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M a t h e m a t i c s

ON REGULAR PARAMEDIAL DIVISION ALGEBRAS

D. N. HARUTYUNYAN *

Chair of Algebra and Geometry, YSU, Armenia

In this paper n-ary regular division algebras are discussed, which are satisfying the hyperidentity of paramediality. It is shown that every operation in n-ary regular paramedial division algebra will be linearly represented over the same Abelian group. Similar results already obtained for regular medial division algebras in [1].

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Introduction and Preliminary Notions. A (Q, f) n-ary groupoid is called medial, if it satisfies the mediality identity:

$$f(f(x_{11},\ldots,x_{1n}),\ldots,f(x_{n1},\ldots,x_{nn})) = f(f(x_{11},\ldots,x_{n1}),\ldots,f(x_{1n},\ldots,x_{nn})).$$

Algebra (Q,Σ) is called medial, if it satisifies the mediality hyperidentity [2–4]:

$$X(Y(x_{11},\ldots,x_{1n}),\ldots,Y(x_{m1},\ldots,x_{mn})) = Y(X(x_{11},\ldots,x_{m1}),\ldots,g(x_{1n},\ldots,x_{mn})).$$

The (Q, f) n-ary groupoid is called paramedial, if it satisfies the paramediality identity:

$$f(f(x_{11},\ldots,x_{n1}),\ldots,f(x_{1n},\ldots,x_{nn})) = f(f(x_{nn},\ldots,x_{n1}),\ldots,f(x_{1n},\ldots,x_{11})).$$

Algebra (Q,Σ) is called paramedial, if it satisifies the paramediality hyperidentity:

$$X(Y(x_{11},\ldots,x_{n1}),\ldots,Y(x_{1m},\ldots,x_{nm})) = Y(X(x_{nm},\ldots,x_{n1}),\ldots,g(x_{1m},\ldots,x_{11})).$$

Some types for paramedial n-ary groupoids are described in [5], and some types for binary paramedial algebras are described in [6].

A non empty set Q with n-ary operation A is called n-groupoid.

The sequence $x_n, x_{n+1},\ldots,x_m$ is denoted by x_n^m, where $n,\ m$ are natural numbers, $n \le m$. If $n = m$, then x_n^m is the element x_n. The sequence $x_m, x_{m-1},\ldots,x_n$ is denoted by ${}_n^m x$, where $n,\ m$ are natural numbers, $n \le m$. If $n = m$, then ${}_n^m x$ is the element x_n. The sequence $a, a, \ldots, a$ (m times) is denoted by a_m.

* E-mail: david.harutyunyan96@gmail.com

Definition 1. *Let (Q,A) be n-groupoid and (Q,B) be m-groupoid. We will say that (Q,B) is retract of (Q,A), if $m \leq n$ and there are $a_1,...,a_{n-m} \in Q$ and $k_1,...,k_{n-m} \in 1,...,n$, such that $B\left(x_1^m\right) = A\left(x_1^{k_1-1}, a_1, x_{k_1+1}^{k_2-1}, ..., x_{k_{n-m-1}+1}^{k_{n-m}-1}, a_{n-m}, x_{k_{n-m}+1}^n\right)$.*

Let (Q,A) be an n-groupoid. Denote by $L_i\left(a_1^n\right)$ a mapping from Q to Q such that

$$L_i\left(a_1^n\right)x = A\left(a_1^{i-1} x a_{i+1}^n\right),$$

for all $x \in Q$. The mapping $L_i(a_1^n)$ is called the i-translation with respect to a_1^n.

Definition 2. *Let (Q,A) be an n-groupoid. We will say (Q,A) is division n-groupoid if $L_i(a_1^n)$ is a surjection for all $a_1^n \in Q$ and $i = 1,...,n$.*

It's easy to see that every retract of paramedial division n-groupoid is also paramedial.

Let denote by $L_i^A\left(a_1^{|A|}\right)$ the i-translation of the algebra (Q,Σ) with respect to element $a_1^{|A|} \in Q^{|A|}$, where $|A|$ is the arity of the operation A.

Definition 3. *The algebra (Q,Σ) is called division algebra, if every $L_i^A\left(a_1^{|A|}\right)$ is a surjection for all $a_1^{|A|} \in Q^{|A|}$, $A \in \Sigma$ and $i = 1,...,n$.*

An n-groupoid is called i-regular if

$$L_i(a_1^n)c = L_i(b_1^n)c \implies L_i(a_1^n) = L_i(b_1^n),$$

for all $a_1^n, b_1^n, c \in Q$. An n-groupoid is called regular if it's regular for all $i = 1,...,n$. It's easy to see that every retract of regular n-groupoid is also regular.

