PROBLEM OF THERMOELASTICITY FOR AN ORTHOTROPIC PLATE-STRIP OF VARIABLE THICKNESS WITH REGARD FOR TRANSVERSE SHEAR

R. M. Kirakosyan\(^1\) and S. P. Stepanyan\(^2\)

We solve a plane problem of thermoelasticity and a problem of bending for an orthotropic plate-strip with linearly varying thickness under restrained boundary conditions. The results of numerical analyses of displacements, forces, and bending moments in the plate-strip are presented.

Introduction

The contemporary technological progress is connected with extensive applications of various thin-walled structures with elements in the form of plates and shells, which suffer the action of mechanical loads and temperature fields. The problems of thermal stresses are encountered in various fields of modern engineering in which the analyses of strength performed with regard for the thermal actions may be of significant and often of crucial importance.

The extensive bibliographic material on the theory of thermoelasticity of plates and shells and on the determination and investigation of the corresponding temperature fields and stresses in elements of this kind can be found, e.g., in [2–4, 6–9, 12, 13, 21, 22, 24–27].

In numerous works, the Ritz and Bubnov–Galerkin methods, as well as the methods of finite differences and finite and boundary elements are extensively used and rapidly developed [10–28].

The number of works devoted to the strength and stability analyses of smooth thin-walled structures and structures of variable thickness placed in temperature fields with regard for transverse shear is relatively small. In connection with the discussed situation, the investigations of the influence of transverse shear on the thermal stressed state of orthotropic plates carried out in the present work should be regarded as quite actual.

1. In [2], by using the method of representation of the solutions by power polynomials in the transverse coordinate and F. Neumann’s hypothesis [5, p. 330], the equations and boundary conditions of the problem of thermoelasticity were deduced for an orthotropic plate of variable thickness with regard for transverse shears and temperature fields by analogy with [3]. In the present work, these relations are regarded as basic.

Consider an orthotropic plate-strip of width \( \ell \) with linearly varying thickness \( h \). We refer the plate-strip to a system of Cartesian coordinates \( x, y, z \), whose axes are parallel to the principal directions of the material orthotropy. The coordinate plane \( xOy \) coincides with the middle plane and the \( Oz \)-axis is directed perpendicularly to this plane in order to form the right-handed system. We assume that the plate-strip is symmetric about the middle plane and its thickness varies only along the \( Ox \)-axis according the law

---

\(^1\) Institute of Mechanics, Armenian National Academy of Sciences, Yerevan, Armenia.

\(^2\) Yerevan State University, Yerevan, Armenia.

where \( h_0 \) and \( h_1 \) are given parameters. It is also assumed that the surface forces are absent and the surface temperatures of the plate-strip are given. For the temperature \( \theta \), we accept a linear distribution law along the transverse coordinate \( z \): 

\[
\theta = \frac{\theta^+ + \theta^-}{2} + \frac{z}{h} (\theta^+ - \theta^-),
\]

where \( \theta^+ \) and \( \theta^- \) are the values of temperatures on the surfaces

\[
z = \frac{h}{2} \quad \text{and} \quad z = -\frac{h}{2}
\]

of the plate-strip, respectively.

We now introduce the dimensionless variables

\[
u = h_0 \bar{u}, \quad w = h_0 \bar{w}, \quad x = \ell \bar{x}, \quad s = \frac{h_0}{\ell}, \quad h = h_0 H, \quad h_1 = \gamma s,
\]

\[
H = 1 + \gamma \bar{x}, \quad B_{12} = m B_{11}, \quad \varphi_1 = B_{11} \bar{\varphi}, \quad a_{55} B_{11} = \chi,
\]

\[
T_x = B_{11} \bar{T}_x, \quad N = B_{11} h_0 \bar{N}_x, \quad M_x = B_{11} h_0^2 \bar{M}_x,
\]

where \( u \) is an axial displacement of points of the middle plane, \( w \) is a deflection, \( B_{ij} \) are mechanical parameters expressed via the elastic constants of the material by the well-known formulas [1], \( \varphi_1 \) is a function characterizing the distribution of transverse tangential stresses \( \tau_{xx}, \) \( T_x, \) and \( N_x \) are the tangential and transverse forces, and \( M_x \) is a bending moment in the plate-strip.

