The spherical sector of the Calogero model as a reduced matrix model

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Abstract

We investigate the matrix-model origin of the spherical sector of the rational Calogero model and its constants of motion. We develop a diagrammatic technique which allows us to find explicit expressions of the constants of motion and calculate their Poisson brackets. In this way we obtain all functionally independent constants of motion to any given order in the momenta. Our technique is related to the valence-bond basis for singlet states.

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1. Introduction and summary

One of the best known multi-particle integrable systems is the Calogero model

\[ H = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + \sum_{i<j} g_i^2 \frac{(q_i - q_j)^2}{(q_i - q_j)^2}, \quad \{ p_i, q_j \} = \delta_{ij}. \] (1)

Being introduced four decades ago \cite{1}, it continues to attract much interest due to its rich internal structure and numerous applications. So far, various integrable extensions have been constructed and studied, in particular, for the trigonometric potentials \cite{2}, for particles with spins \cite{3}, for...
supersymmetric systems [4], and for other Lie algebras [5]. An important feature of all rational Calogero models is the dynamical conformal symmetry \( so(1, 2) \equiv sl(2, \mathbb{R}) \), defined by the Hamiltonian (1) together with the dilatation \( D = \sum_i p_i q_i \) and conformal boost \( K = \sum_i q_i^2 / 2 \) generators,

\[
\{ H, D \} = 2H, \quad \{ K, D \} = -2K, \quad \{ H, K \} = D.
\]

Due to this symmetry one can give an elegant explanation of the superintegrability property of the conformal invariant integrable systems [6] (initially observed by Wojciechowski in Calogero model [7]). The “radial” and “spherical” parts of rational Calogero models can be separated in the Hamiltonian

\[
H = \frac{p_r^2}{2} + \frac{\mathcal{I}(u)}{r^2}, \quad r \equiv \sqrt{2K}, \quad p_r \equiv \frac{D}{\sqrt{2K}}: \quad \{ p_r, r \} = 1, \quad \{ p_r, u^\alpha \} = \{ r, u^\alpha \} = 0,
\]

with the “spherical part” corresponding to the Casimir element of the conformal algebra. Hence, the whole information about the conformal mechanics is encoded in its “spherical part”, given by the Hamiltonian system

\[
\mathcal{I}(u) \quad \text{with} \quad \{ u^\alpha, u^\beta \} = \omega^\alpha{}_{\beta}(u).
\]

This system is of its own interest since it describes a multi-center generalization of the \((N-1)\)-dimensional Higgs oscillator [8]. In the quantum case and for special discrete values of the coupling constant it can be mapped to free-particle systems on the sphere [9]. However, the connection between the constants of motion of the initial conformal mechanics and its spherical sector is highly complicated [6,10]. In particular, it is unclear up to now how to construct the Liouville constants of motion of the spherical sector from the ones of the full conformal mechanics.

On the other hand, the rational Calogero model can be easily constructed from the free Hermitian matrix model via a Hamiltonian reduction [11]. In this way, we get a transparent explanation of its integrability property and the Lax pair formulation. Hence, it is natural to try to explore the matrix origin of the spherical sector of the Calogero model in order to find the matrix-model origin of its constants of motion. In that case we shall immediately get the constants of motion of the spin–Calogero model as well. One may expect that the matrix model formulation of the spherical sector of the Calogero model can simplify the study of its constants of motion. Moreover, such a formulation, being purely algebraic, might establish new relations between the spherical sector of the Calogero model and other algebraic integrable systems, for instance lattice spin systems.

The investigation of the spherical sector of the rational Calogero model and of its constants of motion at the matrix-model level is the goal of the present paper. First we recall the formulation of the Liouville integrals of the initial Calogero model at the matrix-model level in terms of \( U(N) \)-invariant polynomials (and of \( SU(N) \)-invariant polynomials for the Calogero model with the center of mass excluded) corresponding to the highest states of the conformal algebra. Then we observe that the constants of motion of the spherical system are described by \( SU(N) \times SL(2, \mathbb{R}) \) singlets. This allows us to reduce the study of the algebra of invariants of the spherical sector to a purely algebraic computation of \( SU(N) \) invariant tensors. To simplify the calculations, we develop an appropriate diagrammatic technique illustrated by numerous examples. We present explicit expressions for all functionally independent constants of motion up to sixth order in momenta as well as recover the results obtained in [10] by the use of standard methods. Finally, we establish a relation of the developed diagrammatic technique with the valence-bond basis introduced by Temperley and Lieb [12].
The paper is arranged as follows. In Section 2 we give a brief description of the matrix-model formulation of the rational Calogero model, including the description of the reduction procedure and the exclusion of the center of mass at the matrix-model frame. Then we develop a similar formulation for the spherical sector of the Calogero model. In Section 3 we develop the diagrammatic technique for the formulation of the constants of motion of the spherical sector of the Calogero model and find by its use all functionally independent constants of motion up to sixth order in momenta. In Section 4, considering the free-particle limit, we rederive, by the use of our technique, the constants of motion obtained in [10] by standard methods. In Section 5 we establish a correspondence between our technique and the valence-bond basis developed by Temperley and Lieb. An interesting future task concerns the relation of the symmetries of the spherical sector of the Calogero model with \( W \) and Hecke algebras.

