ converges at any point (x,y) to the value $\frac{1}{4}\sum f(x \pm 0, y \pm 0)$.

Convergence of d-dimensional trigonometric Fourier series of functions of bounded Λ-variation was investigated in details by Sahakian [19], Dyachenko [2, 3, 4], Bakhvalov [1], Sablin [18], Goginava, Sahakian [10].

For the d-dimensional Walsh-Fourier series the convergence of partial sums of functions Harmonic bounded fluctuation and other bounded generalized variation were studied by Moricz [15, 16], Onnewer, Waterman [17], Goginava [8, 9].

For two-dimensional functions of bounded Harmonic variation Sargsyan [24] has proved the following

Theorem S (Sargsyan [24])**.** *If $f \in BV_H(I^2)$, then the 2-dimensional Walsh-Fourier series of f converges to $f(x_1,x_2)$ at any point $(x_1,x_2) \in I^2$, where f is continuous .*

Now we formulate the main results of this paper.

Theorem 2. *Let $f \in CV_H(I^d)$, $d \geq 2$. Then the d-dimensional Walsh-Fourier series of f converges to $f(x)$ at any point $x \in I^d$, where the function f is continuous.*

The next theorem shows that Theorem S is not true for $d > 2$.

Theorem 3. *Let $d > 2$. Then there exists a continuous function $f \in BV_H(I^d)$ such that the d-dimensional Walsh-Fourier cubic partial sums of f diverge at some point.*

Note that analogical results for the trigonometric system were proved by Bakhvalov [1].

In the next theorem we consider the behaviour of the multidimensional Walsh-Fourier series of functions of bounded partial Λ-variation.

Theorem 4. *Let $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ and $d \geq 2$.*

a) If

$$\sum_{n=1}^{\infty} \frac{\lambda_n \log^{d-2} n}{n^2} < \infty, \tag{15}$$

then the d-dimensional Walsh-Fourier series of a function $f \in PBV_\Lambda(I^d)$ converges to $f(x)$ at any point $x \in I^d$, where f is continuous.

b) If

$$\frac{\lambda_n}{n} = O\left(\frac{\lambda_{[n^\delta]}}{[n^\delta]}\right) \tag{16}$$

for some $\delta > 1$, *and*

$$\sum_{n=1}^{\infty} \frac{\lambda_n \log^{d-2} n}{n^2} = \infty, \tag{17}$$

then there exists a continuous function $f \in PBV_{\Lambda}(I^d)$ *such that the d-dimensional cubic partial sums of its Walsh-Fourier series diverge at some point.*

Theorem 4 implies

Corollary 1. *a) If* $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ *with*

$$\lambda_n = \frac{n}{\log^{d-1+\varepsilon} n}, \quad n = 2, 3, \ldots, \qquad d \geq 2,$$

for some $\varepsilon > 0$, *then the d-dimensional Walsh-Fourier series of a function* $f \in PBV_{\Lambda}(I^d)$ *converges to* $f(x)$ *at any point* x, *where* f *is continuous.*

b) If $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ *with*

$$\lambda_n = \frac{n}{\log^{d-1} n}, \quad n = 2, 3, \ldots, \qquad d \geq 2,$$

then there exists a continuous function $f \in PBV_{\Lambda}(I^d)$ *such that the d-dimensional cubic partial sums of its Walsh-Fourier series diverge at some point.*

4. Proofs of Main Results

Proof of Theorem 2. Let $n_i := 2^{N_i} + n_i',\ 0 \leq n_i' < 2^{N_i},\ i = 1, 2, ..., d.$ Since

$$D_{2^{N_i}+n_i'} = D_{2^{N_i}} + w_{2^{N_i}} D_{n_i'}$$

we can write

$$\begin{aligned} &S_{n_1,\ldots,n_d} f(x_1, ..., x_d) - f(x_1, ..., x_d) \\ &= \int_{I^d} [f(x_1 \dot{+} s_1, ..., x_d \dot{+} s_d) - f(x_1, ..., x_d)] \prod_{j=1}^{d} D_{n_j}(s_j)\, ds_1 \cdots ds_d \\ &= \sum_{\alpha \subset D} \int_{I^d} [f(x_1 \dot{+} s_1, ..., x_d \dot{+} s_d) - f(x_1, ..., x_d)] \\ &\times \prod_{r \in D \backslash \alpha} D_{2^{N_r}}(s_r) \prod_{l \in \alpha} w_{2^{N_l}}(s_l) D_{n_l'}(s_l)\, ds_1 \cdots ds_d =: \sum_{\alpha \subset D} A_{\alpha}. \end{aligned} \tag{18}$$