By using notation (1), we obtain the following formulas for the dimensionless forces and bending moment within the framework of the theory presented in [2]:

\[
\bar{T}_x = H \left[ s \frac{d \bar{u}}{d \bar{x}} - \frac{1}{2} \left( \alpha_x + m \alpha_y \right) (\theta^+ + \theta^-) \right],
\]

\[
\bar{N}_x = \frac{2}{3} H \bar{\varphi} \frac{ysH}{12} \left[ s H \left( \frac{d^2 \bar{w}}{d \bar{x}^2} - \chi \frac{d \bar{\varphi}_1}{d \bar{x}} \right) + (\alpha_x + m \alpha_y) (\theta^+ + \theta^-) \right],
\]

\[
\bar{M}_x = -\frac{s H^3}{12} \left[ s \left( \frac{d^2 \bar{w}}{d \bar{x}^2} - \chi \frac{d \bar{\varphi}_1}{d \bar{x}} \right) + \frac{1}{s H} (\alpha_x + m \alpha_y) (\theta^+ + \theta^-) \right],
\]
where \( \alpha_x \) and \( \alpha_y \) are the coefficients of thermal expansion of the material in the directions of the \( x \)- and \( y \)-axes, respectively.

2. The differential equations of the plane problem and the problem of bending for the analyzed plate-strip have the form

- **equation of the plane problem:**

\[
\frac{d}{dx} \left( H \frac{d\bar{u}}{dx} \right) = \frac{\gamma}{2s} (\alpha_x + m\alpha_y)(\theta^+ + \theta^-);
\]

(2)

- **equations of the problem of bending:**

\[
\frac{d}{dx} (H^2 \bar{\theta}_1) = 0,
\]

\[
\frac{d}{dx} \left( H^2 \frac{d^2 \bar{w}}{dx^2} \right) - \frac{\alpha}{s} \frac{d}{dx} \left( H^2 \frac{d\bar{\theta}_1}{dx} \right) + \frac{8\bar{\theta}_1}{s^3} + \frac{\gamma}{s^2} (\alpha_x + m\alpha_y)(\theta^+ + \theta^-) = 0.
\]

(3)

The solution of Eq. (2) of the plane problem takes the form

\[
\bar{u} = c_2 + c_1 \frac{\ln H}{\gamma} + \frac{1}{2\gamma s} (\alpha_x + m\alpha_y)(\theta^+ + \theta^-)(\gamma x - \ln H).
\]

(4)

We determine the constants of integration \( c_1 \) and \( c_2 \) from the boundary conditions for the plate-strip. In the case where both edges are restrained, these conditions take the form

\[
\bar{u} \bigg|_{x=0, x=1} = 0.
\]

(5)

Hence, the constants \( c_1 \) and \( c_2 \) are given by the formulas

\[
c_1 = \frac{-(\alpha_x + m\alpha_y)(\theta^+ + \theta^-)}{2s \ln (1 + \gamma)} [\gamma - \ln (1 + \gamma)],
\]

\[
c_2 = 0.
\]

(6)

Substituting (6) in (4), for the dimensionless axial displacements, we obtain

\[
\bar{u} = \frac{1}{2s} (\alpha_x + m\alpha_y)(\theta^+ + \theta^-) \left[ \bar{x} - \frac{\ln H}{\ln (1 + \gamma)} \right].
\]

(7)
In view of (7), for the dimensionless tangential force, we find

\[ \bar{T}_x = -\frac{\gamma}{2 \ln(1+\gamma)} (\alpha_x + m\alpha_y)(\theta^+ + \theta^-) . \]  

(8)

As could be expected, the solution of the plane problem for a plate-strip is independent of the shear characteristic of its material (parameter \( \chi \)).

For a plate-strip of constant thickness \((\gamma = 0)\), relations (7) and (8) turn into

\[ \bar{u} = 0, \quad \bar{T}_x = -\frac{1}{2} (\alpha_x + m\alpha_y)(\theta^+ + \theta^-) . \]  

(9)

