On Characterization of Belousov Quasigroups

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The quasigroup $Q(\circ)$ is called a Belousov quasigroup,if the identities

$$x \circ (x \circ y) = y \circ x,$$

$$(x \circ y) \circ y = x,$$

$$x \circ (y \circ x) = (y \circ x) \circ y$$

are valid. A non-trivial Belousov quasigroup is not a Stein quasigroup and not commutative.

The set $O_p^{(2)}Q$ of all binary operations on the set Q is a monoid under the following operations:

$$f \cdot g(x, y) = f(x, g(x, y)), \quad (1)$$

$$f \circ g(x, y) = f(g(x, y), y). \quad (2)$$

Theorem 1. *If $Q(A)$ is a non-trivial Belousov quasigroup, then it is idempotent and $A \cdot A = A^*$, $A \cdot A^* = A \circ A^*$, $A \circ A = \delta_2^1$, $A^* \cdot A^* = \delta_2^2, A^* \circ A^* = A$. So if $Q(A)$ is a non-trivial Belousov quasigroup, then the set $\{\delta_2^1, \delta_2^2, A, A^*, A \cdot A^* = A \circ A^*\}$ is a bigroup of operations (on the set Q), where $A^*(x,y) = A(y,x)$ for every $x, y \in Q$.*

Theorem 2. *In every Belousov quasigroup $Q(\circ)$ the identities $(x \circ y) \circ (y \circ x) = y, (x \circ y) \circ (x \circ (y \circ x)) = y \circ x, (y \circ x) \circ (x \circ (y \circ x)) = x \circ y$ are valid. In a non-trivial Belousov quasigroup $Q(\circ)$, for any $a \neq b$ in Q the set $\{a, b, a \circ b, b \circ a, a \circ (b \circ a)\}$ is a five-element subquasigroup, which is isomorphic to the five-element quasigroup with the following multiplication table:*

	0	1	2	3	4
0	0	2	4	1	3
1	4	1	3	0	2
2	3	0	2	4	1
3	2	4	1	3	0
4	1	3	0	2	4

If we take such subquasigroups as blocks, we obtain a block design on the set Q.

It follows from the Theorem 2 that the non-trivial Belousov quasigroup has at least five elements. The variety of Belousov quasigroups is called a Belousov variety,which is a subvariety of the Mikado variety ([1]). Hence, the Belousov variety has a solvable word problem and is congruence-permutable. Every Belousov quasigroup of prime order is a simple algebra.

The applications of similar quasigroups in cellular automata see in [2].

To solution of the following problem is open.

To which loops are Belousov quasigroups isotopic?

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References

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