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On an Approach to the Problems of Control of Dynamic Systems with Nonseparated Multipoint Intermediate Conditions

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Abstract—Consideration was given to the problems of control of the linear dynamic systems with given initial, final, and nonseparated (nonlocal) multipoint intermediate conditions and the problems of optimal control with the performance index defined over the entire time interval. An explicit form of the control action was constructed, and a method to solve the problem of optimal control was proposed. Solutions of particular problems were presented.

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1. INTRODUCTION

The study of numerous applied control problems generates a need for considering the multipoint boundary problems for the ordinary differential equations. These problems are distinguished for the intermediate conditions at some points of the interval under consideration. Attention of the researchers was attracted by very interesting boundary control problems where the nonseparated (nonlocal) multipoint intermediate conditions are also given along with the classical boundary (initial and final) conditions [1–5]. The nonseparatedness of the multipoint conditions is due, in particular, to the practical impossibility to carry out instantaneous measurements of the object state parameters in large or at individual points. Problems of this kind arise in the control of motion of the mechanical systems, flight vehicles, manipulator robots, technological processes, and so on [1–6]. They are of significance for theoretical and applied issues, and it is only natural that a need arises for studying them in various formulations. In particular, a numerical method to solve systems of linear ordinary differential equations with nonseparated multipoint and integral conditions was proposed in [1].

The present paper sets forth a constructive approach to the problems of control and optimal control of the linear dynamic systems with nonlocal multipoint intermediate conditions.

2. FORMULATION OF THE PROBLEM

Let us consider a control process obeying a system of linear differential equations

$$\dot{x} = A(t)x + B(t)u + f(t), \tag{1}$$

where $x \in R^n$ is the phase vector, $A(t)$ and $B(t)$ are, respectively, $(n \times n)$ and $(n \times r)$ matrices whose elements are bounded functions for $t_0 \leqslant t \leqslant T$, t_0 and T being the given time instants, $u(t)$ is the r-dimensional column vector of the control actions with the components regarded as measurable bounded functions, and $f(t)$ is an n-dimensional vector of external actions (maybe a measurable bounded function).

Let given be the initial,

$$x(t_0) = x_0, \tag{2}$$

and final,

$$x(T) = x_T, \tag{3}$$

states of system (1), and at some fixed intermediate time instants

$$0 \leqslant t_0 < t_1 < \ldots < t_m < t_{m+1} = T$$

the nonseparated (nonlocal) multipoint intermediate conditions

$$\sum_{k=1}^{m} F_k x(t_k) = \alpha, \tag{4}$$

be given. Here, α is a q-dimensional ($q \leqslant n$) column vector, and F_k are $(q \times n)$ matrices ($k = 1, \ldots, m$) whose elements are real numbers.

Generally speaking, for some applied problems such as control of manipulation robots, flight vehicles, and so on one can assume that at the intermediate time instants t_k $(k = 1, \ldots, m)$ condition (4) is satisfied not by all coordinates of the phase vector $x(t_k)$ but only by some of them. In such cases, we assume that the corresponding elements of the matrix F_k are zeros.

We assume that under the phase constraints (2)–(4) system (1) is fully controllable over the time interval $[t_0, T]$ [4–6], which means that over this interval one can select the control action $u(t)$ and corresponding motion $x(t) = x(t, u(t))$ satisfying system (1) and conditions (2)–(4).

We consider the following problems.

Problem 1. Needed is to determine the conditions under which there exists a program control action $u = u(t)$ and program motion $x = x(t)$ satisfying system (1) and conditions (2)–(4), as well as to construct the control action and motion.

Let a performance index æ$[u]$ having the sense of the norm of some normalized space be given for selection of the optimal decisions over the time interval $[t_0, T]$. The problem of optimal control is formulated as follows.

Problem 2. Needed is to determine the optimal control action $u^0(t)$, $t \in [t_0, T]$, driving system (1) from the initial state (2) to the final state (3) meeting condition (4) and having the least possible value of the performance index æ$[u^0]$.