The algebra (Q,Σ) is called i-regular, if $L_i^A\left(a_1^{|A|}\right)c = L_i^A\left(b_1^{|A|}\right)c$ implies that $L_i^A\left(a_1^{|A|}\right) = L_i^A\left(b_1^{|A|}\right)$. If (Q,Σ) is i-regular for all $i = 1,...,|A|$, then it's called regular.

Definition 4. *A groupoid (Q,A) is homotopic to a groupoid (Q,B), if there exist such mappings α,β,γ from Q to Q that the equality $\gamma A(x,y) = B(\alpha x, \beta y)$ is valid for any $x,y \in Q$. Then the triad (α,β,γ) is a homotopy from (Q,A) to $Q,B)$. If $\gamma = id_Q$, then we say that these groupoids are principally homotopic.*

Definition 5. *A mapping γ from Q to Q is called a homotopy of a groupoid (Q,A), if there exist such mappings α,β from Q to Q that the triad (α,β,γ) is a homotopy from (Q,A) to (Q,A).*

Definition 6. *A mapping ϕ from Q to Q is a quasiendomorphism of a group $(Q,\cdot)$, if*

$$\phi(x \cdot y) = \phi x \cdot (\phi 1)^{-1} \cdot \phi y$$

or all $x,y \in Q$, where 1 is the identity of the group $(Q,\cdot)$.

Lemma 1. *If the group $(Q,\cdot)$ is principally homotopic to the group $(Q,+)$, then they are isomorphic and $x \cdot y = x + y + l$ for all $x,y \in Q$, where $l \in Q$.*

Lemma 2. *Let ϕ be a quasiendomorphism of the group $(Q,\cdot)$, then ϕ is endomorphism of the group $(Q,\cdot)$ if and only if $\phi e = e$, where $e \in Q$ is the identity of the group $(Q,\cdot)$.*

Lemma 3. *Any quasiendomorphism ϕ of a group $(Q,\cdot)$ has the form $\phi = L_a\phi'$, where $L_a x = a\cdot x$, $a \in Q$, and ϕ' is an endomorphism of the group $(Q,\cdot)$.*

Lemma 4. *Any homotopy α of a group $(Q,\cdot)$ is a quasiendomorphism of $(Q,\cdot)$.*

The following results for regular paramedial division binary groupoids and regular paramedial division algebras were proved in [6].

Theorem 1. *A groupoid $(G,\cdot)$ is a regular paramedial division binary groupoid if and only if there exists an abelian group $(G,+)$, two surjective endomorphisms f,g of $(G,+)$ and an element $c \in G$ such that $f^2 = g^2$ and $x\cdot y = f(x)+g(y)+c$ for all $x,y \in G$.*

Theorem 2. *Let $(Q;\Sigma)$ be a regular paramedial division binary algebra. Then there exists an abelian group $(Q,+)$ such that every operation $A \in \Sigma$ has the following representation:*

$$A(x,y) = \phi_A x + \psi_A y + t_A,$$

where ϕ_A,ψ_A are surjective endomorphisms of the group $(Q,+)$ such that $\phi_A\phi_B = \psi_B\psi_A$, $\phi_A\psi_B = \phi_B\psi_A$ and $\psi_A\phi_B = \psi_B\phi_A$ for all $A,B \in \Sigma$ and $t_A \in Q$.

In this paper we generalized those results for n-ary regular paramedial division groupoids and regular paramedial devision algebras.

Main Results.

Theorem 3. *Let (Q,A) be a regular paramedial division n-groupoid. Then there exists an Abelian group $Q(+)$ and surjective endomorphisms $\alpha_1,\dots,\alpha_n$, and a fixed element $b \in Q$ such that*

$$A(x_1^n) = \alpha_1 x_1 + \cdots + \alpha_n x_n + b$$

for all $x_i \in Q, i = 1,\dots,n$, and where $\alpha_i\alpha_j = \alpha_{n+1-j}\alpha_{n+1-i}$ for all $i,j = 1,\dots,n$.

Proof. The proof is by induction on n.

For $n=2$ the assumption follows from Theorem 1. Suppose the assumption satisfied for natural numbers less than n.