If $\alpha = \varnothing$, then from (9) we have

$$A_{\alpha} = o(1), \quad \text{as} \quad \min\{n_1, \ldots, n_d\} \to \infty. \tag{19}$$

If $\alpha = D$, then we can write

$$\begin{aligned}
A_D &= \sum_{i_1=0}^{2^{N_1}-1} \cdots \sum_{i_d=0}^{2^{N_d}-1} \int\limits_{I_{N_1}^{i_1}\times\cdots\times I_{N_d}^{i_d}} \left[f\left(x_1 \dot{+} s_1, ..., x_d \dot{+} s_d\right) - f\left(x_1, ..., x_d\right)\right] \\
&\quad \times \prod_{l=1}^{d} w_{2^{N_l}}\left(s_l\right) D_{n_l'}\left(s_l\right) ds_1 \cdots ds_d \\
&= \sum_{i_1=0}^{2^{N_1-1}} \cdots \sum_{i_d=0}^{2^{N_d-1}} \prod_{r=1}^{d} D_{n_l'}\left(\frac{i_r}{2^{N_r}}\right) \\
&\quad \times \int\limits_{I_{N_1}^{i_1}\times\cdots\times I_{N_d}^{i_d}} \left[f\left(x_1 \dot{+} s_1, ..., x_d \dot{+} s_d\right) - f\left(x_1, ..., x_d\right)\right] \\
&\quad \times \prod_{l=1}^{d} w_{2^{N_l}}\left(s_l\right) ds_1 \cdots ds_d.
\end{aligned}$$

Since

$$\begin{aligned}
&\int\limits_{I_{N_1}^{i_1}\times\cdots\times I_{N_d}^{i_d}} \left[f\left(x_1 \dot{+} s_1, ..., x_d \dot{+} s_d\right) - f\left(x_1, ..., x_d\right)\right] \\
&\times \prod_{l=1}^{d} w_{2^{N_l}}\left(s_l\right) ds_1 \cdots ds_d. \\
= &\int\limits_{I_{N_1+1}^{2i_1}} \int\limits_{I_{N_2}^{i_2}\times\cdots\times I_{N_d}^{i_d}} \Delta^{N_1+1} f\left(x_1 \dot{+} s_1, ..., x_d \dot{+} s_d\right) \prod_{l=2}^{d} w_{2^{N_l}}\left(s_l\right) ds_1 \cdots ds_d \\
= &\cdots \\
= &\int\limits_{I_{N_1+1}^{2i_1}\times\cdots\times I_{N_d+1}^{2i_d}} \Delta^{N_d+1}\left(\Delta^{N_{d-1}+1} \cdots \Delta^{N_1+1}\right. \\
&\qquad \left. f\left(x_1 \dot{+} s_1, ..., x_d \dot{+} s_d\right)_1 \cdots\right)_d ds_1 \cdots ds_d,
\end{aligned}$$

where

$$\begin{aligned}
\Delta^N f\left(x_1, ..., x_d\right)_j \quad : \quad &= f\left(x_1, ..., x_d\right) \\
&- f\left(x_1, ..., x_{j-1}, x_j \dot{+} 2^{-N}, x_{j+1}, ..., x_d\right).
\end{aligned}$$

From (10) we have

$$\begin{aligned} |A_D| &\leq c\prod_{r=1}^{d}2^{N_r}\int\limits_{I_{N_1+1}\times\cdots\times I_{N_d+1}}\sum_{i_1=0}^{2^{N_1}-1}\cdots\sum_{i_d=0}^{2^{N_d}-1}\prod_{r=1}^{d}\frac{1}{i_r+1} \\ &\times \left|\Delta^{N_d+1}\left(\cdots\Delta^{N_1+1}\right.\right. \\ &\left.\left. f\left(x_1\dotplus s_1\dotplus\frac{i_1}{2^{N_1}},...,x_d\dotplus s_d\dotplus\frac{i_d}{2^{N_d}}\right)_1\cdots\right)_d\right| ds_1\cdots ds_d \end{aligned}$$