3. SOLUTION OF THE PROBLEMS

Solution of Eq. (1) is set down as follows:

$$x(t) = X[t, t_0]x(t_0) + \int_{t_0}^{t} X[t, \tau]B(\tau)u(\tau)d\tau + \int_{t_0}^{t} X[t, \tau]f(\tau)d\tau, \tag{5}$$

where $X[t, \tau]$ denotes the normalized fundamental matrix of the solution of the uniform part of Eq. (1).

Assuming that the desired control actions are known, the integral relations

$$\sum_{k=1}^{m} F_k X[t_k, t_0]x(t_0) + \sum_{k=1}^{m} \int_{t_0}^{t_k} F_k X[t_k, \tau]B(\tau)u(\tau)d\tau + \sum_{k=1}^{m} \int_{t_0}^{t_k} F_k X[t_k, \tau]f(\tau)d\tau = \alpha \tag{6}$$

are established from (5) by substituting the values of $x(t_k)$ in (4) for the time instants $t = t_k$ ($k = 1, \ldots, m$), and for the final time instant $t = T$ according to (3) we establish from (5) that

$$x(T) = X[T, t_0]x(t_0) + \int_{t_0}^{T} X[T, \tau]B(\tau)u(\tau)d\tau + \int_{t_0}^{T} X[T, \tau]f(\tau)d\tau. \quad (7)$$

By denoting

$$F_k(t) = \begin{cases} F_k X[t_k, t] & \text{for } t_0 \leqslant t \leqslant t_k \\ 0 & \text{for } t_k < t \leqslant t_{m+1} = T \quad (k = 1, \ldots, m+1), \end{cases} \quad (8)$$

Eq. (6) is representable as

$$\int_{t_0}^{T} F(t)B(t)u(t)dt = \alpha - Fx(t_0) - \int_{t_0}^{T} F(t)f(t)dt, \quad (9)$$

where

$$F(t) = \sum_{k=1}^{m} F_k[t] = \sum_{k=1}^{m} F_k X[t_k, t], \quad F = \sum_{k=1}^{m} F_k X[t_k, t_0] = F(t_0), \quad (10)$$

the matrices $F(t)$ and F having dimension $(q \times r)$.

We introduce the $((q + n) \times r)$ block matrix

$$H[t] = \begin{pmatrix} F(t) \\ X[T, t] \end{pmatrix} B(t), \quad (11)$$

with its use the integral relations (7) and (9) are representable as

$$\int_{t_0}^{T} H[t]u(t)dt = \eta(t_0, \ldots, T), \quad (12)$$

where η is a $(q + n)$-dimensional block column vector

$$\eta(t_0, \ldots, T) = \begin{pmatrix} \alpha - Fx(t_0) - \int_{t_0}^{T} F(t)f(t)dt \\ x(T) - X[T, t_0]x(t_0) - \int_{t_0}^{T} X[T, t]f(t)dt \end{pmatrix}. \quad (13)$$

For any control problem, the question of its solvability which comes down to the analysis of controllability of the dynamic system is a matter of principle. It follows from the integral relation (12) that the dynamic system (1) is quite controllable if and only if for any vector $\eta(t_0, \ldots, T)$ (13) from the space R^{q+n} one can find a control $u(t, \eta) = (u_1(t, \eta), \ldots, u_r(t, \eta))^{\mathrm{T}}$ satisfying condition (12), "T" denoting here and below the operation of transposition.

The condition for complete controllability of the dynamic system (1) can be formulated as follows.

Theorem 1. *For the dynamic system* (1) *to be completely controllable over the interval* $[t_0, T]$, *it is necessary and sufficient that the column vectors of the matrix* $H[t]$ (11) *be linearly independent over this interval.*

We note that the formulation of Theorem 1 is well known for the linear controllable systems and can be found, for example, in [6–8]. The proof follows the same lines as that in [6].

Now, the function $u(t)$ satisfying the integral relation (12) is sought as [7, 8]

$$u(t) = H^{\mathrm{T}}[t]C + V(t), \tag{14}$$

where $H^{\mathrm{T}}[t]$ is the transposed matrix (11), C is a constant vector to be determined, $V(t)$ is some vector function (maybe a measurable bounded function over the time interval $[t_0, T]$) such that

$$\int_{t_0}^{T} H[t]V(t)dt = 0. \tag{15}$$

Equality (15) expresses the condition for orthogonality of the vector functions $V(t)$ to all rows (blocks) of the matrix $H[t]$.