Let us consider the following matrix:

$$\begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \dots & \dots & \dots & \dots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{pmatrix},$$

and define:

$$A\Big(\big\{x_{n+1-in+1-j}\big\}_{j=1}^{n}\Big) = y_i,\ A\Big(\big\{x_{ij}\big\}_{i=1}^{n}\Big) = z_j.$$

Then we can write paramdeial identity as

$$A(y_1^n) = A(z_1^n). \tag{1}$$

Now let us consider the following matrix:

$$\begin{pmatrix} a & a & a & \dots & a \\ \dots & \dots & \dots & \dots & \dots \\ a & a & a & \dots & a \\ a & x_2 & x_3 & \dots & x_n \\ x_1 & a & a & \dots & a \\ a & a & a & \dots & a \end{pmatrix},$$

and suppose z_i and y_j from Eq. (1) have the forms

$$\begin{cases} z_1 = (a^{n-2}x_1a) = \beta x_1, \\ z_i = (a^{n-3}x_ia^2) = \mu x_i, \ i \neq 1, \end{cases} \qquad \begin{cases} y_i = (a^n) = b, \ i \neq 2,3, \\ y_2 = (a^{n-1}x_1) = \alpha x_1, \\ y_3 = (\overset{n}{\underset{2}{}}xa), \end{cases}$$

where α, β, μ are surjections. Thus, from Eq. (1) we get

$$A\Big(b, \alpha x_1, \big(\overset{n}{\underset{2}{}}xa\big), b^{n-3}\Big) = A\Big(\beta x_1, \{\mu x_i\}_{i=2}^n\Big).$$

Define a binar groupoid $B(u,v) = A\big(b,u,v,b^{n-3}\big)$, and so (Q,B) will be regular paramedial division groupoid, because it's a retract of (Q,A).

Define $(n-1)$-ary groupoid $C\big(\overset{n}{\underset{2}{}}u\big) = A\big(\overset{n}{\underset{2}{}}u, a\big)$, thus (Q,C) will be regular paramedial division $(n-1)$-ary groupoid, because it's a retract of (Q,A).

From the assumption it follows that there exists $(Q,*)$ and $(Q,\oplus)$ an Abelian groups such that:

$$B(u,v) = \gamma u \oplus \delta v \oplus d, \ C\big(\overset{n}{\underset{2}{}}u\big) = \lambda_n u_n * \lambda_{n-1}u_{n-1} * \dots * \lambda_2 u_2 * c,$$

where γ, δ are surjective endpmorphisms of the group $(Q,\oplus)$ such that $\gamma^2 = \delta^2$ and $\lambda_i, \ i = 2,\dots,n$ are surjective endomorphisms of the group $(Q,*)$ such that $\lambda_i\lambda_j = \lambda_{n+2-j}\lambda_{n+2-i}$.

Making replacements in Eq. (1), we get

$$B\Big(\alpha x_1, C\big(\overset{n}{\underset{2}{}}x\big)\Big) = A\Big(\beta x_1, \{\mu x_i\}_{i=2}^n\Big)$$

or

$$\gamma\alpha x_1 \oplus \delta\left(\lambda_n x_n * \lambda_{n-1}x_{n-1} * \dots * \lambda_2 x_2 * c\right) \oplus d = A\Big(\beta x_1, \{\mu x_i\}_{i=2}^n\Big).$$

Let h_μ be the right inverse of μ, by replacements we obtain

$$\gamma\alpha x_1 \oplus \delta(\lambda_n h_\mu x_n * \lambda_{n-1}h_\mu x_{n-1} * \dots * \lambda_2 h_\mu x_2 * c) \oplus d = A\big(\beta x_1, x_2^n\big). \tag{2}$$

There exists an element $a_1 \in Q$ such that $\gamma\alpha a_1 \oplus d = 0_\oplus$, where $0_\oplus$ is the identity element of the group $(Q,\oplus)$. By taking $x_1 = a_1$, we get

$$\delta(\lambda_n h_\mu x_n * \lambda_{n-1}h_\mu x_{n-1} * \dots * \lambda_2 h_\mu x_2 * c) = A\big(\beta a_1, x_2^n\big). \tag{3}$$

The retract of (Q,A) groupoid $D(x_2^n) = A\big(\beta a_1, x_2^n\big)$ is also a regular paramedial division of the $(n-1)$-ary groupoid, so from the assumption we get that there exists an Abelian group $(Q,+)$ and surjective endomorphisms $\phi_i, i = 2,...,n$, $\phi_i\phi_j = \phi_{n+2-j}\phi_{n+2-i}$ such that $D\big(x_2^n\big) = \phi_2 x_2 + \phi_3 x_3 + \cdots + \phi_n x_n + t$. So Eq. (3) will look like

$$\delta(\lambda_n h_\mu x_n * \cdots * \lambda_2 h_\mu x_2 * c) = \phi_2 x_2 + \cdots + \phi_n x_n + t = \phi_2 x_2 + \cdots + \phi_n' x_n, \qquad (4)$$

where $\phi_n' x_n = \phi x_n + t$.

Now let put $x_1 = h_\beta x_1$ in Eq. (2), where h_β is the right inverse of β:

$$\gamma\alpha h_\beta x_1 \oplus \delta(\lambda_n h_\mu x_n * \lambda_{n-1} h_\mu x_{n-1} * \cdots * \lambda_2 h_\mu x_2 * c) \oplus d = A\big(x_1^n\big), \qquad (5)$$

and by using Eq. (4) we can rewrite Eq. (5) in the following way:

$$A\big(x_1^n\big) = \nu x_1 \oplus (\phi_2 x_2 + \cdots + \phi_n x_n + t) = \nu x_1 \oplus \big(\phi_2 x_2 + \cdots + \phi_n' x_n\big), \qquad (6)$$

where $\nu x_1 = \gamma\alpha h_\beta x_1 \oplus d$.

Now consider the retract $E(x_1^{n-1}) = A(x_1^{n-1}, a)$, and from the assumption we have that there exists an Abeilan group $(Q,\otimes)$ and μ_i, $i = 1,...,n-1$, surjective endomorphisms such that $\mu_i\mu_j = \mu_{n-j}\mu_{n-i}$ and

$$E(x_1^{n-1}) = \mu_1 x_1 \otimes \cdots \otimes \mu_{n-1} x_{n-1} \otimes l, \qquad (7)$$

where $l \in Q$.

Let us fix $x_n = a$ in Eq. (6) using Eq. (7), we get

$$\nu x_1 \oplus (\phi_2 x_2 + \cdots + \phi_{n-1}' x_{n-1}) = \mu_1 x_1 \otimes \cdots \otimes \mu_{n-1}' x_{n-1}, \qquad (8)$$

where $\phi_{n-1}' x_{n-1} = \phi_{n-1} x_{n-1} + \phi_n' a$ and $\mu_{n-1}' x_{n-1} = \mu n - 1 x_{n-1} \otimes l$.

Put $x_3^{n-1} = a_3^{n-1}$. Such that $\phi_3 a_3 + \cdots + \phi_{n-1}' a_{n-1} = 0_+$, where 0_+ is the identity element of the group $(Q,+)$, we obtain

$$\nu x_1 \oplus \phi_2 x_2 = \mu_1 x_1 \otimes \mu_2' x_2$$

or

$$x_1 \otimes x_2 = \nu h_{\mu_1} x_1 \oplus \phi_2 h_{\mu_2'} x_2,$$

where $\mu_2' x_2 = \mu_2 x_2 \otimes \mu_3 a_3 \otimes \cdots \otimes \mu_{n-1} a_{n-1}$ and $h_{\mu_1}, h_{\mu_2'}$ are right inverses of μ_1, μ_2' respectively. Thus we have that the group $(Q,\otimes)$ is pricnipally homotopic to the group $(Q,\oplus)$, so from Lemma 1, we have

$$x \oplus y = x \otimes y \otimes f'. \qquad (9)$$

Now let replace $x_1 = a_1$ and $x_4^{n-1} = a_4^{n-1}$ in Eq. (8) such that $\nu a_1 = 0_\oplus$ and $\phi_4 a_4 + \cdots + \phi_{n-1}' a_{n-1} = 0_+$, we get

$$\phi_2 x_2 + \phi_3 x_3 = \mu_2 x_2 \otimes \mu_3' x_3.$$

Then again from Lemma 1 we obtain

$$x \otimes y = x + y + f'', \qquad (10)$$

so from Eqs. (9) and (10) we obtain

$$x \oplus y = x + y + f. \qquad (11)$$

Using Eq. (11) in Eq. (6), we obtain

$$A(x_1^n) = \nu x_1 + \phi_2 x_2 + \cdots + \phi_n' x_n + f = \psi_1 x_1 + \cdots + \psi_n x_n + h, \tag{12}$$

where $\psi_1, \ldots, \psi_n$ are surjections and $h \in Q$, and we can assume that $\psi_i 0 = 0, i = 1, \ldots, n$.

