A characterization of hyperidentities of the variety of weakly idempotent lattices

Yu.M. Movsisyan
Yerevan State University
Yerevan, Armenia
e-mail: yurimovsisyan@yahoo.com

D.S.Davidova
European Regional Academy
Yerevan, Armenia
e-mail: di.davidova@yandex.ru

ABSTRACT
A hyperidentity is a universal second order formula, i.e. universal formula from the second order language (in the sense of A.Church and A.I.Mal'tsev). The hyperidentities of the variety of weakly idempotent lattices are characterized in this paper. The existence of a finite base of hyperidentities for this variety is proved as a consequence.

Keywords
Second order formula, hyperidentity, hyperequational theory, consequence, lattice, weakly idempotent lattice, variety, nilpotent shift.

1. INTRODUCTION AND PRELIMINARIES

For the second order formulae (and the second order languages) see [1, 2, 3, 4, 5].

Let us recall that a hyperidentity is a second-order formula of the following type:

$$\forall X_1, \dots, X_m \forall x_1, \dots, x_n (w_1 = w_2),$$

where $X_1, \dots, X_m$ are functional variables, and $x_1, \dots, x_n$ are object variables in the words (terms) of w_1, w_2. Hyperidentities are usually written without the quantifiers: $w_1 = w_2$. We say that in the algebra $(Q; F)$ the hyperidentity $w_1 = w_2$ is satisfied if this equality is valid when every object variable and every functional variable in it is replaced by any element from Q and by any operation of the corresponding arity from F (supposing the possibility of such replacement) [6, 7, 8, 9, 10, 11, 12, 13].

The class (variety) V of algebras is said to satisfy a certain hyperidentity, if this hyperidentity is satisfied in any algebra of this class (variety). In this case the hyperidentity is called a hyperidentity of the class (variety) V. The set of all hyperidentities of the given class (variety) V is called $\forall(\forall)$-theory or hyperequational theory of the class (variety) V.

Characterizations of hyperidentities of varieties of lattices, modular lattices, distributive lattices, Boolean and De Morgan algebras were given in [8, 14, 15, 16].

Theorem 1. *The variety of lattices has a finite base of hyperidentities. Namely the variety of lattices satisfies the following hyperidentities:*

$$X(x, x) = x, \tag{1}$$

$$X(x, y) = X(y, x), \tag{2}$$

$$X(x, X(y, z)) = X(X(x, y), z), \tag{3}$$

$$X(Y(X(x, y), z), Y(y, z)) = Y(X(x, y), z). \tag{4}$$

And conversely, every hyperidentity of the variety of lattices is a consequence of the hyperidentities: (1), (2), (3), (4).

Theorem 2. *The variety of modular lattices has a finite base of hyperidentities. Namely the variety of modular lattices satisfies the following hyperidentities: (1), (2), (3), (4) and the hyperidentity of modularity*

$$\begin{aligned} &X(Y(x, X(y, z)), Y(y, z)) = \\ &= Y(X(x, Y(y, z)), X(y, z)). \end{aligned} \tag{5}$$

And, conversely, every hyperidentity of the variety of modular lattices is a consequence of the hyperidentities: (1), (2), (3), (4), (5).

Theorem 3. *The variety of distributive lattices has a finite base of hyperidentities. Namely the variety of distributive lattices satisfies the following hyperidentities: (1), (2), (3) and the hyperidentity of distributivity*

$$X(x, Y(y, z)) = Y(X(x, y), X(x, z)). \tag{6}$$

And, conversely, every hyperidentity of the variety of distributive lattices is a consequence of the hyperidentities: (1), (2), (3), (6).

Theorem 4. *The variety of Boolean algebras satisfies the following hyperidentities: (1), (2), (3), (6) and*

$$F(F(x)) = x, \tag{7}$$

$$X(F(x), y) = X(F(X(x, y)), y), \tag{8}$$

$$F(X(F(X(x, y)), F(X(x, F(y))))) = x. \tag{9}$$

And, conversely, every hyperidentity of the variety of Boolean algebras is a consequence of the hyperidentities: (1), (2), (3), (6), (7), (8), (9).

All hyperidentities of the variety of Boolean algebras are consequences of one of its hyperidentities, i.e. the hyperequational theory of the variety of Boolean algebras is one-based.

