УДК 517.9

ON THE THEOREM OF AMBARZUMIAN FOR DIRAC SYSTEM

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SUMMARY

We prove that in general case the analogue of famous theorem of Ambarzumian for canonical Dirac system is not true. In the same time we describe the particular cases, when some analogues are true.

Keywords: the inverse problem, theorem of Ambarzumian, Dirac canonical system.

Let us denote by $L(p,q,\alpha,\beta) = L(\Omega,\alpha,\beta)$ the boundary value problem for the canonical Dirac system ([1], [2])

$$\ell y \equiv \left\{B\frac{d}{dx} + \Omega(x)\right\} y = \lambda y, \qquad 0 < x < \pi, y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{1}$$

$$y_1(0)\cos\alpha + y_2(0)\sin\alpha = 0, \qquad \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right], \tag{2}$$

$$y_1(\pi)\cos\beta + y_2(\pi)\sin\beta = 0, \qquad \beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right], \tag{3}$$

where $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and the matrix-function

$$\Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix} \tag{4}$$

usually called a potential. About p and q we assume that $p, q \in L^1_{\mathbb{R}}[0,\pi]$, i.e. they are real, summable on $[0,\pi]$ functions. By the same $L(\Omega,\alpha,\beta)$ we denoted the selfadjoint operator, generated by problem (1)-(3) in Hilbert space of two-component vector-functions $L^2([0,\pi];\ \mathbb{C}^2)$ (see [3]).

It is well known ([4], [5], [6]) that the spectra of operator $L(\Omega,\alpha,\beta)$ is discrete and consists of real, simple eigenvalues λ_n ($n \in \mathbb{Z}$),

$$\ldots\lambda_{-n} < \lambda_{-n+1} < \cdots < \lambda_0 \le 0 < \lambda_1 < \cdots < \lambda_n < \cdots,$$

which we denote also by $\lambda_n = \lambda_n(\Omega,\alpha,\beta) = \lambda_n(p,q,\alpha,\beta)$, $n = 0, \pm1, \pm2, \ldots$, emphasizing the dependence λ_n on p, q, α, β, and which have the asymptotics

$$\lambda_n(p,q,\alpha,\beta) = n + \frac{\beta-\alpha}{\pi} + r_n, \tag{5}$$

where $r_n = r_n(p,q,\alpha,\beta) = o(1)$, when $n \to \pm\infty$, uniformly by $\alpha,\beta \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and p, q from bounded subsets of $L^1_{\mathbb{R}}[0,\pi]$.

In particular, it is also known that

$$\lambda_n(0,0,\alpha,\beta) = n + \frac{\beta-\alpha}{\pi}, \quad n \in \mathbb{Z}. \tag{6}$$

Inverse spectral problem for operator $L(\Omega,\alpha,\beta)$ is to reconstruct the potential and parameters α and β from spectral data. The spectra of an operator is its spectral data. There are also other spectral data (see [7]-[13]).

If denote by $\mu_n(q,\alpha,\beta)$, $n = 0,1,2,\ldots$, the eigenvalues of Sturm-Liouville problem

$$-y'' + q(x)y = \mu y, \qquad x \in (0,\pi), q \in L^1_{\mathbb{R}}[0,\pi], \mu \in \mathbb{C},$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \qquad \alpha \in (0,\pi],$$

$$y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \qquad \beta \in [0,\pi),$$

then the famous Ambarzumian theorem asserts:

Theorem 1 ([14]). If

$$\mu_n\left(q,\frac{\pi}{2},\frac{\pi}{2}\right) = n^2, \qquad n = 0,1,2,\ldots,$$

then $q(x) = 0$ almost everywhere (a.e.) on $[0,\pi]$.

Is there an analogue of Theorem of Ambarzumian for problem $L(p,q,\alpha,\beta)$? The same: are there α_0 and β_0 such that if

$$\lambda_n(p,q,\alpha_0,\beta_0) = \lambda_n(0,0,\alpha_0,\beta_0) = n + \frac{\beta_0 - \alpha_0}{\pi}, \qquad n \in \mathbb{Z}, \tag{7}$$

then $p(x) = q(x) = 0$ a.e. on $[0,\pi]$?

The answer is negative! More presise, there is infinite set of canonical potentials of the form (4), for which the set of eigenvalues $\{\lambda_n(p,q,\alpha_0,\beta_0)\}_{n\in\mathbb{Z}}$ coinside with the set (7). This infinite set of isospectral potentials described in paper [15].

Example 1. Let

$$\Omega_{m,t}(x) = \frac{\pi(e^t - 1)}{\pi + (e^t - 1)x}\begin{pmatrix} -\sin 2mx & \cos 2mx \\ \cos 2mx & \sin 2mx \end{pmatrix} \tag{8}$$

where t is an arbitrary real parameter ($t \in \mathbb{R}$) and m is an arbitrary integer ($m \in \mathbb{Z}$). Then

$$\lambda_n(\Omega_{m,t},0,0) = \lambda_n(0,0,0) = n, \qquad n \in \mathbb{Z},$$

it est there are infinite (continuum) canonical potintial matrix, differ from zero matrix, which have the same spectra as zero potential.