Substitution of (14) in (12) provides

$$Q(t_0, \ldots, T)C = \eta(t_0, \ldots, T), \tag{16}$$

where

$$Q(t_0, \ldots, T) = \int_{t_0}^{T} H[t]H^{\mathrm{T}}[t]dt. \tag{17}$$

Equation (16) is a system of $q + n$ algebraic equations in the unknowns C_j $(j = 1, \ldots, q + n)$. Q is a $(q + n) \times (q + n)$ matrix having solution if $\det Q \neq 0$ or the rank of the matrix Q coincides with the rank of the extended matrix $\{Q, \eta\}$.

Under the assumption that $\det Q \neq 0$, the solution of Eq. (16) is given by

$$C = Q^{-1}\eta. \tag{18}$$

Consequently, we establish from (14) that

$$u(t) = H^{\mathrm{T}}[t]Q^{-1}\eta + V(t). \tag{19}$$

Therefore, solution of Problem 1 can be formulated as a theorem similar to that proved in [6].

Theorem 2. *For existence of the program control* (19) *and its corresponding solution of system* (1) *meeting conditions* (2)–(4)*, it is necessary and sufficient that matrix* (17) *be nonsingular or that the ranks of the matrix Q and $\{Q, \eta\}$ coincide.*

With regard for (8), (10) and (11) under $V(t) = 0$, formula (19) is representable as

$$u(t) = \begin{cases} B^{\mathrm{T}}(t)\left(\left(\sum\limits_{k=1}^{m} F_k X[t_k, t]\right)^{\mathrm{T}}, (X[T,t])^{\mathrm{T}}\right) Q^{-1}\eta \quad \text{for} \quad t \in [t_0, t_1] \\ B^{\mathrm{T}}(t)\left(\left(\sum\limits_{k=2}^{m} F_k X[t_k, t]\right)^{\mathrm{T}}, (X[T,t])^{\mathrm{T}}\right) Q^{-1}\eta \quad \text{for} \quad t \in (t_1, t_2] \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ B^{\mathrm{T}}(t)\left((F_m X[t_m, t])^{\mathrm{T}}, (X[T,t])^{\mathrm{T}}\right) Q^{-1}\eta \quad \text{for} \quad t \in (t_{m-1}, t_m] \\ B^{\mathrm{T}}(t)\left(0,\ (X[T,t])^{\mathrm{T}}\right) Q^{-1}\eta \quad \text{for} \quad t \in (t_m, T]. \end{cases} \tag{20}$$

By substituting (20) in (5), we obtain the program motion of system (1) satisfying conditions (2)–(4) over the time interval $[t_0, T]$.

For solution of Problem 2, we notice that for the given performance index æ$[u]$ the problem of optimal control with integral conditions (12) can be regarded as the variational problem of conditional extremum where it is required to determine the minimum of functional æ$[u]$ under conditions (12). However, as can be seen from notation (8), the subintegral functions in (12) are discontinuous. Therefore, the classical theorems of the variational calculus cannot be used to consider this problem.

The left side of condition (12) is a linear operation generated by the function $u(t)$ over the time interval $[t_0, T]$.

Consequently, if the functional æ$[u]$ is the norm of some linear normalized space, then the solution of Problem 2 must be sought with the use of the problem of moments [4–6]. In this case, the solution of Problem 2 is given by the optimal control action $u^0(t),\ t \in [t_0, T]$, minimizing the functional æ$[u]$ and satisfying condition (12). Therefore, Problem 2 is reduced to that of moments whose solution is known from [6].

4. EXAMPLES

By way of illustrating the solution of Problem 1, we consider the problem of control of a material point moving in the vertical plane under the action of the reactive and gravity forces. Then, its motion is representable by the vector equation

$$m\frac{d\bar{v}}{dt} = \overline{P} + \overline{f}, \tag{21}$$

where $m = m_0 + m_1(t)$, $m_0 = \text{const}$, $m_1(t)$ is the reactive mass of the point, $\overline{f} = \overline{a}\frac{dm_1}{dt}$ is the reactive force, $\bar{a}$ is the vector of relative velocity of the separating particle, and $\overline{P} = m\overline{g}$.

Projection of Eq. (21) on the horizontal and vertical coordinate axes provides

$$m\ddot{\xi} = \dot{m}a_\xi, \quad m\ddot{\zeta} = \dot{m}a_\zeta - mg,$$

where a_ξ and a_ζ are the projections of the vector $\bar{a}$ on the axes ξ and ζ. By admitting that the absolute value of the vector $\bar{a}$ is given and denoting

$$x_1 = \xi, \quad x_2 = \dot{\xi}, \quad x_3 = \zeta, \quad x_4 = \dot{\zeta},$$
$$u_1 = |\bar{a}|\cos\alpha_\xi\frac{\dot{m}}{m}, \quad u_2 = |\bar{a}|\cos\alpha_\zeta\frac{\dot{m}}{m},$$

where α_ξ and α_ζ are the angles between the vector $\bar{a}$ and the axes ξ and ζ, the motion equation is represented in the normal form [6] as

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u_1, \quad \dot{x}_3 = x_4, \quad \dot{x}_4 = u_2 - g. \tag{22}$$

We regard the reactive force, that is, u_1 and u_2, as the control action.

Let $t_0 = 0$ and $T = 3$. The initial and final states of the phase vector $x = (x_1, x_2, x_3, x_4)^T$ are selected as $x(0) = (0, 0, 0, 0)^T$ and $x(3) = (3, 2, 2, 1)^T$.

Let the intermediate time instants $t_0 < t_1 < t_2 < T$ be given, and the nonseparated intermediate conditions like (4) be given by

$$x_1(t_1) + x_3(t_1) + x_1(t_2) + x_3(t_2) = \alpha_1,$$
$$x_2(t_1) + x_4(t_1) + x_2(t_2) + x_4(t_2) = \alpha_2,$$

that is, according to (4) the matrices F_1 and F_2 are given by

$$F_1 = F_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

According to (10) and (11), for system (22) we get

$$F(\tau)=\begin{pmatrix} x_{11}(t_1,\tau)+x_{11}(t_2,\tau) & x_{12}(t_1,\tau)+x_{12}(t_2,\tau) & x_{33}(t_1,\tau)+x_{33}(t_2,\tau) & x_{34}(t_1,\tau)+x_{34}(t_2,\tau) \\ 0 & x_{22}(t_1,\tau)+x_{22}(t_2,\tau) & 0 & x_{44}(t_1,\tau)+x_{44}(t_2,\tau) \end{pmatrix},$$

$$H[\tau] = \begin{pmatrix} x_{12}(t_1,\tau)+x_{12}(t_2,\tau) & x_{34}(t_1,\tau)+x_{34}(t_2,\tau) \\ x_{22}(t_1,\tau)+x_{22}(t_2,\tau) & x_{44}(t_1,\tau)+x_{44}(t_2,\tau) \\ x_{12}(T,\tau) & 0 \\ x_{22}(T,\tau) & 0 \\ 0 & x_{34}(T,\tau) \\ 0 & x_{44}(T,\tau) \end{pmatrix},$$

where

$$x_{11}(t,\tau)=x_{22}(t,\tau)=x_{33}(t,\tau)=x_{44}(t,\tau)=1, \quad x_{12}(t,\tau)=x_{34}(t,\tau)=t-\tau \tag{23}$$

are the elements of the normalized fundamental matrix of the solution of the uniform part of system (22).

According to (13), for $t_1 = 1$, $t_2 = 2$, $\alpha_1 = 2$, and $\alpha_2 = 1$ we get

$$\eta = (\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6)^T = \left(2 + \frac{5g}{2},\ 1 + 3g,\ 3,\ 2,\ 2 + \frac{9g}{2},\ 1 + 3g\right)^T,$$

and according to (17) the matrix Q is given by

$$Q = \begin{pmatrix} \frac{28}{3} & 9 & 6 & \frac{5}{2} & 6 & \frac{5}{2} \\ 9 & 10 & \frac{13}{2} & 3 & \frac{13}{2} & 3 \\ 6 & \frac{13}{2} & 9 & \frac{9}{2} & 0 & 0 \\ \frac{5}{2} & 3 & \frac{9}{2} & 3 & 0 & 0 \\ 6 & \frac{13}{2} & 0 & 0 & 9 & \frac{9}{2} \\ \frac{5}{2} & 3 & 0 & 0 & \frac{9}{2} & 3 \end{pmatrix}.$$

