On the Controllability and Observability of Linear Dynamic Systems with Variable Structure

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***Abstract*—The paper considers the questions of controllability and observability of linear dynamic systems with variable structure and with conditions of relation on intermediate moments of time. Necessary and sufficient conditions of complete controllability and observability of compounded (and stage by stage changing) linear stationary system are derived. The conditions are expressed directly through the initial parameters of the system and comparisons with the well-known conditions of Kalman are made. It is shown that in separate periods of time the compounded system composed of not completely controllable (or observable) systems with the corresponding intermediate conditions of relations can be completely controllable (or observable) on the entire interval of time. For such a stationary system a conjugate system is considered and an analogy of principle of duality is established, linking the concepts of controllability and observability.**

I. Introduction

The investigation of many applied problems of process of control and observation is reduced to dynamic systems with variable structure (in particular, the compounded systems, stage by stage changing systems, etc.). As in conventional problems of control and observation [1-3], also in the control and observation problems of dynamical systems with variable structure the questions of controllability and observability of these systems are fundamental. Some questions of controllability and observability of systems with variable structure and piecewise linear impulse systems are investigated in particular in [3-10].

In this paper, necessary and sufficient conditions for complete controllability and observability of linear non-stationary and stationary dynamic systems with variable structure are obtained. It is shown that in separate periods of time stationary compounded system composed of not completely controllable (or observable) systems with the corresponding intermediate conditions of relations can be completely controllable (or observable) on the entire interval of time.

II. The Problem Setting

This paper considers a compounded linear dynamic system, the motion of which on the time interval $t_{k-1} \leq t \leq t_k$ $(k = 1, ..., m)$ is described by n_k-dimensional linear differential equations:

$$\dot{x}^{(k)} = A_k(t)x^{(k)} + B_k(t)u^{(k)} \tag{1}$$

with intermediate conditions of relation:

$$E_k x^{(k)}(t_k) + F_k x^{(k+1)}(t_k) = \beta_k. \tag{2}$$

The moments of time t_k, $0 \leq t_0 < t_1 < ... < t_{m-1} < < t_m = T$ are assumed to be given.

Here $x^{(k)}(t) \in R^{n_k}$, $x^{(k)}$ is the phase vector of the system; $A_k(t)$, $B_k(t)$ $(k = 1, ..., m)$ are matrices of the parameters of the system, $u^{(k)}(t)$ is the control action, the dimensions of which are $A_k(t) - (n_k \times n_k)$, $B_k(t) - (n_k \times r_k)$, $u_k(t) - (r_k \times 1)$, respectively. E_k is $(n_{k+1} \times n_k)$ dimensional matrix, F_k is $(n_{k+1} \times n_{k+1})$ matrix, β_k is $(n_k \times 1)$ dimensional column vector. It is assumed that the elements of matrix function $A_k(t)$, $B_k(t)$, and those of column vector $u^{(k)}(t)$ are continuous functions. It is also assumed that the matrices E_k, F_k and the vectors β_k are known and that the matrices F_k are such that the inverse matrices exist F_k^{-1}, e.g. $det\ F_k \neq 0$. It is assumed that by means of measurement devices some values of compounded system are measured and the result of the measurement (e.g. output) has the following form:

$$y^{(k)}(t) = G_k(t)x^{(k)}(t) + D_k(t)u^{(k)}, \tag{3}$$

where $y^{(k)}(t)$ is known s_k-dimensional vector function of output variables of the system (the measurements or indications of observation device); $G_k(t), D_k(t)$ $(k = 1, ..., m)$ are matrices with $(s_k \times n_k)$ and $(s_k \times r_k)$ dimensions, respectively, with real continuous elements.

Stage by stage changing controllable and observable systems are also considered, e.g. when the dimensions of phase vector, control vector and observation vector do not change. The concepts of complete controllability of compounded systems (1), (2) on time interval $[t_0, T]$ and complete observability of the systems (1), (2) based on (3) measurements are defined. The considered problems are to find the conditions under which the system (1), (2) will be completely controllable on the time interval $[t_0, T]$ and to find conditions under which compounded system (1), (2) based on the measurement results (3) will be completely observable.

III. The main results

The paper formulates the necessary and sufficient conditions for complete controllability of linear non-stationary compounded systems (1), (2) and for complete observability of systems (1), (2) based on (3) measurements.

In that sense they give an answer to the questions of complete controllability and complete observability of stationary compounded systems. However, for the practical application of these conditions it is necessary to build impulse transition

978-1-4673-9997-5/16/$31.00

matrices for the systems under consideration, which is inconvenient. In general, the judgements on existence of the solution to the problems of control or observation of compounded system are reasonable to make, based only on the initial data of the problem. Therefore, it is desirable to have the conditions that allow to judge about complete controllability or complete observability based on the elements of the matrices, present in the systems being investigated.

Necessary and sufficient conditions of complete controllability and observability of compounded and stage by stage changing linear stationary systems, expressed directly through the original parameters of the system, are obtained. These conditions are expressed through the ranks of the matrices of controllability and observability, respectively.

The following theorem holds.