This example follows from the results of paper [15], where was described all possible canonical Dirac operators, which have the same spectra (isospectral) as an fixed Dirac operator.

But if we take additional conditions, then it is possible to find the analogues of Ambarzumian theorem. For example in paper [16] there is such a theorem:

Theorem 2 ([16]). Let for potential matrix Ω of the form (4) satisfied the conditions:

$$\lambda_n(\Omega,0,0) = \lambda_n(0,0,0) = n \text{ for all } n \in \mathbb{Z}$$

and

$$q(0) = q(\pi). \tag{9}$$

Then $\Omega(x) = 0$ a.e. on $[0,\pi]$.

If we compare this result with example 1, we see that condition (9) require that $e^t - 1 = 0$ i.e. $t = 0$. That means that additional condition (9) convert the infinite set of isospectral potential (8) to unique zero potential.

In paper [17] was described many additional conditions when the inverse problems for Dirac canonical operators can be solved by less spectral data, than it required in general case. In particular, in [17] formulated such analogues of theorem of Ambarzumian.

Theorem 3 ([17]).

1) If $\lambda_n(0, q, \alpha, 0) = n - \frac{\alpha}{\pi}$, $\forall n \in \mathbb{Z}$ and some $\alpha \in \left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right)$, then $q(x) = 0$ a.e. on $[0, \pi]$.
2) If $\lambda_n\left(p, 0, \alpha, \frac{\pi}{4}\right) = n + \frac{1}{4} - \frac{\alpha}{\pi}$, $\forall n \in \mathbb{Z}$ and some $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\alpha \neq \frac{\pi}{4}$, then $\mathrm{p}(x) = 0$ a.e. on $[0, \pi]$.

There are also the analogues of theorem of Ambarzumian for non canonical Dirac operators. In paper [18] was considered the eigenvalue problem

$$\left\{B\frac{d}{dx} + \begin{pmatrix} V(x) + m & 0 \\ 0 & V(x) - m \end{pmatrix}\right\} y = \lambda y, \qquad x \in (0, \pi), \tag{10}$$

$$y_1(0) = y_1(\pi) = 0. \tag{11}$$

In case $V(x) \equiv 0$ we can count the eigenvalues (and eigenfunctions) of (10)-(11):

$$\lambda_0 = -m, \qquad \lambda_k = \sqrt{m^2 + k^2}, \qquad \lambda_{-k} = -\sqrt{m^2 + k^2}, \qquad k = 1, 2, \ldots \tag{12}$$

In paper proved the following

Theorem 4 ([18]). Let $0 < m \leq \frac{1}{2}$ and $V \in C\,[0, \pi]$.

Suppose that (10) and (11) produce the same spectrum as $V(x) \equiv 0$, i.e. that the eigenvalues are $-m$ and $\pm\sqrt{m^2 + k^2}$, $k = 1, 2, \ldots$. Then $V(x) \equiv 0$ on $[0, \pi]$.

This result of Horvath M. was generalized by Marton Kiss in [19] for n-dimentional Dirac operator of (10) form.

Acknowledgment. This work was supported by the RA MES State Committee of Science, in the frames of the research project № 15T-1A392.

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О ТЕОРЕМЕ АМБАРЦУМЯНА ДЛЯ СИСТЕМЫ ДИРАКА

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АННОТАЦИЯ

Мы доказываем что в общем случае для системы Дирака известная теорема Амбарцумяна не имеет места. В тоже время мы описываем частные случаи, для которых имеют место аналоги теоремы Амбарцумяна.

Ключевые слова: обратная задача, теорема Амбарцумяна, каноническая система Дирака.

ՀԱՄԲԱՐՁՈՒՄՅԱՆԻ ԹԵՈՐԵՄԻ ՄԱՍԻՆ ԴԻՐԱԿԻ ՀԱՄԱԿԱՐԳԻ ԴԵՊՔՈՒՄ

Տ.Ն. Հարությունյան

ԱՄՓՈՓՈՒՄ

Մենք ապացուցում ենք որ ընդհանուր դեպքում Դիրակի համակարգի համար Համբարձումյանի հայտնի թեորեմը տեղի չունի: Միաժամանակ մենք նկարագրում ենք մասնավոր դեպքեր, որոնց համար տեղի ունեն Համբարձումյանի թեորեմի անալոգները (նմանակները):

Հիմնաբառեր՝ հակադարձ խնդիր, Համբարձումյանի թեորեմ, Դիրակի կանոնական համակարգ: