Runge-Lenz vector in Calogero-Coulomb problem

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We construct the Runge-Lenz vector and symmetry algebra of rational Calogero-Coulomb problem, formulated in terms of Dunkl operators. We find that they are proper deformations of their Coulomb counterpart. Together with similar correspondence between the Calogero-oscillator and oscillator models, this observation permits to claim that most of properties of Coulomb and oscillator systems can be lifted to their Calogero-extended analogs by the proper replacement of momenta by Dunkl momenta operators.

I. INTRODUCTION

The N-dimensional oscillator and Coulomb problem are the most known bound systems with maximal number, $2N-1$, of functionally independent constants of motion (such systems are called maximally superintegrable). The free particle is obviously the best known superintegrable system among unbounded ones. It seems, that all other superintegrable systems can be obtained somehow from those listed above.

Among the most non-trivial unbounded superintegrable systems is the rational Calogero model [1]

$$\mathcal{H}_0 = \sum_{i=1}^{N} \frac{\hat{p}_i^2}{2} + \sum_{i<j} \frac{g(g-1)}{(x_i - x_j)^2}. \tag{1}$$

Its generalization associated with arbitrary Coxeter group [2] is also superintegrable. The superintegrability of these models was established in the classical [3] and quantum [4, 5] cases.

The rational Calogero model with additional oscillator potential (we will refer it as the *Calogero-oscillator* model) is as known, as usual Calogero model and is also superintegrable system [6]. The similarity between Calogero model and free particle, as well as between the Calogero-oscillator model and oscillator can be easily understood from the viewpoint of matrix model reduction and from the operator exchange formalism (see for the review [2]). Let us briefly describe the second approach, independently invented by Polychronakos and by Brink, Hanson and Vasiliev [7], which then has been found to be related with seminal work by Dunkl [8]. Following these authors, we take into account the Calogero interaction, replacing the momenta $\hat{p}_i = -\imath\partial_i$ by the "Dunkl momenta" $-\imath\nabla_i$, where ∇_i are the Dunkl operators [8],

$$\begin{aligned} &\nabla_i = \partial_i - \sum_{j\neq i} \frac{g}{x_i - x_j} s_{ij} : \qquad [\nabla_i, \nabla_j] = 0, \\ &[\nabla_i, x_j] = S_{ij} = \begin{cases} -g s_{ij} & \text{for } i \neq j, \\ 1 + g\sum_{k\neq i} s_{ik} & \text{for } i = j. \end{cases} \end{aligned} \tag{2}$$

Here s_{ij} is the exchange operator between ith and jth coordinates. The original Hamiltonian can be obtained by the restriction of the extended Hamiltonian to the symmetric wavefunctions.

In these terms, the superinegrability of oscillator and Calogero-oscillator models has visible similarity. Namely, the latter, formulated in terms of the Dunkl operators, has overcompleted number of symmetry generators given by the Dunkl angular momentum [4, 9]

$$\begin{aligned} M_{ij} &= x_i\nabla_j - x_j\nabla_i, \\ [M_{ij}, M_{kl}] &= S_{k[j}M_{i]l} + S_{l[i}M_{j]k}, \end{aligned} \tag{3}$$

and by the generators of hidden symmetry

$$I_{ij} = -\nabla_i\nabla_j + \alpha^2 x_i x_j. \tag{4}$$

The constants of motion of the Calogero-oscillator model can be obtained from them by symmetrization over all indices. In the $g = 0$ limit, the generators M_{ij} and I_{ij} result in the symmetry generators of oscillator, which form $u(N)$ algebra. The symmetries of Calogero model without oscillator are related with the symmetries of free particles in the same way.

Hence, the symmetries of rational Calogero model without and with the oscillator potential, formulated in terms of the Dunkl operators, are in one-to-one correspondence with those of the free particle and of the oscillator. This holds also for the generalized rational Calogero model (rational Calogero model associated with arbitrary Coxeter group) [9].

On the other hand, it is known for years that the rational Calogero model, extended by Coulomb and by any other central potential, is an integrable system [10, 11] (we will refer the former as the *Calogero-Coulomb* system). However, the integrability of these systems is more or less obvious, and straightly follows from the integrability of the angular part of the generalized rational Calogero model [12]. However, in the recent paper with O. Lechtenfeld, we have observed that among rational Calogero models with central potentials there are two superintegrable systems, that are Calogero models with oscillator and Coulomb potentials [13]. Moreover, we showed that the Calogero-oscillator and Calogero-Coulomb models are, in fact, the only isospectral deformations of the oscillator and Coulomb systems.

