ON LEBESGUE CONSTANTS OF VILENKIN SYSTEMS

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In the paper some properties of Lebesgue constants \( \{ L_n(W) \}_{n=1}^{\infty} \) of Vilenkin system are investigated. Non almost convergence property for the sequence \( \{ L_n(W) \}_{n=2}^{\infty} \log_2 n \) is obtained.

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Introduction. Let \( \Psi = \{ \psi_k(x) \}_{k=1}^{\infty} \) be an orthonormal system of functions defined on \([a, b]\). The Lebesgue constants of \( \Psi \) is defined as follows:

\[
L_n(\Psi, x) := \int_a^b \left| \sum_{k=1}^{n} \psi_k(x) \bar{\psi}_k(t) \right| dt \quad \text{(here } \bar{x} \text{ means the complex conjugate of } x). \]

If this functions are independent on \( x \), then they are called Lebesgue constants \( \{ L_n(\Psi) \}_{n=1}^{\infty} \) of system \( \Psi \). Recall the definition of Vilenkin systems. Consider an arbitrary sequence of natural numbers \( P = \{ p_1, p_2, \ldots, p_k, \ldots \} \), where \( p_j \geq 2 \) for all \( j \in \mathbb{N} \).

Let denote \( m_0 = 1 \), \( m_k = \prod_{j=1}^{k} p_j \) \( (p_j \geq 2) \).

It is easy to see that for each \( x \in [0, 1) \) and for each \( n \in \mathbb{N} \) there exist integers \( x_j, \alpha_j \in \{ 0, 1, \ldots, p_j - 1 \} \) \( (\text{in the case } x = \frac{l}{m_k}, l \in \mathbb{N}, 0 \leq l \leq m_k - 1, \text{we take } x_j = 0 \) for all \( j > k \) \), so that \( n = \sum_{j=1}^{\infty} \alpha_j m_{j-1} \) and \( x = \sum_{j=1}^{\infty} \frac{x_j}{m_j} \) \( (P\text{-adic expansions}) \).

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Vilenkin or multiplicative system with respect to the sequence \( P \) is defined as follows:

\[
W_0(x) \equiv 1; \quad W_n(x) = \exp \left( 2\pi i \sum_{j=1}^{k} \alpha_j \frac{x_j}{p_j} \right).
\]

Obviously the \( n \)th function can be represented by

\[
W_n(x) = \prod_{j=1}^{k} (W_{m_j-1}(x))^{\alpha_j}.
\]

Note that systems corresponding to distinct sequences \( P = \{p_k\} \) are different, in particular if \( P \equiv \{2, 2, \ldots, 2, \ldots\} \) the Vilenkin system coincides with the Walsh one (see \([1]\)). The theory of these systems was developed by N. Ya. Vilenkin in 1946 (see \([2, 3]\)).

The Lebesgue constants of Vilenkin system have the form

\[
L_n(W) = \int_0^1 |D_n(t)| \, dt, \tag{1}
\]

where \( D_n(t) = \sum_{k=0}^{n-1} W_k(t) \) is the \( n \)th Dirichlet kernel of Vilenkin system.

It is known \([4]\) that

\[
\lim_{n \to \infty} \frac{L_n(T)}{\log_2 n} = \frac{4}{\pi} \quad \text{where} \quad T \quad \text{is the trygonometric system.}
\]

Note that, in contrast to this, for the Vilenkin system it was proved \([3]\) that

\[
0 = \liminf_{n \to \infty} \frac{L_n(W)}{\log_2 n} < \limsup_{n \to \infty} \frac{L_n(W)}{\log_2 n} < \infty. \tag{2}
\]

Recall that a bounded sequence \( \{x_n\}_{n=1}^{\infty} \) is called almost convergent, if for some \( a \in \mathbb{R} \) we have \( \lim_{n \to \infty} \frac{1}{n} \sum_{k=m+1}^{m+n} x_k = a \) uniformly by \( m \). Denote

\[
q(x_n) = \lim \inf_{n \to \infty} \frac{1}{n} \sum_{k=m+1}^{m+n} x_k \quad \text{and} \quad p(x_n) = \lim \sup_{n \to \infty} \frac{1}{n} \sum_{k=m+1}^{m+n} x_k.
\]

This limits exist for every bounded sequence, and obviously the almost convergence of a sequence \( \{x_n\}_{n=1}^{\infty} \) is equivalent to the condition \( q(x_n) = p(x_n) \) (see \([5, 6]\)).

In this paper we prove the following

**Theorem 1.** For any Vilenkin system the following equivalencies are true:

1. \( q \left( \frac{L_n(W)}{\log_2 n} \right) = 0 \left( = \liminf_{n \to \infty} \frac{L_n(W)}{\log_2 n} \right) \);

2. \( p \left( \frac{L_n(W)}{\log_2 n} \right) = \limsup_{n \to \infty} \frac{L_n(W)}{\log_2 n} < \infty \).

From this Theorem as a direct consequence we obtain that the sequence \( \left\{ \frac{L_n(W)}{\log_2 n} \right\}_{n=2}^{\infty} \) is not almost convergent. Note that the analogues result for the Walsh system is formulated in \([6]\).
Auxiliary Propositions. We will use the following properties of Vilenkin system
\[ L_m(W) = 1, \ k = 0, 1, \ldots, \] (3)
\[ W_{l_0 + \beta}(x) = W_{l_0}(x)W_{\beta}(x) \text{ if } \beta < m_k \ (k, l, \beta \in \mathbb{N}). \] (4)

From (4) we get
\[ D_{m+\varepsilon}(t) \equiv \sum_{j=0}^{m-1} W_j(t) + W_{m}(t) \sum_{j=0}^{r-1} W_j(t) \equiv \] (5)
\[ \equiv D_m(t) + W_m(t)D_r(t) \text{ for all } 1 < r \leq m_k. \]

Proof of Main Result. Let us begin with a proof of first equation.