2. Matrix-model formulation

We recall that the Calogero model (1) has \( N \) Liouville constants of motion [13–15]

\[ I_k = (L^k) := \text{tr} L^k, \quad 1 \leq k \leq N, \tag{5} \]

given in terms of the Lax matrix

\[ L_{jk} = \delta_{jk} p_k + (1 - \delta_{jk}) \frac{ig}{q_j - q_k}. \tag{6} \]

For convenience, hereafter the trace of a matrix is denoted by round brackets. The Calogero model can be obtained from the free Hermitian matrix model

\[ H = \frac{1}{2} (P^2), \quad P^+ = P, \quad Q^+ = Q, \quad \{P_{ij}, Q_{j'k'}\} = \delta_{ii'} \delta_{jj'} \tag{7} \]

by the reduction corresponding to the \( SU(N) \) group action

\[ P \rightarrow U P U^+, \quad Q \rightarrow U Q U^+, \tag{8} \]

which also preserves the canonical brackets. The related conserved current is given by the traceless Hermitian matrix [14]

\[ J = i [Q, P]. \tag{9} \]

Using this symmetry, one can diagonalize the coordinate matrix

\[ Q_{jk} \rightarrow q_j \delta_{jk}. \tag{10} \]

Then, according to the relation (9), the diagonal matrix elements of \( J \) vanish while the off-diagonal ones define the related elements of the reduced matrix \( P \):

\[ P_{jk} \rightarrow -i \frac{J_{jk}}{q_j - q_k}, \quad j \neq k. \]

The diagonal elements \( p_j \) of the reduced matrix remain independent and are conjugate to the respective coordinates, i.e. \( \{p_j, q_k\} = \delta_{jk} \). The simplest choice \( J_{jk} = -g \) for all \( j \neq k \) recovers the Lax matrix (6) of the usual Calogero model:

\[ P_{jk} \rightarrow L_{jk}. \tag{11} \]

After the gauge fixing, the commutator (9) reduces to
\[ [Q, P]_{jk} = ig(\delta_{jk} - u_j u_k), \quad \text{where } u_i = 1. \]  

(12)

Let us choose an orthogonal basis \( \{ T_a \}, a = 0, 1, \ldots, N^2 - 1 \), for \( u(N) \) with

\[ (T_a T_b) = \delta_{ab}, \quad [T_a, T_b] = \sum_c i f_{abc} T_c \]  

(13)

and real antisymmetric structure constants \( f_{abc} \) and expand the Hermitian matrices

\[ Q = \sum_a Q_a T_a, \quad P = \sum_a P_a T_a. \]  

(14)

According to the decomposition \( U(N) = U(1) \times SU(N) \), we take the identity matrix as a \( U(1) \) generator \( T_0 \), and the remaining \( T_a \) are given by traceless Hermitian matrices. The standard form of the basis is given by (A.1), (A.2), and (A.3) in Appendix A.

The coefficients \( P_a \) and \( Q_a \) form pairs of conjugate momenta and coordinates [14,16]

\[ \{ P_a, Q_b \} = \delta_{ab}, \]  

as follows from the Poisson brackets in (7) and the completeness relation among the \( U(N) \) generators (A.5). We define the angular momentum tensor \( M \) for the matrix model by

\[ M_{ab} = P_a Q_b - P_b Q_a. \]  

(15)

The angular momentum tensor components \( M_{ab} \) form an \( SO(N^2) \) algebra

\[ \{ M_{ab}, M_{a'b'} \} = \delta_{ab} M_{a'b'} + \delta_{ba'} M_{a'b'} - \delta_{aa'} M_{bb'} - \delta_{bb'} M_{aa'}. \]  

(16)

The matrix form of angular momentum tensor is

\[ M = P \wedge Q \equiv P \otimes Q - Q \otimes P = \sum_{a,b} M_{ab} T_a \otimes T_b. \]  

(17)

The components \( J_c = (J T_c) \) of the \( SU(N) \) conserved current (9) are expressed in terms of the angular momentum components as

\[ J_c = -i \sum_{a,b} f_{abc} M_{ab}. \]  

(18)

They obey, of course, the \( su(N) \) algebra

\[ \{ J_a, J_b \} = i \sum_c f_{abc} J_c. \]  

(19)

as can be verified independently using (16).

There is an additional \( U(1) \) symmetry of the matrix system given by the translations

\[ Q \rightarrow Q + \epsilon 1, \quad P \rightarrow P. \]  

(20)

The related conserved current is (see (A.4) in Appendix A)

\[ (P) \sim P_0. \]  

(21)

The \( SU(N) \) shifts preserve it, so it is in involution with the \( SU(N) \) currents \( J_a \), which can be verified also using the relations (7) and (9). Together these currents generate the \( U(1) \times SU(N) = U(N) \) group. The \( U(1) \) reduction eliminates of the center-of-mass momenta and coordinates of the Calogero system,
\[(P) = \sum_i p_i = 0, \quad (Q) = \sum_i q_i = 0.\]

The action of the matrix model, \(S = \frac{1}{2} \int \mathcal{Q}^2 \, dt\), remains invariant under the conformal transformations forming the \(sl(2, R)\) algebra \((2)\) generated by the quantities

\[D = (P Q) = \sum_a Q_a P_a, \quad K = \frac{1}{2} (Q^2) = \frac{1}{2} \sum_a Q_a^2,\]

\[H = \frac{1}{2} (P^2) = \frac{1}{2} \sum_a P_a^2.\]

The standard basis of this algebra has the form

\[J_1 = H + K, \quad J_2 = D, \quad J_3 = H - K, \quad \{J_\alpha, J_\beta\} = -2 \epsilon_{\alpha\beta\gamma} J_\gamma,\]

\[(23)\]

where the indices are raised by the conformal metric \(\text{diag}(1, -1, -1)\). The conformal algebra is in involution with the angular momentum tensor

\[{M_{ab}, sl(2, R)} = 0 \quad (24)\]

and, hence, with the \(su(N)\) algebra \((18), (19)\). The last fact follows also from the invariance of the traces \((22)\) under the gauge transformations \((8)\).