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Let us remind that hidden symmetries of the Coulomb problem are given by the Runge-Lenz vector, forming, with the generators of angular momentum, a quadratic algebra [14]. On the negative-energy states, this algebra is reduced to $so(N+1)$, while on the states with positive energy it gives $so(N,1)$. Having in mind the similarity between symmetries of oscillator and Calogero-oscillator models, and that Calogero-oscillator and Calogero-Coulomb systems are highlighted from the integrability viewpoint [13], we can ask: *Can symmetries of conventional Coulomb problem be deformed to the symmetries of Calogero-Coulomb model formulated in terms of Dunkl operators?*

In this note we will show that it is a case, viz,

- Symmetry generators of Calogero-Coulomb system formulated in terms of Dunkl operator are given by the deformed angular momentum tensor (3) and by the deformed Runge-Lenz vector.
- The symmetry algebra of Calogero-Coulomb model is a deformation of Coulomb symmetry algebra.
- Functional relation between Coulomb Hamiltonian, Runge-Lenz vector and angular momentum has straight analog in the Calogero-Coulomb problem.

In fact, this means that Calogero-Coulomb problem is as fundamental as the Calogero-oscillator one. Because of such profound similarity with conventional Coulomb problem, we expect, that most of applications and deformations of Coulomb system can be extended, somehow, to the Calogero-Coulomb system.

II. SYMMETRIES

In this section we show, that in terms of Dunkl operator symmetries of Calogero-Coulomb model look as deformation of those of conventional Coulomb problem. Calogero-Coulomb model is given by Hamiltonian

$$\mathcal{H} = -\frac{\nabla^2}{2} - \frac{\gamma}{r}, \qquad r^2 = \sum_i x_i^2. \tag{5}$$

Like in Calogero-oscillator case, it commutes with the Dunkl angular momentum operators (3).

The hidden symmetry operators are given, as proven in Appendix, by the Hermitian operators, generalizing the Runge-Lenz vector,

$$A_i = -\frac{1}{2}\sum_j \{M_{ij}, \nabla_j\} + \frac{1}{2}[\nabla_i, S] - \gamma\frac{x_i}{r}, \tag{6}$$

where

$$S = \sum_{i<j} S_{ij}. \tag{7}$$

The operator S is invariant with respect to the permutations and takes the value $\frac{1}{2}gN(1-N)$ on the symmetric wavefunctions. In the absence of inverse-square interaction, $g=0$, the operators S_{ij} are reduced to δ_{ij}, and the second term in (6) vanishes. The integrals A_i then are reduced to the usual Runge-Lenz vector of the N-dimensional Coulomb system. They obey the following commutation relations (see Appendix):

$$[A_i, M_{kl}] = A_{[l}S_{k]i}, \qquad [A_i, A_j] = -2\mathcal{H}M_{ij}. \tag{8}$$

These relations together with (3), lead in $g=0$ limit to the symmetry algebra of the N-dimensional Coulomb problem. Note that in this limit, the algebra (3) reduces to the usual $so(N)$ algebra of angular momentum [9]. Let us also note that the first commutator in (8) is a deformation of the infinitesimal rotation of the vector generated by M_{kl}.

The coordinates ($u_i = x_i$) and Dunkl operators ($u_i = \nabla_i$) obey the same relation, as it follows from (2) and (3):

$$[u_i, M_{kl}] = S_{i[k}u_{l]} = u_{[l}S_{k]i}. \tag{9}$$

The first relation in (8) becomes apparent upon expressing A_i in terms of the coordinates and Dunkl operators (see Appendix),

$$A_i = \Big(r\partial_r + \frac{N-1}{2}\Big)\nabla_i - x_i\Big(\nabla^2 + \frac{\gamma}{r}\Big). \tag{10}$$

Indeed, the operator valued coefficients of x_i and ∇_i in above expressions commute with M_{kl} and S_{kl}. Then the first commutation relation in (8) follows directly from the identities (9).

Like in the oscillator case, we are forced to combine them into symmetric polynomials in order to get the well defined constants of motion, acting on the symmetric wavefunctions,

$$\mathcal{A}_k = \sum_{i=1}^N A_i^k, \qquad \mathcal{M}_{2k} = \sum_{i<j} M_{ij}^{2k}. \tag{11}$$

These quantities define the whole set of constants of motion of Calogero-Coulomb model.

The first member of this family is independent from S-term and is given by the expression [13]

$$\mathcal{A}_1 = \sum_i x_i\left(2\mathcal{H} + \frac{\gamma}{r}\right) + (r\partial_r + \tfrac{N-1}{2})\sum_i \partial_i. \tag{12}$$

The constant of motion $\mathcal{M}_2$ does not commute with M_{ij}, but is related with the Casimir of algebra (3) in highly simple way [9],

$$\begin{aligned}\mathcal{M}_2' &= \mathcal{M}_2 - S(S-N+2)\\ &= r^2\nabla^2 - r^2\partial_r^2 - (N-1)r\partial_r.\end{aligned} \tag{13}$$

It describes the angular part of the Calogero model, studied from various viewpoints in [15, 16].