Therefore, according to (20) we obtain for the vector of control actions the following expressions: for $\tau \in [t_0, t_1]$,

$$u(\tau) = \begin{pmatrix} \frac{1}{18}(115x_{12}(T,\tau) - 3(9x_{12}(t_1,\tau) + 9x_{12}(t_2,\tau) + 13x_{22}(T,\tau) + 33(x_{22}(t_1,\tau) + x_{22}(t_2,\tau)))) \\ \frac{1}{18}(119x_{34}(T,\tau) - 3(9x_{34}(t_1,\tau) + 9x_{34}(t_2,\tau) + (17-6g)x_{44}(T,\tau) + 33(x_{44}(t_1,\tau) + x_{44}(t_2,\tau)))) \end{pmatrix},$$

for $\tau \in (t_1, t_2)$,

$$u(\tau) = \begin{pmatrix} \frac{1}{18}(115x_{12}(T,\tau) - 3(9x_{12}(t_2,\tau) + 13x_{22}(T,\tau) + 33x_{22}(t_2,\tau))) \\ \frac{1}{18}(119x_{34}(T,\tau) - 3(9x_{34}(t_2,\tau) + (17-6g)x_{44}(T,\tau) + 33x_{44}(t_2,\tau))) \end{pmatrix},$$

and for $\tau \in [t_2, T]$,

$$u(\tau) = \begin{pmatrix} \frac{1}{18}(115x_{12}(T,\tau) - 39x_{22}(T,\tau)) \\ \frac{1}{18}(119x_{34}(T,\tau) - 3(17-6g)x_{44}(T,\tau)) \end{pmatrix}.$$

Taking into consideration (23) and substituting $t_1 = 1$ and $t_2 = 2$, we obtain

$$u(t) = \begin{pmatrix} \frac{3}{2} - \frac{61t}{18} \\ \frac{3}{2} + g - \frac{65t}{18} \end{pmatrix} \quad \text{for} \quad t \in [0,1],$$

$$u(t) = \begin{pmatrix} \frac{17}{2} - \frac{44t}{9} \\ \frac{17}{2} + g - \frac{46t}{9} \end{pmatrix} \quad \text{for} \quad t \in [1,2], \tag{24}$$

$$u(t) = \begin{pmatrix} 17 - \frac{115t}{18} \\ 17 + g - \frac{119t}{18} \end{pmatrix} \quad \text{for} \quad t \in [2,3].$$

By substituting in (22) the established expressions for the control $u(t)$ (24) and integrating these equations over each time interval, we get that the object motion is given by

$$x(t) = \begin{pmatrix} \frac{3t^2}{4} - \frac{61t^3}{108} \\ \frac{3t}{2} - \frac{61t^2}{36} \\ \frac{3t^2}{4} - \frac{65t^3}{108} \\ \frac{3t}{2} - \frac{65t^2}{36} \end{pmatrix} \quad \text{for} \quad t \in [0,1],$$

$$x(t) = \begin{pmatrix} 3 - \frac{25t}{4} + \frac{17t^2}{4} - \frac{22t^3}{27} \\ -\frac{25}{4} + \frac{17t}{2} - \frac{22t^2}{9} \\ 3 - \frac{25t}{4} + \frac{17t^2}{4} - \frac{23t^3}{27} \\ -\frac{25}{4} + \frac{17t}{2} - \frac{23t^2}{9} \end{pmatrix} \quad \text{for} \quad t \in [1,2],$$

$$x(t) = \begin{pmatrix} 16 - \frac{81t}{4} + \frac{17t^2}{2} - \frac{115t^3}{108} \\ -\frac{81}{4} + 17t - \frac{115t^2}{36} \\ 16 - \frac{81t}{4} + \frac{17t^2}{2} - \frac{119t^3}{108} \\ -\frac{81}{4} + 17t - \frac{119t^2}{36} \end{pmatrix} \quad \text{for} \quad t \in [2,3].$$