Theorem 1. Linear stationary compounded system:

$$\dot{x}^{(k)} = A_k x^{(k)} + B_k u^{(k)} \tag{4}$$

with intermediate conditions of relation (2) is completely controllable on time interval $t_0 \leq t \leq T$ if and only if the matrix

$$\begin{aligned} K = \{C_1(B_1, A_1B_1, ..., A_1^{p_1-1}B_1), ... \\ ..., C_m(B_m, A_mB_m, ..., A_m^{p_m-1}B_m)\} \end{aligned} \tag{5}$$

has rank equal to n_m, where

$$\begin{aligned} &C_i = (-1)^{m-i} e^{A_m(T-t_{m-1})} F_{m-1}^{-1} W_i^{(m-1)} E_i e^{A_i t_i}, \\ &i = 1, ..., m-1; \\ &C_m = e^{A_m T}, \\ &W_i^{(k)} = \Pi_{j=1}^{k-i} E_{k+1-j} e^{A_{k+1-j}(t_{k+1-j}-t_{k-j})} F_{k-j}^{-1}, \\ &k = 2, 3, ..., m;\ j = 1, 2, ..., k-1, \end{aligned}$$

and the numbers p_j are specified by (algebraic) multiplicity of eigenvalues of matrix A_j $(j = 1, ..., m)$.

For stage by stage changing stationary system:

$$\dot{x} = \begin{cases} A_1x + B_1u, & when \quad t \in [t_0, t_1) \\ A_2x + B_2u, & when \quad t \in [t_1, t_2) \\ \quad .... & \\ A_mx + B_mu, & when \quad t \in [t_{m-1}, T] \end{cases} \tag{6}$$

with intermediate conditions on time intervals t_k

$$x(t_k - 0) = x(t_k + 0) = x(t_k) \quad (k = 1, ..., m-1)$$

the following theorem holds.

Theorem 2. Stage by stage changing linear stationary system is completely controllable on time interval $t_0 \leq t \leq T$, if and only if, the matrix

$$\begin{aligned} K = \{B_1, A_1B_1, ..., A_1^{p_1-1}B_1, ... \\ ..., B_m, A_mB_m, ..., A_m^{p_m-1}B_m\} \end{aligned} \tag{7}$$

has a rank equal to n, where the numbers p_j are specified by the multiplicity of eigenvalues of matrx A_j $(j = 1, ..., m)$.

For homogeneous equation (6) (e.g. when $u(t) \equiv 0$) with observable value:

$$y = \begin{cases} G_1x, \ when \ t \in [t_0, t_1) \\ G_2x, \ when \ t \in [t_1, t_2) \\ ... \\ G_mx, \ when \ t \in [t_{m-1}, T] \end{cases} \tag{8}$$

the following theorem holds.

Theorem 3. For the stage by stage changing linear stationary system (6) (when $u(t) \equiv 0$) with observable value (8) to be completely controllable on time interval $t_0 \leq t \leq T$, it is necessary and sufficient, that the rank of the matrix

$$\begin{aligned} S = \{G_1^T, A_1^TG_1^T, ..., (A_1^T)^{p_1-1}G_1^T, ... \\ ..., G_m^T, A_m^TG_m^T, ..., (A_m^T)^{p_m-1}G_m^T\} \end{aligned} \tag{9}$$

is equal to n, where the numbers p_j are specified by the multiplicity of eigenvalues of matrix A_j $(j = 1, ..., m)$.

The matrices (5) and (7) are the matrices of controllability of the systems and (9) is the matrix of observability.

If all the eigenvalues of matrices A_k $(k = 1, ..., m)$ are prime, then the matrix of controllability (7) and the matrix of observability (9) have the following form:

$$K = \{B_1, A_1B_1, ..., A_1^{n-1}B_1, ..., B_m, A_mB_m, ..., A_m^{n-1}B_m\}$$

$$\begin{aligned} S = \{G_1^T, A_1^TG_1^T, ..., (A_1^T)^{n-1}G_1^T, ... \\ ..., G_m^T, A_m^TG_m^T, ..., (A_m^T)^{n-1}G_m^T\} \end{aligned}$$

respectively.

A comparison with the well-known Kalman conditions of complete controllability and observability [1-2] is made and it is shown, that the obtained conditions are the generalization of those conditions.

It is shown that on separate time intervals all subsystems, that form the compounded system, individually can be not completely controllable (observable), but as a whole on the entire time interval the compounded system can be completely controllable (observable). But if at least one subsystem of compounded system on its time interval of functioning is completely controllable (observable), then the compounded system will also be completely controllable (observable).

For the stage by stage changing linear stationary system the article considers a conjugate system and shows an analogy of duality principle of Kalman, connecting the concepts of controllability and observability for stage by stage changing linear stationary systems.

The examples are built, that illustrate the obtained results.

IV. Conclusion

Derived necessary and sufficient conditions of complete controllability and observability of compounded (and stage by stage changing) linear stationary systems and the principle of duality are of important theoretical and applied value. Necessary and sufficient conditions expressed only through initial parameters of the systems have widespread practical application for complete controllability and observability of such stationary systems.

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