Set

$$\tau(N_1,...,N_d):=\left[\min\left\{N_1-2,...,N_d-2,(\theta(N_1,...,N_d))^{-1}\right\}\right],$$

where

$$\theta(N_1,...,N_d) = \sup_{0<s_i<N_i 2^{-N_i},\ i=1,...,d}\left|f(x_1\dotplus s_1,...,x_d\dotplus s_d)-f(x_1,...,x_d)\right|.$$

Then we can write

$$\begin{aligned} |A_D| &\leq c\prod_{r=1}^{d}2^{N_r}\int\limits_{I_{N_1+1}\times\cdots\times I_{N_d+1}}\sum_{i_1,...,i_d=0}^{\tau(N_1,...,N_d)}\prod_{r=1}^{d}\frac{1}{i_r+1} \\ &\times \left|\Delta^{N_d+1}\left(\cdots\Delta^{N_1+1}\right.\right. \\ &\left.\left. f\left(x_1\dotplus s_1\dotplus\frac{i_1}{2^{N_1}},...,x_d\dotplus s_d\dotplus\frac{i_d}{2^{N_d}}\right)_1\cdots\right)_d\right| ds_1\cdots ds_d \\ &+ c\prod_{r=1}^{d}2^{N_r}\sum_{l=1}^{d}\int\limits_{I_{N_1+1}^{2i_1}\times\cdots\times I_{N_d+1}^{2i_d}}\sum_{i_1=0}^{2^{N_1}-1}\cdots\sum_{i_{l-1}=0}^{2^{N_{l-1}}-1} \\ &\sum_{i_l=\tau(N_1,...,N_d)}^{2^{N_l}-1}\sum_{i_{l+1}=0}^{2^{N_{l+1}}-1}\cdots\sum_{i_d=0}^{2^{N_d}-1}\prod_{r=1}^{d}\frac{1}{i_r+1} \qquad (20) \\ &\times \left|\Delta^{N_d+1}\left(\cdots\Delta^{N_1+1}\right.\right. \\ &\left.\left. f\left(x_1\dotplus s_1\dotplus\frac{i_1}{2^{N_1}},...,x_d\dotplus s_d\dotplus\frac{i_d}{2^{N_d}}\right)_1\cdots\right)_d\right| ds_1\cdots ds_d \\ &\leq c\theta(N_1,...,N_d)\log^d\left(\frac{1}{\theta(N_1,...,N_d)}\right) \\ &+ c\sum_{l=1}^{d}V^D_{\{i_1\}...\{i_{l-1}\}\{i_l+\tau(N_1,...,N_d)\}\{i_{l+1}\}...\{i_d\}}(f)=o(1), \end{aligned}$$

as $\min(n_1,...,n_d)\to\infty$.

If $\alpha\subset D,\ \alpha\neq\varnothing,\ \alpha\neq D$, then we can prove similarly, that

$$A_\alpha = o(1) \text{ as } \min(n_1,...,n_d)\to\infty. \qquad (21)$$

Combining (18)-(21) we complete the proof of Theorem 2. $\square$

Proof of Theorem 3. Let $\{A_k : k \geq 1\}$ be an increasing sequence of positive integers, satisfying

$$A_k > 2A_{k-1}, \tag{22}$$

$$\frac{A_k 2^{2dA_{k-1}}}{2^{A_k}} < \frac{1}{k^2}, \tag{23}$$

$$\frac{A_{k-1}^d}{A_k} < \frac{1}{k}. \tag{24}$$

Set

$$\varphi_k(x) := \begin{cases} 2\left(2^{2A_k}x - j\right), & \text{if } x \in [j2^{-2A_k}, (2j+1)\,2^{-2A_k-1}) \\ -2\left(2^{2A_k}x - j - 1\right), & \text{if } x \in [(2j+1)\,2^{-2A_k-1}, (j+1)\,2^{-2A_k}) \\ & \qquad j = 1, 2, ..., 2^{2A_k-2A_{k-1}} - 1 \\ 0, & \text{otherwise} \end{cases},$$

$$\psi_k(x) := \begin{cases} 2\left(2^{2A_k}x - 1\right), & \text{if } x \in [2^{-2A_k}, 3\cdot 2^{-2A_k-1}) \\ -2\left(2^{2A_k}x - 2\right), & \text{if } x \in [3\cdot 2^{-2A_k-1}, 2^{-2A_k+1}) \\ 0, & \text{otherwise} \end{cases}.$$