The integrating problem of several circuits into a single hypercircuit, using the lattice hyperidentities, was considered in [17].

The algebra $Q(+,\cdot,')$ with two binary and one unary operations is called a De Morgan algebra if $Q(+,\cdot)$ is a distributive lattice and $Q(+,\cdot,')$ satisfies the following identities:

$$(x+y)' = x' \cdot y',$$

$$x'' = x,$$

where $x'' = (x')'$. De Morgan algebras (and De Morgan bisemilattices) have applications in multi-valued simulations of digital circuits ([18, 19]).

Theorem 5. *The variety of De Morgan algebras has a finite base of hyperidentities. Namely the variety of De Morgan algebras satisfies the following hyperidentities: (1), (2), (3), (6), (7) and*

$$X(x) = Y(x), \tag{10}$$

$$F(\overline{F}(x,y),\overline{x}) = \overline{x}, \tag{11}$$

$$\overline{G}(\overline{F}(x_1,x_2)x_3) = F(\overline{G}(\overline{x_1},x_3),\overline{G}(\overline{x_2},x_3)), \tag{12}$$

$$F(x,F(y,z)) = F(G(x,G(y,z)),\overline{G}(\overline{x},G(\overline{y},\overline{z}))), \tag{13}$$

$$\begin{gathered} F(\overline{F}(G(x,e),G(x,G(y,e))),\overline{F}(G(z,e),G(z,\\ G(t,e)))) = F(\overline{F}(G(x,G(z,e)),G(x,G(y,G(z,\\ G(u,e))))),\overline{F}(\overline{G}(\overline{G}(x,G(y,e)),\overline{G}(z,G(t,e))),e)). \end{gathered} \tag{14}$$

And conversely, every hyperidentity of the variety of De Morgan algebras is a consequence of the hyperidentities: (1), (2), (3), (6), (7), (10),(11), (12), (13), (14).

For hyperidentities in term (polynomial) algebras, see [20, 21, 22, 23, 14, 24, 25]. Varieties of varieties are characterized by hyperidentities ([26]).

The concept of hyperidentity gives rise to new concepts of discrete mathematics, computational models, which also are applicable in quantum computation, quantum information theory, quantum artificial intelligence, quantum logic and the theory of quantum computers [27, 28].

2. MAIN RESULT

There exist various extensions of the concept of lattice. For example, in works [29, 30] weakly associative lattices were introduced. In [31] an algebra with a system of identities was introduced which we call weakly idempotent lattices (see also [32, 33]). The variety of weakly idempotent lattices is a nilpotent shift of the variety of lattices.

In this talk we characterize hyperidentities of the variety of weakly idempotent latices.

Definition. The algebra $(L;\wedge,\vee)$ with two binary operations is called weakly idempotent lattice, if it satisfies the following identities:

$a\wedge b = b\wedge a, a\vee b = b\vee a;$ (commutativity),
$(a\wedge b)\wedge c = a\wedge(b\wedge c), (a\vee b)\vee c = a\vee(b\vee c);$ (associativity)
$a\wedge(b\wedge b) = a\wedge b, a\vee(b\vee b) = a\vee b;$ (weakly idempotency)
$a\wedge(a\vee b) = a\wedge a, a\vee(a\wedge b) = a\vee a;$ (weakly absorption)
$a\wedge a = a\vee a.$ (equalization)

Adding the identities of idempotency $a\wedge a = a$ and $a\vee a = a$ we obtain a lattice. Each weakly idempotent lattice corresponds to the quasiorder $\leq$, defined in the following way:

$$a \leq b \iff a\wedge b = a\wedge a \iff a\vee b = b\vee b.$$

Every weakly idempotent lattice $L = (L;\wedge,\vee)$ is interlaced, i.e. the operations of the weakly idempotent lattice preserve the corresponding quasiorder.

Theorem 6. *The variety of weakly idempotent lattices has a finite base of hyperidentities. Namely, the variety of weakly idempotent lattices satisfies the following hyperidentities: (2), (3), (4) and*

$$X(x,x) = Y(x,x), \tag{15}$$

$$X(x,X(y,y)) = X(x,y), \tag{16}$$

And, conversely, each hyperidentity of the variety of weakly idempotent lattices is a consequence of the hyperidentities (2), (3), (4), (15), (16).

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