The Casimir element of the algebra \((2)\) looks as follows,

\[I_\alpha = \sum_\alpha J_\alpha J_\alpha = 4 KH - D^2 = (P^2)(Q^2) - (P Q)^2,\]

\[(25)\]

and defines the Hamiltonian of the matrix-model origin of the spherical part of the Calogero model \((3)\), which can be considered as a separate system and hereafter will be called “spherical mechanics”. It can also be expressed in terms of the angular momentum \((15)\),

\[I = \frac{1}{2} (\text{tr} \otimes \text{tr}) M^2 = \sum_{a<b} M_{ab}^2.\]

\[(26)\]

We note that the Casimir maps the Liouville constants of motion \((5)\) to \(N - 1\) additional ones \([6,10]\), which are responsible for the superintegrability of the Calogero model \([7]\):

\[G_k = \{I, I_k\}, \quad k = 1, 3, 4, \ldots, N.\]

\[(27)\]

Note that the conformal generators \((22)\) are composed from independent parts, each specified by one coordinate and momentum component. However, as was mentioned above, the \(SU(N)\) reduction to the Calogero model \((10), (11)\) mixes together all components apart from the first one, which corresponds to the center of mass. Therefore, we have a well defined decomposition of mutually involutive conformal algebras,

\[J_\alpha = J_\alpha^{\text{red}} + J_\alpha^0,\]

\[(28)\]

where \(J_\alpha^0\) is defined by the \(a = 0\) term in the sums \(22)\)

\[D_0 = P_0 Q_0, \quad K_0 = \frac{1}{2} Q_0^2, \quad H_0 = \frac{1}{2} P_0^2,\]

while \(J_\alpha^{\text{red}}\) is determined by the others. For the Casimir element, we have

\[I = I^{\text{red}} + 2 J_\alpha^{\text{red}} J_0^\alpha.\]

\[(29)\]

This is the relation between the spherical systems with and without center of mass.
3. The constants of motion of the spherical mechanics

Before discussing the constants of motion of the spherical mechanics, we rewrite the Liouville constants of motion of the Calogero system (5) in terms of matrix model generators,

$$I_k = (P^k) = \sum_{a_1,...,a_k} d_{a_1...a_k} P_{a_1} \cdots P_{a_k}, \quad 0 \leq a_i \leq N^2 - 1,$$

(30)

where the coefficients $d_{a_1...a_k}$ are $U(N)$ invariant tensors defined by the expressions

$$d_{a_1...a_k} = (T_{a_1} \cdots T_{a_k}).$$

(31)

In particular,

$$d_a = \delta_{a0}, \quad d_{ab} = \delta_{ab}. \quad (32)$$

The first two constants of motion are proportional to the total momentum and Hamiltonian of the system, respectively. Setting $P_0 = 0$ in (30), we obtain the constants of motion for the system with excluded center of mass,

$$I^{\text{red}}_k = \sum_{b_1,...,b_k} d_{b_1...b_k} P_{b_1} \cdots P_{b_k}, \quad 1 \leq b_i \leq N^2 - 1.$$

(33)

Since the first constant of motion vanishes due to (32), only $N - 1$ independent Liouville integrals remain. Then the relation between the constants of motion of the Calogero system with and without mass center reads

$$I_k \to \sum_{i=0}^{k} N^{\frac{i-k}{2}} \binom{k}{i} P_0^{k-i} I^{\text{red}}_i,$$

(34)

where we set $I^{\text{red}}_0 = 1$ and $I^{\text{red}}_1 = 0$. It is a consequence of the relation between the invariant tensors of the $SU(N)$ and $U(N)$ groups,

$$d_{a_1...a_k} = N^{-\frac{k-k'}{2}} d_{a_1'...a_{k'}}.$$  

(35)

where $r_1, \ldots, r_{k'}$ are the positions of the indices with nonzero values taken in ascending order, i.e. $a_{r_j} > 0$. Eq. (35) follows from (A.1), (A.2) and (31). A similar relation can be derived between the additional integrals $G_k$ of both systems using their expressions (27) and (29).

The constants of motion of the spherical mechanics (25) have to be in involution with the whole conformal algebra (2) since they are expressed in terms of the angular coordinates and momenta, while the conformal algebra generators depend on the radial coordinate and momentum only. Therefore, they are $sl(2, R)$ singlets. On the other hand, the integrals must be also $SU(N)$ scalars in order to assure a valid reduction map. So, the algebra of integrals of the spherical mechanics is formed by $SU(N) \times SL(2, R)$ singlets. In this section, we construct them from $SL(2, R)$ invariants by combining them in an appropriate way in order to obtain an $SU(N)$ invariant. This approach provides the integrals of the spherical Hamiltonian with a simple graphical picture.

As was mentioned above, any $SL(2, R)$ invariant can be expressed in terms of angular momentum tensor components $M_{ab}$ with indices belonging to the adjoint representation of $SU(N)$. At the same time, an $SU(N)$ invariant can be constructed by contraction of the monomials $M_{a_1 b_1} \cdots M_{a_k b_k}$ with a number of invariant tensors (31). This observable will be a polynomial...
constant of motion of \( \mathcal{I} \) of \( k \)th order both in \( M_{ab} \) and momenta \( P_a \). It can be presented in graphical form by drawing the angular momentum tensor as a vector with the endpoints endowed with the corresponding indices. The aligned endpoints inside a cycle mean the contraction of the related indices with the invariant tensor as shown in Fig. 1. The entire diagram consists of vectors with endpoints distributed along legs of such type. Among these diagrams, some are expressed in terms of others or vanish. In particular, the quadratic bond-crossing relations among the components of the momentum tensor,

\[
M_{a'b'}M_{ab} = M_{ab}M_{a'b'} - M_{aa'}M_{bb'},
\]

which is a consequence of the definition \( (17) \) and presented diagrammatically in Fig. 2, reduce significantly the number of functionally independent integrals.

It is clear that an invariant corresponding to a disconnected diagram is just the product of the invariants corresponding to its connected parts. Although the order of indices inside the leg is relevant up to cyclic permutations, any permutation changes the observable by integrals of similar type carrying lower orders in the momenta. This follows from the crossing relations \( (36) \), the antisymmetry of the angular momentum tensor and the definition of the \( SU(N) \) current \( J_c \) \((18)\), which becomes a number upon the reduction:

\[
\sum_{a,b}(\ldots T_a, T_b \ldots)M_{aa'}M_{bb'} = \frac{i}{2} \sum_{a,b,e} f_{abe}(\ldots T_e \ldots)M_{ab}M_{a'b'} = -\frac{1}{2} \sum_e J_c d_{\ldots e \ldots}M_{a'b'}.
\]

Moreover, if two indices of an invariant tensor are attached to the same \( M_{ab} \), the corresponding integral is reduced to a combination of lower-order integrals. Indeed, for contracted adjacent indices we have

\[
\sum_{a,b} d_{ab \ldots}M_{ab} = \sum_{a,b}(T_a T_b \ldots)M_{ab} = \sum_{a,b,c} \frac{i}{2} f_{abc}d_{c \ldots}M_{ab} = -\sum_e J_c d_{c \ldots}.
\]

Therefore, without loss of generality, one may consider a diagram topologically equivalent to one with invariant tensors located along a single line or cycle with mutually nonintersecting angular momentum bonds.

As an example, consider the analogue of the Liouville constants of motion \( (30) \) of the original Calogero Hamiltonian. For the spherical Hamiltonian, one must use the angular momentum instead of the moment:

\[
\mathcal{I}_k = (M^{2k}) := (\text{tr} \otimes \text{tr})M^{2k} = \sum_{a_1, b_1} d_{a_1 \ldots a_{2k}} d_{b_1 \ldots b_{2k}} M_{a_1 b_1} \ldots M_{a_{2k} b_{2k}}.
\]
Fig. 3. Diagrammatic representations for the integrals $I_k$ (a) and $I'_k$ (b) and the spherical Hamiltonian (c).

Fig. 4. A double dot leg can be replaced by a single dot.

Note that due to the antisymmetry of $M_{ab}$, the odd powers vanish. The related diagram is shown in Fig. 3(a). In contrast to their analogue (30), these spherical integrals are not in involution. Their bracket equals

$$\{I_n, I_m\} = 4nmI_{n,m},$$

where $I_{n,m}$ is a combined diagram glued in a way presented in Figs. 3(b) and 3(c). The above relation is a consequence of (16), the cyclic symmetry of the invariant tensors (31) and the completeness relation among them,

$$\sum_{a=0}^{N^2-1} d_{a_1 a_2 ... a_{n+m-1}} = d_{a_1 a_2 ... a_{n+m-1}},$$

which follows from (A.5). For the system with reduced center of mass, the last relation reads, according to (A.6),

$$\sum_{b=1}^{N^2-1} d_{b_1 b_2 ... b_{n+m-1}} = d_{b_1 b_2 ... b_{n+m-1}} + \frac{1}{N} d_{b_1 b_2 ... b_{n+m-1}} d_{b_2 ... b_{n+m-1}}.$$

As a consequence, the commutator (38) acquires the following form:

$$\{I_{n,m}^{\text{red}}, I_{n,m}^{\text{red}}\} = 4nm \left( I_{n,m}^{\text{red}} - \frac{1}{N} I_{n,m}^{\text{red}} \right),$$

where $I_{n,m}^{\text{red}}$ is derived from $I_{n,m}$ by splitting its longest leg as shown in Fig. 5.

Since $d_{ab} = \delta_{ab}$, a double-dot leg can be replaced by a single dot as shown in Fig. 4. So, the first integral from the set (37) just coincides with the spherical Hamiltonian itself: $I_1 = I$ (see Fig. 5(c)). The single-dot leg exists only in the presence of the center of mass, because $d_{a} = \delta_{0a}$ (see the same figure). This property reduces significantly the number of independent invariants for the spherical mechanics without mass center and simplifies their classification.
Fig. 5. Diagrammatic representations of some spherical invariants. The invariants of type $I_{n,m}$ and $I'_{n,m}$ are obtained from the commutators of $I_n$ and $I_m$ drawn in Fig. 3.