So, the explicit expressions for the program control (24) and the corresponding program motion were established for system (22) with the given initial and final values of the phase vector and nonseparated intermediate conditions. By the direct substitution we make sure that the resulting motion satisfies the condition

$$F_1 x(1) + F_2 x(2) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

To illustrate solution of Problem 2, we consider the linear control system

$$\begin{aligned} \dot{x}_1 &= x_2 + u_1, \\ \dot{x}_2 &= u_2. \end{aligned} \tag{25}$$

Given are the time instants $t_0 < t_1 < t_2 < T$, and let $t_0 = 0$, $T = 3$. The initial and final states of the phase vector are chosen for simplicity as

$$x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad x(3) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Let for the intermediate time instants $t_1 < t_2$ we have according to (4) the following nonseparated intermediate conditions

$$\begin{aligned} x_1(t_1) + x_1(t_2) &= \alpha_1, \\ x_2(t_1) + x_2(t_2) &= \alpha_2, \end{aligned} \tag{26}$$

that is, the matrices F_1 and F_2 are given by

$$F_1 = F_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let the performance index of control be given by

$$æ[u] = \int_{t_0}^{\mathrm{T}} (u_1^2 + u_2^2) dt. \tag{27}$$

According to (10) and (11), we get for system (25)

$$F(\tau) = \begin{pmatrix} x_{11}(t_1,\tau) + x_{11}(t_2,\tau) & x_{12}(t_1,\tau) + x_{12}(t_2,\tau) \\ 0 & x_{22}(t_1,\tau) + x_{22}(t_2,\tau) \end{pmatrix},$$

$$H[\tau] = \begin{pmatrix} x_{11}(t_1,\tau) + x_{11}(t_2,\tau) & x_{12}(t_1,\tau) + x_{12}(t_2,\tau) \\ 0 & x_{22}(t_1,\tau) + x_{22}(t_2,\tau) \\ x_{11}(T,\tau) & x_{12}(T,\tau) \\ 0 & x_{22}(T,\tau) \end{pmatrix},$$

where the functions $x_{ij}(t,\tau)$, $i,j = 1,2$, are the elements of the normalized fundamental matrix of the solution of the uniform part of system (25) given by

$$x_{11}(t,\tau) = x_{22}(t,\tau) = 1, \quad x_{12}(t,\tau) = t - \tau. \tag{28}$$

At generating the matrix of $F(\tau)$, notation similar to (8) is taken into consideration.

According to (12), we get the following integral relations

$$\begin{aligned} &\int_{t_0}^{T} [h_{11}(\tau)\, u_1 + h_{12}(\tau)\, u_2]\, d\tau = \eta_1, \quad \int_{t_0}^{T} h_{22}(\tau)\, u_2\, d\tau = \eta_2, \\ &\int_{t_0}^{T} [h_{31}(\tau)\, u_1 + h_{32}(\tau)\, u_2]\, d\tau = \eta_3, \quad \int_{t_0}^{T} h_{44}(\tau)\, u_2\, d\tau = \eta_4, \end{aligned} \tag{29}$$

where the following notation is used:

$$h_{11}(\tau) = x_{11}(t_1, \tau) + x_{11}(t_2, \tau), \quad h_{12}(\tau) = x_{12}(t_1, \tau) + x_{12}(t_2, \tau),$$
$$h_{22}(\tau) = x_{22}(t_1, \tau) + x_{22}(t_2, \tau), \quad h_{31}(\tau) = x_{11}(T, \tau),$$
$$h_{32}(\tau) = x_{12}(T, \tau), \quad h_{44}(\tau) = x_{22}(T, \tau).$$

In consideration of $t_1 = 1$, $t_2 = 2$, $\alpha_1 = 2$, $\alpha_2 = 1$, according to (13) we establish that $\eta_1 = 2$, $\eta_2 = 1$, $\eta_3 = 3$, $\eta_4 = 2$.