Let

$$g_k(x) := \varphi_k(x)\operatorname{sgn}\left(D_{m_{A_k}}(x)\right), \qquad g_k(x+l) = g_k(x), \quad l = 0, \pm 1, \pm 2, \ldots,$$

$$h_k(x) := \psi_k(x)\operatorname{sgn}\left(D_{m_{A_k}}(x)\right), \qquad h_k(x+l) = h_k(x), \quad l = 0, \pm 1, \pm 2, \ldots$$

Consider the function f defined by

$$f(x_1, ..., x_d) := \sum_{k=1}^{\infty} f_k(x_1, ..., x_d), \qquad f(0, ..., 0) = 0, \tag{25}$$

where

$$f_k(x_1, ..., x_d) := \frac{g_k(x_1)}{A_k} \prod_{j=2}^{d} h_k(x_j).$$

First, we prove that $f \in BV_H$. We consider several cases:

a) If $\alpha := \{\alpha_1, ..., \alpha_p\} \subset D\backslash\{1\}$ then by the construction of f we can write

$$V_H^{\alpha}(f) \leq \frac{c}{A_k} \sum_{i_{\alpha_1}, ..., i_{\alpha_p}} \frac{\left|h_k\left(I_{i_{\alpha_1}}^{\alpha_1}, ..., I_{i_{\alpha_p}}^{\alpha_p}\right)\right|}{i_{\alpha_1}\cdots i_{\alpha_p}} \leq c < \infty, \qquad k = 1, 2, ... \tag{26}$$

b) If $\alpha := \{1, \alpha_2, ..., \alpha_p\} \subset D$ and $p < d - 1$, then we have

$$V_H^{\alpha}(f) \leq \frac{c}{A_k} \sum_{i_1, i_{\alpha_2}, ..., i_{\alpha_p}} \frac{\left|g_k\left(I_{i_1}^1\right)\right|}{i_1} \frac{\left|h_k\left(I_{i_{\alpha_2}}^{\alpha_2}, ..., I_{i_{\alpha_p}}^{\alpha_p}\right)\right|}{i_{\alpha_2}\cdots i_{\alpha_p}}, \qquad k = 1, 2, ... \tag{27}$$

On the other hand,

$$\sum_{i_1} \frac{\left|g_k\left(I_{i_1}^1\right)\right|}{i_1} \leq cA_k \tag{28}$$

and

$$\sum_{i_{\alpha_2},...,i_{\alpha_p}} \frac{\left|h_k\left(I_{i_{\alpha_2}}^{\alpha_2},...,I_{i_{\alpha_p}}^{\alpha_p}\right)\right|}{i_{\alpha_2}\cdots i_{\alpha_p}} \leq c < \infty. \tag{29}$$

From (27) – (29) we obtain

$$V_H^{\alpha}(f) \leq c < \infty. \tag{30}$$

c) Let $\alpha = D$. Then by the construction of f we get

$$\begin{aligned} V_H^{\alpha}(f) &\leq c\sum_{k=1}^{\infty}\frac{1}{A_k}\sum_{i_1,i_2,...,i_d}\frac{\left|g_k\left(I_{i_1}^1\right)\right|}{i_1}\frac{\left|h_k\left(I_{i_2}^2,...,I_{i_d}^d\right)\right|}{i_2\cdots i_d} \\ &\leq c\sum_{k=1}^{\infty}\frac{1}{A_k}\frac{1}{k^{d-1}}\sum_{i_1=1}^{2^{2A_k-2A_{k-1}}}\frac{1}{i_1} \\ &\leq c\sum_{k=1}^{\infty}\frac{1}{k^{d-1}} < \infty, \quad d > 2. \end{aligned} \tag{31}$$

Combining (26), (30) and (31) we conclude that $f \in BV_H$.