Fig. 6. The two first diagrams correspond to vanishing invariants: they undergo a sign change under the reflection with respect to the symmetry axis indicated by the dotted line. The invariants (43) are given by an $n$-sided polygon.

Another family of constants of motion can be obtained by contracting the first indices of two adjacent angular momentum tensors (via the invariant $\delta_{aa'}$) and contracting their second indices with some $(2k)$th-order invariant tensor:

$$I'_k = \left( R^k \right) = \sum_{a_1, b_1, b'_1} d_{b_1 b'_1 \ldots b'_k} M_{a_1 b_1} M_{a_1 b'_1} \ldots M_{a_k b_k} M_{a_k b'_k}, \quad R = (\text{tr} \otimes 1) M^2. \quad (42)$$

This set has been considered in [17] as constants of motion for the system with the Hamiltonian $HK$. Note that the first constant of motion from this set also coincides with the spherical Hamiltonian, $I'_1 = I$.

More general invariants may contain nontrivial loops of angular momentum bonds, which include more invariant tensors. The number of bonds yields the order in momenta or angular momenta of the related invariant. In Figs. 5, 6 and 7 some other examples of constants of motion of the spherical Hamiltonian are presented. Let us write down, for example, the invariant corresponding to the sixth-order diagram in Fig. 7(c):

$$\sum_{\text{all indices}} d_{ab'c} d_{a'b'c'} M_{ad} M_{a'd} M_{bb'} M_{cc'},$$

Of course, only a finite number of diagrams are functionally independent. In the presence of a symmetry altering the overall sign, the related observable vanishes. Two such examples are shown in Fig. 6. Consider, for instance, the invariant

$$\sum_{a_1 \ldots a_n} M_{a_1 a_2} M_{a_2 a_3} \ldots M_{a_n a_1} = \text{tr} M^n \quad (43)$$

where, in contrast to $M$ defined in (17), $M = (M_{ab})$ is the angular momentum tensor treated as a matrix. It is described by an $n$-sided polygon as shown in Fig. 6. For odd values of $n$, they vanish due to the antisymmetry with respect to the inversion $M_{ab} \rightarrow M_{ba}$ of all arrows. For even values
of \( n \), they correspond to the Casimir invariants of \( SO(N^2) \). As can be easily verified, the crossing relation (36) implies that

\[
\mathcal{M}^n = \frac{1}{2} (\text{tr} \mathcal{M}^2) \mathcal{M}^{n-2} = \mathcal{I} \mathcal{M}^{n-2}.
\]

Hence, the invariant (43) is just a power of the spherical Hamiltonian (see Fig. 6(a)),

\[
\text{tr} \mathcal{M}^n = \mathcal{T}^{n/2}.
\]

Using the relation (35), which expresses the \( U(N) \) invariant tensors in terms of \( SU(N) \) ones, one can extend to the spherical invariants the relation between the invariants of the conformal mechanics with and without center of mass (34). For the spherical Hamiltonians this relation is given by (29). The general case can be treated using the split sum relation

\[
\sum_{a_1, a_2 = 0}^{N^2-1} d^{(k_1)}_{...a_1...} d^{(k_2)}_{...a_2...} M_{a_1a_2} = \sum_{b_1, b_2 = 1}^{N^2-1} d^{(k_1)}_{...b_1...} d^{(k_2)}_{...b_2...} M_{b_1b_2} + \frac{1}{\sqrt{N}} \sum_{b=1}^{N^2-1} (d^{(k_1)}_{...b...} d^{(k_2-1)}_{...} - d^{(k_1-1)}_{...b...} d^{(k_2)}_{...}) M_{b0},
\]

where \( k_1, k_2 \) denote the order of the two invariant tensors. As a result, a spherical invariant decomposes into \( 3 \sum k \) parts and can be combined to a polynomial in \( M_{0b} \). Its free term is just the version of the original integral with excluded center of mass.

4. Independent invariants and free-particle limit

From the above observation, hereafter, we consider the spherical mechanics without center of mass. It is a superintegrable system on the \( (N - 2) \)-dimensional sphere with \( 2N - 5 \) functionally independent constants of motion. Apart from the spherical Hamiltonian itself (see Fig. 3(c)), there are no invariants of second order in momenta (or angular momenta). There is no nontrivial third-order invariant, but we have three independent invariants of fourth order and a single fifth-order invariant, which are depicted respectively in Figs. 7 and 8. Here, functional independence is understood in the \( N \to \infty \) limit. For fewer particles, additional algebraic relations restricting the number of independent integrals appear.

There are sixteen sixth-order invariants. In order to simplify the graphics and save space, we collapse each leg into a single dot. For example, the fourth-order diagrams in Fig. 7 are
equivalent to those depicted in Fig. 9. Then all independent sixth-order invariants are shown in Fig. 10. The others either vanish or are expressed in terms of second- and fourth-order invariants. This result can be established by computer algebra and applying the free-particle limit of the spherical mechanics.