In Figs. 1 and 2, we present the plots of \( \bar{u} \) and \( \bar{T}_x \) for some values of the parameter \( \gamma \) characterizing the variations of thickness of the plate-strip. It is easy to see that, for \( \gamma > 0 \), the plots of \( \bar{u} \) are asymmetric about the middle of the width \( \bar{x} = 0.5 \).
Consider the problem of bending. The solution of the system of equations (3) takes the form

$$\bar{\varphi} = \frac{c_3}{H^2},$$  \hspace{1cm} \tag{10}$$

$$\bar{w} = c_6 + c_5 \bar{x} - \frac{c_4}{\gamma^2} \ln H + \frac{c_3}{s^3 \gamma^2 H} \left(4 - \chi \gamma^2 s^2\right) - \frac{1}{s^2 \gamma^2} \left(\alpha_x + m \alpha_y\right)(\theta^+ - \theta^-) \left[(1 + H) \ln H - \gamma \bar{x}\right].$$  \hspace{1cm} \tag{11}$$

We now determine the constants of integration $c_3 + c_6$ from the conditions of restraining of both edges of the plate-strip

$$\bar{w} \bigg|_{\bar{x}=0, \bar{x}=1} = 0, \hspace{1cm} \left(\frac{d\bar{w}}{dx} - \chi \bar{\varphi}_1\right) \bigg|_{\bar{x}=0, \bar{x}=1} = 0, \hspace{1cm} u_x = 0.$$  \hspace{1cm} \tag{12}$$

In view of (10) and (7), conditions (12) take the form

$$\frac{4 - \chi \gamma^2 s^2}{\gamma^2 s^3 (1 + \gamma)} c_3 + \frac{\ln (1 + \gamma)}{\gamma^2} c_4 - c_5 = -\frac{\left(\alpha_x + m \alpha_y\right)(\theta^+ - \theta^-)}{\gamma^2 s^2} \left[(2 + \gamma) \ln (1 + \gamma) - \gamma\right],$$

$$\frac{4}{\gamma^2 s^2} c_3 + \frac{s}{\gamma} c_4 - s c_5 = -\frac{1}{\gamma s} \left(\alpha_x + m \alpha_y\right)(\theta^+ - \theta^-),$$

$$\frac{4}{\gamma^2 s^2 (1 + \gamma)} c_3 + \frac{s}{\gamma (1 + \gamma)} c_4 - s c_5 = -\frac{\left(\alpha_x + m \alpha_y\right)(\theta^+ - \theta^-)}{\gamma s} \left[\frac{1}{1 + \gamma} + \ln (1 + \gamma)\right],$$

$$\frac{4 - \chi \gamma^2 s^2}{\gamma^3 s^3} c_3 + c_6 = 0.$$  \hspace{1cm} \tag{13}$$

As a result of the numerical solution of system (13) and finding the constants of integration $c_3 + c_6$, we determine the values of the dimensionless deflection $\bar{w}$, transverse force $\bar{N}_x$, and bending moment $\bar{M}_x$.

In Figs. 3–5, we present the plots of these characteristics for the following numerical values of parameters of the problem:

$s = 0.15, \hspace{0.5cm} \alpha_x = 10^{-5} \text{1/deg}, \hspace{0.5cm} \alpha_y = 2 \alpha_x, \hspace{0.5cm} m = 0.3, \hspace{0.5cm} \theta^+ = 300 \text{deg},$\n
$$\theta^- = 0, \hspace{1cm} \chi = 0, 5, 10, \hspace{1cm} \gamma = 1.$$

CONCLUSIONS

We pose and solve the plane problem and the problem of bending for an orthotropic plate-strip with linearly varying thickness, which suffers the action of temperature fields, with regard for transverse shears.
The results of the performed calculations are used to analyze the dependences of the basic characteristics (displacements, forces, and bending moments) on the parameter $\gamma$ characterizing the variations of thickness of the plate and the parameter $\chi$ characterizing the transverse shear. The corresponding plots are presented.
It is shown that the influence of shears leads to an increase in the value of deflection \( \tilde{\omega} \) and a decrease in the transverse force \( \tilde{N}_x \). As the coefficient \( \chi \) increases, the point of maximum deflection approaches the edge of the plate with the smallest thickness.

It is demonstrated that the horizontal displacements of points of the middle surface are equal to zero for the plate with constant thickness (\( \gamma = 0 \)). If \( \gamma \neq 0 \), then the horizontal displacements are nonzero, and the maximum values of these displacements increase with the parameter \( \gamma \).

As the parameter \( \gamma \) increases, the tangential force \( \tilde{T}_x \) also increases. In this case, the bending moment \( \tilde{M}_x \) weakly varies as a function of the parameter \( \gamma \).

REFERENCES