Now, we prove that the d-dimensional cubic partial sums of Walsh-Fourier series of f diverge at the point $(0,...,0)$. By (25) we can write

$$\begin{aligned} & S_{m_{A_k},...,m_{A_k}}f(0,...,0) = S_{m_{A_k},...,m_{A_k}}f_k(0,...,0) \\ & +\sum_{i=1}^{k-1}S_{m_{A_k},...,m_{A_k}}f_i(0,...,0) + \sum_{i=k+1}^{\infty}S_{m_{A_k},...,m_{A_k}}f_i(0,...,0) \\ & = J_1 + J_2 + J_3. \end{aligned} \tag{32}$$

Since

$$\left|S_{m_{A_k},...,m_{A_k}}f_i(0,...,0)\right| \leq \|f_i\|_C\left(\log m_{A_k}\right)^d \leq \frac{cA_k^d}{A_i},$$

by (22) and (24) for J_3 we obtain

$$J_3 \leq cA_k^d\sum_{i=k+1}^{\infty}\frac{1}{A_i} \leq \frac{cA_k^d}{A_{k+1}} = o(1) \quad \text{as} \quad k \to \infty. \tag{33}$$

It is well-known [6] that

$$\begin{aligned} & \left\|S_{m_{A_k},...,m_{A_k}}f_i - f_i\right\|_C \\ \leq \; & c\sum_{\{\alpha_1,...,\alpha_p\}\subset D}\omega_{\alpha_1,...,\alpha_p}\left(\frac{1}{2^{2A_k}},...,\frac{1}{2^{2A_k}};f_i\right)_C A_k^p. \end{aligned}$$

On the other hand,

$$\omega_{\alpha_1,\dots,\alpha_p}\left(\frac{1}{2^{2A_k}},\dots,\frac{1}{2^{2A_k}};f_i\right)_C \le c\left(\frac{2^{2A_i}}{2^{2A_k}}\right)^p.$$

Consequently, taking into account (23) and the equality $f_i(0,\dots,0)=0$, we obtain

$$\begin{aligned}(34)\qquad J_2 &\le \sum_{i=1}^{k-1}\left|S_{m_{A_k},\dots,m_{A_k}}f_i(0,\dots,0)\right| \\ &\le \frac{cA_k}{2^{2A_k}}\sum_{i=1}^{k-1}2^{2dA_i} \le \frac{cA_k 2^{2dA_{k-1}}}{2^{2A_k}} = o(1), \quad \text{as} \quad k\to\infty.\end{aligned}$$

Finally, by (11) we have

$$\begin{aligned}(35)\qquad &\left|S_{m_{A_k},\dots,m_{A_k}}f_k(0,\dots,0)\right| \\ &= \frac{1}{A_k}\left|\int_{I^d} f_k(x_1,\dots,x_d)\prod_{j=1}^d D_{m_{A_k}}(x_j)\,dx_1\cdots dx_d\right| \\ &= \frac{1}{A_k}\left|\int_I g_k(x_1)\,D_{m_{A_k}}(x_1)\,dx_1\right| \\ &\qquad\times\left|\int_{I^{d-1}}\prod_{j=2}^d h_k(x_j)\,D_{m_{A_k}}(x_j)\,dx_2\cdots dx_d\right| \\ &= \frac{1}{A_k}\int_I \varphi_k(x_1)\left|D_{m_{A_k}}(x_1)\right|dx_1 \\ &\qquad\times\prod_{j=2}^d\int_I \psi_k(x_j)\left|D_{m_{A_k}}(x_j)\right|dx_j \\ &= \frac{1}{A_k}\sum_{j=0}^{2^{2A_k-2A_{k-1}-1}}\int_{j\cdot 2^{-2A_k}}^{(j+1)2^{-2A_k}}\varphi_k(x_1)\left|D_{m_{A_k}}(x_1)\right|dx_1 \\ &\qquad\times\prod_{j=2}^d\int_{2^{-2A_k}}^{2^{-2A_k+1}}\psi_k(x_j)\left|D_{m_{A_k}}(x_j)\right|dx_j\end{aligned}$$