In the $g \to 0$ limit the constants of motion significantly simplify like they do for the full Calogero model. Recall that the reduction of the matrix model to the Calogero system preserves only the diagonal elements of the matrix of coordinates (10). In the free-particle limit, in addition, solely the diagonal elements of $P = L$ survive according to (5). Therefore, only the components $Q_k$, $P_k$ and $M_{kl}$ corresponding to the Cartan subalgebra survive in the free-particle limit. Note that the restriction of the invariant tensor to the Cartan subalgebra are totally symmetric. The $g \to 0$ limit corresponds to the highest-order term in momenta:

$$I_k(g = 0) = \sum_{i_1 \ldots i_k=1}^N d_{i_1 \ldots i_k}^p p_{i_1} \ldots p_{i_k} = \sum_{i=1}^N p_i^k.$$  (45)
Here the usual coordinates of the Calogero model are used for simplicity. The related basis is obtained by replacing \( T_i \rightarrow E_{i+1}^1 \) in (A.1), (A.2), and the decomposition of a diagonal element is \( P = \sum_{i=1}^N p_i E_{ii} \). In this basis, the indices of nonzero entries of the \( U(N) \) invariant tensor coincides:

\[
d_{i_1i_2...i_k} = (E_{i_1i_1} \cdots E_{i_ki_k}) = \delta_{i_1i_2} \delta_{i_1i_3} \cdots \delta_{i_1i_k}, \quad 1 \leq i \leq N. \tag{46}
\]

Coming back to the invariants of the spherical mechanics, we see that the simplified graphs defined in Figs. 9 and 10 acquire a clear meaning in the free-particle limit: a single point is labeled by a single index. For example, the highest-order terms for the invariants (37) and (42) are, respectively,

\[
I_k(g = 0) = \sum_{i,j=1}^N M^2_{ij}^{k}, \quad I'_k(g = 0) = \sum_{i=1}^N (M^2_{ii})^k.
\]

In Fig. 10 the highest-order terms of all sixth-order invariants are presented. For a large enough number of particles they are independent.

As an example, consider the two-dimensional spherical mechanics \( I_{\text{red}} \) inherited from the four-particle Calogero model with reduced center of mass. Apart from the Hamiltonian, it has two independent invariants: one of them can be chosen to be of fourth order in momenta, the other one of sixth order. The remaining constants of motion are expressed in terms of them. In our recent paper [10], we have constructed such type of spherical invariants starting from ones for the conformal mechanics. Now, using computer algebra, we can express them in terms of the diagrammatic invariants:

\[
J_2 \sim \frac{3}{4} \frac{1}{2}, \quad J_{3/2}^{1/2} = \frac{21}{6} \frac{3}{2} + \frac{3}{64}.
\]

Note that the double-dot leg in the last terms of both expressions corresponds to the Hamiltonian.

5. Relation with the valence-bond basis

In the previous sections, we have constructed the constants of motion of the spherical mechanics by combining the conformal algebra invariants \( M_{ab} \) with the help of unitary invariant tensors. Alternatively, one can obtain the same constants by proceeding in the opposite order, namely by combining the unitarily invariant multiplets of the conformal algebra into conformal singlets. This method is dual to the previous one. Although the results are similar, the first approach is simpler for the description and for calculations while the second one interprets the constants of motion in terms of the valence-bond basis for spin singlets [12].

First, we consider the observables of the matrix mechanics (7), which are formed by the traces of the strings formed by a product of \( P \) and \( Q \) matrices:

\[
O^{\sigma_1...\sigma_n} = \sum_{a_1...a_n} d_{a_1...a_n} A_{a_1}^{\sigma_1} \cdots A_{a_n}^{\sigma_n} = \left( A^{\sigma_1} \cdots A^{\sigma_n} \right), \quad \sigma_i = \pm, \ A^+ = P, \ A^- = Q. \tag{47}
\]

Here we use \( A^\pm \) instead of the dynamical variables for later convenience. The canonical Poisson brackets now read

\[
\{ A^\sigma_{ij}, A^{\sigma'}_{jj'} \} = \epsilon_{\sigma\sigma'} \delta_{ii'} \delta_{jj'}, \tag{48}
\]
where $\epsilon_{\sigma\sigma'}$ is the antisymmetric tensor with $\epsilon_{++} = 1$. The quantities (47) are $SU(N)$ invariant. Due to the above brackets, they form the linear Poisson algebra [18]

$$\{O^{\sigma_1...\sigma_n}, O^{\sigma'_1...\sigma'_m}\} = \sum_{i,j} \epsilon_{\sigma_i\sigma'_j} O^{\sigma_i+1...\sigma_i-1\sigma'_j+1...\sigma'_j-1}. \tag{49}$$

This relation follows also from the completeness relation among the $U(N)$ invariant tensors (39). Here the cyclic ordering in the last trace is implied. Note that, in the quantum case, polynomial corrections appear due to the ordering issue between the matrix elements [19].

The angular momentum components can be expressed in terms of the newly defined matrices as

$$M_{ab} = \sum_{\sigma_1,\sigma_2} \epsilon_{\sigma_1\sigma_2} A^{\sigma_1}_a A^{\sigma_2}_b. \tag{50}$$

The invariants of spherical mechanics may be expressed in terms of (47) and $\epsilon_{\sigma\sigma'}$. Namely, the $i$th dot of the related diagram (see, for instance, Figs. 3 and 5) is marked now by $\sigma_i$, while the legs and bonds are associated with $O^{\sigma_1...\sigma_n}$ and $\epsilon_{\sigma_1\sigma_2}$ correspondingly, as shown in Fig. 11. This is a “dual” interpretation of the diagram. It can be obtained by substituting (50) into expressions for the invariants, like (37) or (42), with the subsequent use of (47) and (31). Commutation relations between the invariants, analogous to (38) and (41), can then be derived using the necklace relations (49).

