Duality in spaces of harmonic functions

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A positive continuous decreasing function φ on $[0,1)$ is called *weight function* if $\lim \varphi(r)=0$, as $r\to 1$, and a positive finite Borel measure η on $[0,1)$ is called *weighting measure* if it is not supported in any subinterval $[0,\rho)$, $0<\rho<1$. Let $h_\infty(\varphi)$ be the Banach space of functions u, harmonic in the unit ball $B_n\subset\mathbb{R}^n$, with the norm $\|u\|_\varphi=\sup\{|u(x)|\varphi(|x|)\colon\ x\in B_n\}$ and let $h_0(\varphi)$ be the closed subspace of functions u with $|u(x)|=o\big(1/\varphi(|x|\big)$ as $|x|\to 1$.

It has been shown by Rubel and Shields, [1] that $h_\infty(\varphi)$ is isometrically isomorphic to the second dual of $h_0(\varphi)$. In [2], in the case $n=2$, it was posed and solved the duality problem of finding a weighting measure η such that

$$h^1(\eta)=\big\{v\in L^1(d\eta(r)\,d\sigma)\colon\ v \text{ is harmonic in } B_2\big\}$$

represents the intermediate space, the dual of $h_0(\varphi)$ and the predual of $h_\infty(\varphi)$, i.e. $h^1(\eta)\sim h_0(\varphi)^*$ and $h^1(\eta)^*\sim h_\infty(\eta)$. In this duality relations the pairing is given by

$$\langle u,v\rangle=\int_0^1\int_S u(r\zeta)\overline{v(r\zeta)}\varphi(r)\,d\sigma(\zeta)\,d\eta(r),\quad u\in h_\infty(\varphi),\ \ v\in h^1(\eta).$$

In the indicated articles [1] and [2] only the case $n=2$ is considered. It well-known that in this case every harmonic function h has expansion in a series on degrees z and $\overline{z}$ in unit disk $|z|<1$, because real-valued $h\in h(B)$ is a real part of holomorphic function. It allows to apply the methods of complex analysis.

In the report, we consider duality problem in the case of harmonic functions in the unit ball of $\mathbb{R}^n$, $n>2$. The multidimensional case has the specifics in the sense that we can not speak about connection between harmonic and holomorphic functions, and instead of degrees z and $\overline{z}$ we deal with spherical harmonics.

Let the weight functions ϕ has the form $\phi(r)=\psi\left(1/(1-r)\right)^{-1}$ where ψ is positive, continuously differentiable, increasing to $+\infty$ on $[0,\infty)$, and satisfies the following conditions:

I. $\Delta^p(1/\psi(k)) \geq 0, \quad k, p = 0, 1, 2, \ldots;$

II. there exist $a > 0$ and $x_0 > 0$ such that $\psi(x)/x^k \searrow 0$ for $x > x_0$ if $x \to \infty$.

Here $\Delta^p(1/\psi(k))$ is the standard notation for a difference of order p.

Condition II is the restriction of the rate of growth of ϕ stated above, namely, $1/\phi(r)$ does not grow faster than some power of $1/(1-r)$. Condition I allows us to produce a weighting measure solving the Hausdorff moment problem (see [3, Theorem 2.6.4.]) for $1/\psi(k)$.

As examples of functions $\psi(x)$ satisfying the above conditions we have

$$\begin{aligned} \psi(x) &= (x+1)^\alpha, && \alpha > 0, \\ \psi(x) &= [\log(x+2)]^\alpha, && \alpha > 0, \\ \psi(x) &= [\log\log(x+4)]^\alpha, && \alpha > 0. \end{aligned}$$

References

[1] L.A. Rubel, A.L. Shields, *The second duals of certain spaces of analytic functions.* J. Austral. Math. Soc, **11**: 276–280, 1970.

[2] A.L. Shields, D.L. Williams, *Bounded projections, duality, and multipliers in spaces of harmonic functions.* J. Reine Angew. Math. **299–300**: 256–279, 1978.

[3] N.I. Akhiezer, *The classical moment problem,* Edinburg and London, 1965.