$$\geq \frac{1}{16A_k} \sum_{j=0}^{2^{2A_k-2A_{k-1}-1}} \int_{j\cdot 2^{-2A_k}}^{(j+1)2^{-2A_k}} \frac{\varphi_k(x_1)}{x_1} dx_1 \prod_{j=2}^{d} \int_{2^{-2A_k}}^{2^{-2A_k+1}} \frac{\psi_k(x_j)}{x_j} dx_j$$

$$\geq \frac{1}{16A_k} \sum_{j=0}^{2^{2A_k-2A_{k-1}-1}} \frac{2^{2A_k}}{j+1} \int_{j\cdot 2^{-2A_k}}^{(j+1)2^{-2A_k}} \varphi_k(x_1)\, dx_1$$

$$\times \left(2^{2A_k-1}\right)^{d-1} \prod_{j=2}^{d} \int_{2^{-2A_k}}^{2^{-2A_k+1}} \psi_k(x_j)\, dx_j$$

$$= \frac{1}{16A_k} \sum_{j=0}^{2^{2A_k-2A_{k-1}-1}} \frac{2^{2A_k}}{j+1} \frac{2^{(2A_k-1)(d-1)}}{2^{2A_k+1}} \left(\frac{1}{2^{2A_k+1}}\right)^{d-1} \geq c > 0.$$

Combining (32)-(35) completes the proof of Theorem 3. □

Proof of Theorem 4. Part a) immediately follows from Theorem 1, Theorem 2 and Theorem A.

To prove part b) we denote

$$A_{i_1,\ldots,i_d} := \left[\frac{i_1}{2^{2N}}, \frac{i_1+1}{2^{2N}}\right) \times \cdots \times \left[\frac{i_d}{2^{2N}}, \frac{i_d+1}{2^{2N}}\right),$$

$$W := \left\{(i_1,\ldots,i_d) : i_d < i_s < i_d + m_{i_d},\ 1 \leq s < d,\ 1 \leq i_d \leq N_\delta\right\},$$

$$N_\delta = \left[4^{(N-1)/(\delta+1)}\right], \qquad t_j := \left(\sum_{i=1}^{s_j} \frac{1}{\lambda_i}\right)^{-1}, \qquad s_j := \left[j^{1+\delta}\right],$$

where $[x]$ is the integer part of x.

It is not hard to see, that for any sequence $\Lambda = \{\lambda_n\}$ satisfying (1) the class $C(I^d) \cap PBV_\Lambda(I^d)$ is a Banach space with the norm

$$\|f\|_{PBV_\Lambda} := \|f\|_C + PV_\Lambda(f).$$

For $N \in \mathbb{N}$ consider the following function

$$f_N(x_1,\ldots,x_d) := \sum_{(i_1,\ldots,i_d)\in W} t_{i_d} 1_{A_{i_1,\ldots,i_d}}(x_1,\ldots,x_d) \prod_{s=1}^{d} \xi_N(x_s)\, \mathrm{sgn}\left(D_{m_N}(x_s)\right),$$

where $1_A(x_1,\ldots,x_d)$ is the characteristic function of the set $A \subset T^d$, m_a is defined by (12) and

$$\xi_N(x) := \begin{cases} 1, \text{ if } x = (2j+1)\,2^{-(2N+1)},\ j = 1,2,...,2^{2N}-1 \\ 0, \text{ if } x \in [0, 2^{-2N}),\ x = j \cdot 2^{-2N},\ j = 1,2,...,2^{2N} \\ \text{linear and continuous on } [j2^{-(2N+1)}, (j+1)2^{-(2N+1)}],\ j = 2,3\ldots,2^{2N} \end{cases},$$

$$\xi_N(x+l)=\xi_N(x), \qquad l=\pm 1,\pm 2,....$$

First we show that the norms $\|f_N\|_{PBV_\Lambda}$ are uniformly bounded.