Of course, only a finite amount of the described observables are independent. Moreover, upon the reduction to the Calogero system by the gauge fixing (10) and (12), the commutation relations (12) establish some order in a string of matrices $P$ and $Q$. Most natural is either the normal ordering or the Weyl (symmetrized) ordering [20–22], which are defined, respectively, by

$$I_{kl} = \langle P^k Q^l \rangle, \tag{51}$$

$$I_{kl}^{\text{sym}} = \text{Sym}(P^k Q^l) = \left(\begin{array}{c} k+l \\ l \end{array}\right)^{-1} \oint \frac{dz}{2\pi i z^{l+1}} \text{tr}(P + z Q)^{k+l}. \tag{52}$$

The symmetrized traces (52) with $k + l = 2s$ form a $(2s + 1)$-dimensional (nonunitary) $sl(2, R)$-representation of conformal spin $s$ with the highest weight vector given by the Liouville integral $I_{2s}$ of the Calogero model (30). Using the usual notation of representation theory, they read

$$I_{s+m,s-m}^{\text{sym}} = O^{(\sigma_1...\sigma_{2s})} = \left(\begin{array}{c} 2s \\ s-m \end{array}\right)^{-\frac{1}{2}} \psi_{sm}, \quad \text{where} \quad (\sigma_1...\sigma_{2s}) = \underbrace{+ \cdot \cdot \cdot +}_{s+m} - \underbrace{\cdot \cdot \cdot -}_{s-m} \tag{53}$$

and $2m = k - l$. The symmetrization is performed over the bracketed indices. The action of the conformal generators (22) can be verified using the canonical brackets (48).

The quantities $I_{k0} = I_k$ and $I_{k1}$ coincide with their symmetrized counterparts and form a closed Lie algebra. Their quadratic combinations $I_k I_l - I_l I_k$ lead to the additional integrals of the Calogero system [7]. However, the whole set of observables (51) or (52) does not form a closed Poisson algebra. Indeed, in the case of normal ordering, according to (49) we have the following decomposition of the Poisson-bracket structure constants:

$$\{I_{kl}, I_{k'l'}\} = (k' - k')(I_{k+k'-1} + \cdot \cdot \cdot + I_{k+k'-1} + g \cdot \text{lower-length strings}).$$
Here the strings with no more than \( l + l' - 2 \) copies of \( Q \) and \( k + k' - 2 \) copies of \( P \) appear after their rearrangement according to the normal ordering. Apart from \( I_{ij} \), these ordering terms contain polynomials in \( \langle v|P^k Q^l|v \rangle \), which together with the \( I_{ij} \) form a closed nonlinear algebra [21].

A similar structure occurs for the brackets between Weyl-ordered observables,
\[
\{ \psi_{s_1 m_1}, \psi_{s_2 m_2} \} = 2(s_2 m_1 - s_1 m_2) \psi_{s_1 + s_2 - 1 m_1 + m_2} + g \cdot \{ \text{ordering terms} \},
\]
where the lower-order terms, in general, have no symmetrized form like (52) or (53). Since the \( sl(2, R) \) algebra acts additively on the Poisson brackets due to the Jacobi identity, we get some kind of angular momentum sum rule for the \( sl(2, R) \) representations of conformal spins \( s_1 \) and \( s_2 \). Here we have a kind of nonassociative wedge product, where only the symmetric terms survive:
\[
(s_1) \wedge (s_2) = (s_1 + s_2 - 1) \oplus (s_1 + s_2 - 3) \oplus \cdots = \bigoplus_{k=0}^{[s_1 + s_2 - 1]} (s_1 + s_2 - 1 - 2k).
\]
In fact, the above properties of the Weyl-ordered quantities follow from the symmetrized analogue of the necklace relation,
\[
\{ O^{(\sigma_1, \ldots, \sigma_{2s})}, O^{(\sigma'_1, \ldots, \sigma'_{2s'})} \} = \sum_{\sigma_i, \sigma'_j} \epsilon_{\sigma_i \sigma'_j} O^{(\sigma_1, \ldots, \sigma_{2s})(\sigma'_1, \ldots, \sigma'_{2s'})},
\]
The right part of this equation has the structure of a tensor product of two \( sl(2, R) \) multiplets with conformal spins \( s - \frac{1}{2} \) and \( s' - \frac{1}{2} \). Its contraction by \( \epsilon_{\sigma_1 \sigma'_1} \cdots \epsilon_{\sigma_s \sigma'_{s'}} \) with subsequent symmetrization over the remaining spins projects into the spin \( s + s' - k - 1 \) representation. Since the left part of (56) is antisymmetric under the exchange of the two spin sets \( \{ \sigma \} \) and \( \{ \sigma' \} \), only even values of \( k \) survive, which proves (55). Using (53) and the Clebsch–Gordan decomposition, one can derive from (56) the following brackets between the usual spin projection states:
\[
\{ \psi_{s m}, \psi_{s' m'} \} = \sum_{k=0}^{[s + s' - 1]} \left( \sqrt{(s + m)(s' + m')} C^{s + s' - 2k - 1 m + m'}_{s - \frac{1}{2} m - \frac{1}{2}, s' - \frac{1}{2} m' + \frac{1}{2}} - \sqrt{(s - m)(s' + m')} C^{s + s' - 2k - 1 m + m'}_{s - \frac{1}{2} m + \frac{1}{2}, s' - \frac{1}{2} m' - \frac{1}{2}} \right) 2 \sqrt{s s'} \psi_{s + s' - 2k - 1 m + m'}. \]
One may replace all strings (47) in the expression for the spherical invariant by their symmetrized counterparts (53). In the dual description of the previous section, this substitution is equivalent to the use of symmetrized \( SU(N) \)-invariant tensors
\[
d_{a_1 \ldots a_n}^{\text{sym}} = \frac{1}{n!} \sum_{p \in S_n} d_{a p_1 \ldots a p_n}.
\]
Of course, the new set of invariants differs from the old one. Although they appear to be more cumbersome, the related diagrams are simpler. Due to the symmetry, all dots which form a multiplet are equivalent in a particular string. So, it is more natural to use a single dot for a multiplet instead of a leg as in the previous section, where the free-particle limit has been discussed. In contrast to that case, however, the dots (see, for example, Figs. 9 and 10) must be labeled by multiple indices as before.