The constant of motion $\mathcal{A}_2$ does not commute with M_{ij} as well. However supplemented with corrections caused by the Calogero interaction, it becomes commutative with Dunkl angular momentum (see Appendix)

$$\mathcal{A}_2' = \mathcal{A}_2 + 2\mathcal{H}S : \qquad [\mathcal{A}_2', M_{ij}] = 0. \tag{14}$$

This suggests that certain combination of these invariants may commute with A_i too.

Finally, let us present the expression relating Calogero-Coulomb Hamiltonian with Runge-Lenz vector and Dunkle angular momentum, which generalizes similar relation in conventional Coulomb problem (see Appendix)

$$\mathcal{A}_2' = \gamma^2 - 2\mathcal{H}\big(\mathcal{M}_2' - \tfrac{(N-1)^2}{4}\big). \tag{15}$$

Presumably, it can be used for the pure algebraic derivation of the spectrum of Calogero-Coulomb problem.

III. CONCLUDING REMARKS

In this note we found that all relations between symmetry generators of Coulomb problems appear in the Calogero-Coulomb model formulated in terms of Dunkl operators, in a deformed way. To obtain them we should replace the momenta operators $-\imath\partial_i$ by Dunkl momenta $-\imath\nabla_i$, and make proper corrections depending on the permutation operator S_{ij}. We proved it for conventional Calogero-Coulomb model only, but it is straightforward to extend our consideration to Calogero-Coulomb model associated with arbitrary Coxeter group (two-dimensional Calogero-Coulomb problem associated with D_2 dihedral group was investigated by the use of Dunkl operators in [17]). The same correspondence holds for the Calogero-oscillator model [9].

Both Calogero-oscillator and Calogero-Coulomb models have superintegrable counterparts on (pseudo)spheres [13], and we have no doubt that the symmetry algebras of (pseudo)spherical oscillator and Coulomb systems can be lifted, in the same way, to those with Calogero term. However, due to technical difficulties we are unable to complete these calculations.

Pretty similarity between Calogero-oscillator (Calogero-Coulomb) model and oscillator (Coulomb) one permits to claim that *most of the properties of oscillator and Coulomb systems can be extended to their "Calogero-extended" counterparts.* In particular, we expect, that in this way one can solve the problem of $\mathcal{N} = 4, 8$ supersymmetrization of Calogero model, which was treated by many authors (see [18] and refs therein).

We are sure that in this way one can construct the extensions of three- or five-dimensional Calogero-Coulomb problems, specified by the presence, respectively, of Dirac and Yang monopole. Moreover, it seems that acting in the suggested way, we can relate two-, four-, eight-dimensional Calogero-oscillator model with two-, three-, five-dimensional Calogero-Coulomb one, including those specified by the presence of anyon and Dirac, Yang monopoles in the spirit of [19] (for previous treatments see [11]).

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Derivations

1. Conservation of Runge-Lenz vector (6)

Here we prove that the deformed Runge-Lentz vector (6) preserves Calogero-Coulomb Hamiltonian. First we compute the commutator between the Hamiltonian and the operators A_i. The commutator with the first term in (6) can be simplified using the following identity:

$$\Big[\sum_j \{M_{ij}, \nabla_j\}\,,\,\frac{1}{r}\Big] = -\frac{1}{r^3}\sum_j \{M_{ij}, x_j\} = \Big\{\frac{1}{r}, \nabla_i\Big\} - \sum_j \Big\{\frac{x_i x_j}{r^3}, \nabla_j\Big\}. \tag{16}$$

The relations $[\nabla^2, M_{ij}] = [r, M_{ij}] = 0$ [9] are used in the derivation. Next, we can calculate the commutator of the Hamiltonian with the last term in the expression (12) using the following identity:

$$\Big[\frac{x_i}{r}, \nabla^2\Big] = \sum_j \Big\{\frac{x_i x_j}{r^3} - \frac{S_{ij}}{r}, \nabla_j\Big\}. \tag{17}$$

As a result, the contribution of these two terms in the commutator eliminates the second term contribution as soon as

$$\Big[-\frac{1}{2}\sum_j \{M_{ij}, \nabla_j\} - \frac{\gamma x_i}{r}\,,\,\mathcal{H}\Big] = \sum_j \Big\{\frac{\gamma S_{ij}}{2r}, \nabla_i - \nabla_j\Big\} = \frac{1}{2}\big[[S, \nabla_i], \mathcal{H}\big], \tag{18}$$

where S is defined in (7). In the last equation we use the identity

$$\sum_j (\nabla_j - \nabla_i) S_{ij} = [S, \nabla_i].$$

The relation (18) completes the proof of conservation of (6).