By solving the resulting problem of conditional extremum (27), (29) as that of moments [6], we establish the following expressions for the optimal control actions:

$$u_1^0(\tau) = \begin{cases} \frac{15}{91} x_{11}(T, \tau) + \frac{1}{7} x_{11}(t_1, \tau) + \frac{1}{7} x_{11}(t_2, \tau), & \tau \in [t_0, t_1] \\ \frac{15}{91} x_{11}(T, \tau) + \frac{1}{7} x_{11}(t_2, \tau), & \tau \in [t_1, t_2] \\ \frac{15}{91} x_{11}(T, \tau), & \tau \in [t_2, T], \end{cases}$$

$$u_2^0(\tau) = \begin{cases} \frac{15}{91} x_{12}(T, \tau) + \frac{1}{7} x_{12}(t_1, \tau) + \frac{1}{7} x_{12}(t_2, \tau) + \frac{605}{546} x_{22}(T, \tau) \\ \qquad - \frac{21}{26} x_{22}(t_1, \tau) - \frac{21}{26} x_{22}(t_2, \tau), & \tau \in [t_0, t_1] \\ \frac{15}{91} x_{12}(T, \tau) + \frac{1}{7} x_{12}(t_2, \tau) + \frac{605}{546} x_{22}(T, \tau) - \frac{21}{26} x_{22}(t_2, \tau), & \tau \in [t_1, t_2] \\ \frac{15}{91} x_{12}(T, \tau) + \frac{605}{546} x_{22}(T, \tau), & \tau \in [t_2, T]. \end{cases}$$

With regard for notation (28), we obtain by substituting $t_1 = 1$ and $t_2 = 2$ that

$$u_1^0(t) = \begin{cases} \frac{41}{91}, & t \in [0, 1] \\ \frac{4}{13}, & t \in [1, 2] \\ \frac{15}{91}, & t \in [2, 3], \end{cases}$$

$$u_2^0(t) = \begin{cases} \frac{1}{546}(227 - 246t), & t \in [0, 1] \\ \frac{1}{273}(295 - 84t), & t \in [1, 2] \\ \frac{125}{78} - \frac{15t}{91}, & t \in [2, 3]. \end{cases}$$

The minimal value of the performance index is given by

$$æ[u^0] = \frac{1195}{546}.$$

Now, by substituting $u_1^0(t)$ and $u_2^0(t)$ in (25) we establish expressions for the phase coordinates of the optimal motion:

$$x_1^0(t) = \begin{cases} \frac{41t}{91} + \frac{227t^2}{1092} - \frac{41t^3}{546}, & t \in [t_0, t_1] \\ \frac{571}{1092} - \frac{3t}{7} + \frac{295t^2}{546} - \frac{2t^3}{39}, & t \in [t_1, t_2] \\ \frac{813}{364} - \frac{173t}{91} + \frac{125t^2}{156} - \frac{5t^3}{182}, & t \in [t_2, T], \end{cases}$$

$$x_2^0(t) = \begin{cases} \frac{227t}{546} - \frac{41t^2}{182}, & t \in [t_0, t_1] \\ -\frac{67}{91} + \frac{295t}{273} - \frac{2t^2}{13}, & t \in [t_1, t_2] \\ -\frac{188}{91} + \frac{125t}{78} - \frac{15t^2}{182}, & t \in [t_2, T]. \end{cases}$$

We make sure by the direct substitution that the obtained optimal motion satisfies condition (26). Thus, explicit expressions of the program control and program motion were obtained for system (25) with the given initial and final values of the phase vector and nonseparated intermediate conditions.

Remark. If one adds a quadratic functional (similar to (27)) to the formulation of the first example, then the resulting problem can be solved as that of moments. At that, its optimal solution coincides with the constructed solution (24) [6].

5. CONCLUSIONS

Proposed was a constructive approach to studying the problem of control of the linear dynamic systems with nonseparated (nonlocal) multipoint intermediate conditions and optimal control with a performance index defined over the entire time span. The integral relations satisfied by the control action were established with regard for the linear nonseparated multipoint intermediate conditions. Solution of the control problem was constructed, and a method to solve the problem of optimal control was proposed. The conditions for existence of the program control and motion were formulated. Explicit solutions of particular problems were described.

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