Let $(i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_d)$ be fixed, where $k=1,\ldots,d-1$. Then it is easy to show that

$$V_\Lambda^{\{k\}}(f_N)\leq C\cdot t_{i_d}\left(\sum_{i_k=i_d+1}^{i_d+m_{i_d}}\frac{1}{\lambda_{i_k-i_d}}\right)\leq C\cdot t_{i_d}\left(\sum_{i_k=1}^{m_{i_d}}\frac{1}{\lambda_{i_k}}\right)\leq C<\infty.$$

If $(i_1,\ldots,i_{d-1})$ is fixed, the condition $(i_1,\ldots,i_d)\in W$ implies

$$\max\{i_d(i_s):1\leq s\leq d-1\}<i_d<\min\{i_s:1\leq s\leq d-1\},$$

where

$$i_d(i_s):=\min\{i_d:i_d+m_{i_d}>i_s\}.$$

Consequently, by the definition of the function f_N we obtain that for any $s=1,...,d-1$

$$\begin{aligned}V_\Lambda^{\{d\}}(f_N) &\leq C\sum_{i_d=i_d(i_s)+1}^{i_s}\frac{t_{i_d}}{\lambda_{i_d-i_d(i_s)}}\\ &\leq C\cdot t_{i_d(i_s)}\sum_{i_d=i_d(i_s)+1}^{i_s}\frac{1}{\lambda_{i_d-i_d(i_s)}}\\ &= C\cdot t_{i_d(i_s)}\sum_{i_d=1}^{i_s-i_d(i_s)}\frac{1}{\lambda_{i_d}}\leq C\cdot t_{i_d(i_s)}\sum_{i_d=1}^{m_{i_d(i_s)}}\frac{1}{\lambda_{i_d}}=C<\infty.\end{aligned}$$

Hence $f_N\in PBV_\Lambda$ and

$$\|f_N\|_{PV_\Lambda}\leq C,\quad N=1,2,\ldots. \tag{36}$$

Observe, that by (16) we have

$$\frac{1}{t_j}=\sum_{i=1}^{m_j}\frac{1}{\lambda_i}=\sum_{i=1}^{m_j}\frac{1}{i}\cdot\frac{i}{\lambda_i}\leq C\frac{m_j}{\lambda_{m_j}}\log m_j\leq C\frac{j\log j}{\lambda_j}.$$

Hence

$$t_j\log j\geq c\frac{\lambda_j}{j}.$$

Consequently,

$$
\begin{aligned}
(37)\qquad & S_{m_N,\cdots,m_N} q_N\left(0,\cdots,0\right) \\
= & \int\limits_{I^d} q_N\left(x_1,\cdots,x_d\right)\prod_{s=1}^{d} D_{m_N}\left(x_s\right)dx_1\cdots dx_d \\
= & \sum_{(i_1,\cdots,i_d)\in W} t_{i_d} \int\limits_{A_{i_1,\cdots,i_d}} \prod_{s=1}^{d}\left|\xi N(x_s) D_{m_N}\left(x_s\right)\right| dx_1\cdots dx_d \\
\geq & \; c \sum_{(i_1,\cdots,i_d)\in W} t_{i_d} \int\limits_{A_{i_1,\cdots,i_d}} \frac{dx_1\cdots dx_d}{x_1\cdots x_d} \geq c \sum_{(i_1,\cdots,i_d)\in W} t_{i_d}\frac{1}{i_1\cdots i_d} \\
\geq & \; c\sum_{i_d=1}^{N_\delta}\frac{t_{i_d}}{i_d}\sum_{i_1=i_d}^{i_d+m_{i_d}}\cdots\sum_{i_{d-1}=i_d}^{i_d+m_{i_d}}\frac{1}{i_1\cdots i_{d-1}} \\
\geq & \; c\sum_{i_d=1}^{N_\delta}\frac{t_{i_d}}{i_d}\log^{d-1}\left(\frac{i_d+m_{i_d}}{i_d}\right)\geq c\left(\delta\right)\sum_{i_d=1}^{N_\delta}\frac{t_{i_d}\log i_d}{i_d}\log^{d-2} i_d \\
\geq & \; c\left(\delta\right)\sum_{n=1}^{N_\delta}\frac{\lambda_n\log^{d-2}n}{n^2}\to\infty,
\end{aligned}
$$

as $N\to\infty$, according to (17).

Applying the Banach-Steinhaus theorem, from (36) and (37) we obtain that there exists a continuous function $f\in PBV_\Lambda(I^d)$ such that

$$\sup_N\left|S_{N,\cdots,N}f(0,\cdots,0)\right|=\infty.$$

□

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