The relation (36) depicted in Fig. 2 is the well-known valence-bond crossing relation for four spin-1/2 singlet states. In the theory of spin systems, the described states are known as
a valence-bond basis for spin-singlet states. It is overcomplete. A true basis is formed by the Temperley–Lieb noncrossing states: the multiplets are positioned along a single line or circle, and any bond distribution is allowed if it respects the spin of the multiplet and avoids the crossing. The Temperley–Lieb basis is nonorthogonal, even for \( su(2) \) spins. All states presented in Figs. 9 and 10 are elements of the Temperley–Lieb basis. After the \( SU(N) \) reduction the constructed invariants form a functional representation for \( SL(2, R) \) singlet states, like the simplest one considered in [23] for the \( SU(2) \) \( s = 1/2 \) spins.

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Appendix A. Generators of the \( U(N) \) group

Here we present an orthogonal basis of Hermitian matrices, which respects the decomposition \( u(N) = u(1) \oplus su(N) \). Let

\[
T_0 = \frac{1}{\sqrt{N}} 1
\]

be the \( u(1) \) phase generator, and the remaining \( N - 1 \) generators span the \( su(N) \) subalgebra:

\[
T_k = \frac{1}{\sqrt{k(k+1)}} \left( \sum_{i=1}^{k} E_{ii} - k E_{kk} \right), \quad 1 \leq k \leq N - 1.
\]

(A.2)

Here, \( E_{ij} \) are the matrices with vanishing entries except for one in the \( i \)th row and \( j \)th column. The remaining \( T_a \) are given by the off-diagonal matrices

\[
\frac{1}{\sqrt{2}} (E_{jk} + E_{kj}), \quad \frac{i}{\sqrt{2}} (E_{jk} - E_{kj}), \quad j > k.
\]

(A.3)

Together with the orthogonality condition (13), the basic matrices obey

\[
\text{tr} T_a = \delta_{a0}.
\]

(A.4)

In addition, all generators obey the \( u(N) \) completeness relation

\[
\sum_{a=0}^{N-1} T_{ij}^a T_{kl}^a = \delta_{il} \delta_{jk}.
\]

(A.5)

The corresponding relation for the \( su(N) \) generators is

\[
\sum_{a=1}^{N-1} T_{ij}^a T_{kl}^a = \delta_{il} \delta_{jk} - \frac{1}{N} \delta_{ij} \delta_{kl}.
\]

(A.6)

After the \( su(N) \) reduction, the diagonal entries of the reduced matrices (10) and (11) define the coordinates and momenta of the Calogero system (1). Using definitions (A.1) and (A.2), their relation with the new coordinates \( Q_i = (QT_i) \) and \( P_i = (PT_i) \) can be derived [8,24]:
\[ Q_k = \frac{1}{\sqrt{k(k+1)}}(q_1 + \cdots + q_k - kq_{k+1}), \quad 1 \leq k \leq N - 1, \]
\[ Q_0 = \frac{1}{\sqrt{N}}(q_1 + \cdots + q_N), \]
\[ P_k = \frac{1}{\sqrt{k(k+1)}}(p_1 + \cdots + p_k - kp_{k+1}), \quad 1 \leq k \leq N - 1, \]
\[ P_0 = \frac{1}{\sqrt{N}}(p_1 + \cdots + p_N). \]

These are the usual Jacobi coordinates, which are used in scattering theory in order to eliminate the center of mass [25]. Here it corresponds to the \( U(1) \) reduction [16]. The center of mass is excluded simply by imposing \( Q_0 = P_0 = 0 \). Note that \( q_i = (QE_{ii}) \) and \( p_i = (PE_{ii}) \). Since both bases \( E_{ii} \) and \( T_i \) are orthonormal, the transformation (A.7) is orthogonal. Therefore, the kinetic therm of the original Calogero model remains unchanged,

\[ H_{\text{red}} = \frac{1}{2} \sum_{i=1}^{N-1} P_i^2 + \sum_{1 \leq i < j \leq N-1} \frac{g^2}{(\alpha_{ij} Q)^2}, \]

where \( \alpha_{ij} = E_{ii} - E_{jj} \) are the roots of the \( su(N) \) algebra.

References