We put
\[ l_n := \frac{L_n(W)}{\log_2 n}, \ \bar{l}_n = \inf_{m \in \mathbb{N}} \frac{1}{n} \sum_{k=m+1}^{m+n} l_k \text{ and } \hat{l}_n = \sup_{m \in \mathbb{N}} \frac{1}{n} \sum_{k=m+1}^{m+n} l_k. \]

Let \( \varepsilon > 0 \) be an arbitrary positive number and \( n \in \mathbb{N} \). Obviously there exists \( k \in \mathbb{N} \) depending on \( n \) such that
\[ m_k > n \text{ and } \frac{1}{k} \left( \frac{1}{n} \sum_{r=1}^{n} (1 + L_r) \right) < \varepsilon. \] (6)

From (1), (3) and (5) we obtain
\[ L_{m+\varepsilon} \leq 1 + L_r \text{ for all } 1 \leq r \leq n. \]

Combining this with (6) and taking into consideration a relation \( m_k \geq 2^k \), we get
\[ \frac{1}{n} \sum_{j=m+1}^{m+n} l_j \leq \frac{1}{k} \left( \frac{1}{n} \sum_{r=1}^{n} (1 + L_r) \right) < \varepsilon. \]

Therefore we have \( 0 \leq \bar{l}_n < \varepsilon \) for any natural number \( n \), and eventually since \( \varepsilon \) is arbitrary we get the first equation \( q(l_n) = 0 \).

Next we prove that \( \lim_{r \to \infty} \bar{l}_r \geq c \). We put
\[ c = \limsup_{n \to \infty} l_n, \ c_1 = \sup_{n \in \mathbb{N}} l_n. \] (7)

Let \( r \) be any natural number and \( \varepsilon > 0 \). We fix \( k_0 \in \mathbb{N} \) such that \( m_{k_0} \geq r \), then we take \( n_0 \in \mathbb{N} \ (n_0 > m_{k_0}) \), so that
\[ l_{n_0} > c - \varepsilon \frac{\log_2 m_{k_0}}{\log_2 (n_0 - m_{k_0})} < \varepsilon \frac{3}{6c_1} \text{ and } \frac{\log_2 n_0}{\log_2 (n_0 + m_{k_0})} > 1 - \varepsilon \frac{3}{3(c+1)}. \] (8)

The \( p \)-adic expansion of \( n_0 \) has the form \( n_0 = \sum_{j=0}^{k} \alpha_j m_j \). Denote
\[ n_0' := \sum_{j=k_0}^{k} \alpha_j m_j. \] (9)

Let \( n \in [n_0', n_0' + m_{k_0}) \). By the same argument as in (5) we get
\[ D_n(t) \equiv D_{n_0'}(t) + W_{n_0'}(t)D_{n-n_0'}(t). \]
Thus, from (1) and reverse triangle inequality, we obtain
\[ L_n \geq L_{n_0} - L_{n-n_0} \quad \text{and} \quad L_{n_0}' \geq L_{n_0} - L_{n-n_0}'. \]

Hence (see also (7)),
\[ L_n \geq L_{n_0} - 2L_{n-n_0}' \geq L_{n_0} - 2c_1 \log_2 m_{k_0}. \]

From this and (8), (9) we have
\[ l_n \geq \frac{L_{n_0}}{\log_2 n_0} \left( \frac{\log_2 n_0}{\log_2 n} \right) - 2c_1 \log_2 m_{k_0} > c - \varepsilon. \quad (10) \]

From (10) we get
\[ 1_r m + \sum_{k=m+1}^{m+r} l_k < c + \varepsilon \quad \text{for all } r \geq k_0', \]

which implies \( \hat{l}_r \geq c - \varepsilon \), and from arbitrariness of \( r \) and \( \varepsilon \) we obtain that \( \lim_{r \to \infty} \hat{l}_r \geq c. \)

It remains to show that \( \lim_{r \to \infty} \hat{l}_r \leq c. \)

Again we take \( \varepsilon > 0 \) to be an arbitrary number, then we choose natural numbers \( k_0 \) and \( k_0' > k_0 \) such that
\[ l_k < c + \frac{\varepsilon}{2} \quad \text{for all } k \geq k_0 \quad \text{and} \quad \frac{1}{k_0} \sum_{k=1}^{k_0} l_k < \frac{\varepsilon}{2}. \quad (11) \]

If \( m < k_0 \) or all \( r \geq k_0' \), we have
\[ \frac{1}{r} \sum_{k=m+1}^{m+r} l_k = \frac{1}{r} \sum_{k=m+1}^{k_0} l_k + \frac{1}{r} \sum_{k=k_0+1}^{m+r} l_k. \quad (12) \]

From (11) and (12) we get
\[ \frac{1}{r} \sum_{k=m+1}^{m+r} l_k \leq c + \varepsilon \quad \text{for all } r \geq k_0'. \]

If \( m \geq k_0 \), then from (11) for all \( r \in \mathbb{N} \) we obtain
\[ \frac{1}{r} \sum_{k=m+1}^{m+r} l_k \leq c + \frac{\varepsilon}{2}. \]

Hence \( \hat{l}_r \leq c + \varepsilon \) for all \( r \geq k_0' \) and eventually we get \( \lim_{r \to \infty} \hat{l}_r \leq c. \)

\[ \square \]

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