2. Second commutation relation in (8)

Let us derive the commutation relation between the components of the Runge-Lentz vector in (8). For the convenience, we present (10) in the form:

$$A_i = a\nabla_i - x_i b, \quad a \equiv r\partial_r + \frac{N-1}{2}, \quad b \equiv \nabla^2 + \frac{\gamma}{r}. \quad (19)$$

So,

$$[A_i, A_j] = [a\nabla_i, a\nabla_j] + [x_i b, x_j b] + [a\nabla_{[j}, x_{i]} b]. \quad (20)$$

The action $[a, f]$ count the total degree in coordinates of f,

$$[a, \nabla_i] = -\nabla_i, \quad [a, x_i] = x_i, \quad [a, b] = -2\nabla^2 - \frac{\gamma}{r}. \quad (21)$$

The commutators with b are:

$$[b, x_i] = 2\nabla_i, \qquad [b, \nabla_i] = \frac{\gamma x_i}{r^3}. \quad (22)$$

Using the above equations, we obtain:

$$[a\nabla_i, a\nabla_j] = 0, \qquad [x_i b, x_j b] = 2M_{ij} b, \quad (23)$$

$$[a\nabla_j, x_i b] = x_j \nabla_i \nabla^2 + \frac{\gamma x_i x_j}{r^3}(a+1) + aS_{ij} b. \quad (24)$$

The last two terms are symmetric under the exchange of i and j, and disappear in the commutator (20). Substituting (23),(24) into (20), we arrive at the desired relation in (8).

3. Relation (10)

Here we derive the relation (10), which expresses the Runge-Lenz invariant in terms of the coordinates and Dunkl momenta. First we calculate the first term of (6):

$$\sum_j \{M_{ij}, \nabla_j\} = \{\nabla^2, x_i\} - (x\cdot\nabla)\nabla_i - \nabla_i(\nabla\cdot x)$$
$$= \{\nabla^2, x_i\} - \big(2r\partial_r + (N+1)\big)\nabla_i + [\nabla_i, S]. \quad (25)$$

We have used the identities

$$x\cdot\nabla = r\partial_r + S, \qquad \nabla\cdot x = r\partial_r - S + N, \quad (26)$$

Substituting this into (6), we arrive at Eq.(10).

4. Relation (14)

Let us calculate the commutator of M_{ij} with $\mathcal{A}_2$:

$$[M_{ij}, \mathcal{A}_2] = \sum_k \{A_{[i} S_{j]k}, A_k\}. \quad (27)$$

Each term from the r.h.s. of this equation can be presented as

$$\sum_k \{A_i S_{jk}, A_k\} = 2A_i A_j - 2\mathcal{H}\sum_k M_{ki} S_{kj}. \quad (28)$$

Here we take into account the identity

$$\sum_k \{S_{ik}, u_k\} = 2u_i + \sum_{k\neq i}\{S_{ik}, u_k - u_i\} = 2u_i, \quad (29)$$

which is fulfilled for any local operator u_k. Applying above relation, one can further simplify the commutator (27)

$$[M_{ij}, \mathcal{A}_2] = -2\mathcal{H}\Big(2M_{ij} + \sum_k M_{k[i} S_{j]k}$$
$$= -2\mathcal{H}\Big[M_{ij}, \sum_{k\neq i,j}(S_{kj} + S_{ki})\Big] = -2\mathcal{H}[M_{ij}, S]. \quad (30)$$

This completes the proof of (14).

5. Relation (15)

We use the representation (19) for $\mathcal{A}_2$:

$$\mathcal{A}_2 = \sum_i (a\nabla_i - x_i b)^2 = (a+1)a\nabla^2$$
$$+ (r^2 + 2x\cdot\nabla)b^2 - \sum_i \{a\nabla_i, x_i b\} \quad (31)$$
$$= r^2 b^2 + 2a^2\mathcal{H} - 2(x\cdot\nabla)\mathcal{H} - 2a\frac{\gamma}{r}.$$

The commutation relations (21), (22) and (26) are used for the derivation.

The Hamiltonian may be selected from the first term in the last expression,

$$r^2 b^2 = -2r^2\nabla^2\mathcal{H} + \frac{\gamma(N-3)}{r} + 2\gamma\partial_r + \gamma^2. \quad (32)$$

Inserting this into Eq. (31) and simplifying it we arrive at the desired relation (15).

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