Let us proof that ψ_i, $i = 1, \ldots, n$, are surjective endomorphisms and $\psi_i \psi_j = \psi_{n+1-j} \psi_{n+1-i}$. Consider the following matrix:

$$\begin{array}{c} \\ i \end{array} \begin{pmatrix} & \overset{j}{\cdot} & & \overset{k}{\cdot} & \\ & \cdot & & \cdot & \\ & \cdot & & \cdot & \\ \ldots & u & \ldots & v & \ldots \\ & \cdot & & \cdot & \\ & \cdot & & \cdot & \\ & \cdot & & \cdot & \end{pmatrix},$$

where $x_{ij} = u,\ y_{jk} = v$ and all other elements are equal to 0_+. Thus we have

$$\begin{cases} y_{n+1-i} = \psi_{n+1-j} u + \psi_{n+1-k} v + h, \\ y_s = h, s \neq i, \end{cases} \qquad \begin{cases} z_j = \psi_i u + h, \\ z_k = \psi_i v + h, \\ z_s = h, d \neq j, k, \end{cases}$$

hence

$$A(y_1^n) = A\left(h^{n-i}, \psi_{n+1-j} u + \psi_{n+1-k} v + h, h^{i-1}\right),$$
$$A(z_1^n) = A\left(h^{j-1}, \psi_i u + h, h^{k-j-1}, \psi_i v + h, h^{n-k}\right),$$

and

$$A\left(h^{n-i}, \psi_{n+1-j} u + \psi_{n+1-k} v + h, h^{i-1}\right) = A\left(h^{j-1}, \psi_i u + h, h^{k-j-1}, \psi_i v + h, h^{n-k}\right).$$

Thus, using Eq. (12) we obtain

$$\sum_{s=1}^{n-i} \psi_s h + \psi_{n+1-i}(\psi_{n+1-j} u + \psi_{n+1-k} v + h) + \sum_{s=n+2-i}^{n} \psi_s h + h =$$
$$\sum_{s=1}^{j-1} \psi_s h + \psi_j(\psi_i u + h) + \sum_{s=j+1}^{k-1} \psi_s h + \psi_k(\psi_i v + h) + \sum_{s=k+1}^{n} \psi_s h + h.$$

From this identity we obtain

$$\psi_{n+1-i}(\psi_{n+1-j} u + \psi_{n+1-k} v + h) = \psi_j(\psi_i u + h) + \psi_k(\psi_i v + h) + r,$$

where $r \in Q$. By making substitutions $u = h_{\psi_{n+1-j}} u$ and $v = h_{\psi_{n+1-k}} v - h$, where $h_{\psi_{n+1-j}}$ and $h_{\psi_{n+1-k}}$ are the right inverses of ψ_{n+1-j} and ψ_{n+1-k}, we get

$$\psi_{n+1-i}(u+v)=\psi_j(\psi_i h_{\psi_{n+1-j}}u+h)+\psi_k(\psi_i h_{\psi_{n+1-k}}v+h)+r$$

or

$$\psi_{n+1-i}(u+v)=\theta u+\sigma v,$$

where θ and σ are surjections. Thus from Lemma 4 it follows that ψ_i, $i=1,\dots,n$, are quasiendomorphisms. Since $\psi_i 0_+=0_+$, from Lemma 2 it follows that ψ_i is endomorphism of the group $(Q,+)$.

Fixing $v=0_+$, we obtain

$$\psi_{n+1-i}\psi_{n+1-j}u+\psi_{n+1-i}h=\psi_j\psi_i u+\psi_j h+\psi_k h+r, \tag{13}$$

and if we fix $u=0_+$, we get

$$\psi_{n+1-i}h=\psi_j h+\psi_k h+r. \tag{14}$$

Using Eq. (14) in Eq. (13), we get

$$\psi_{n+1-i}\psi_{n+1-j}u=\psi_j\psi_i u$$

for all $i,j=1,\dots,n$. □

Theorem 4. *Let (Q,Σ) be a regular paramedial division algebra. Then there exists an Abelian group $(Q,+)$ such that every operation $A\in\Sigma$ has the representation*

$$A\left(x_1^{|A|}\right)=\phi_1^A x_1+\dots+\phi_{|A|}^A x_{|A|}+b_A,$$

where ϕ_i^A are surjective endomorphisms of the group $(Q,+)$ such that $\phi_i^A\phi_j^A=\phi_{n+1-j}^A\phi_{n+1-i}^A$ for all $i,j=1,\dots,n$ and $b_A\in Q$.

Proof. From Theorem 3 we know that for every $A\in\Sigma$ there exists group $(Q,+_A)$ and surjective endomorphisms such that

$$A(x_1^{|A|})=\phi_1^A x_1+_A\dots+_A\phi_{|A|}^A x_{|A|}+_A b_A.$$

Let $A,B\in\Sigma$. From the hyperidentity of paramediality we have

$$\phi_1^A\left(\phi_1^B x_{11}+_B\dots+_B\phi_{|B|}^B x_{|B|1}+_B b_B\right)+_A\dots+_A\phi_{|A|}^A\left(\phi_1^B x_{1|A|}+_B\dots+_B\phi_{|B|}^B x_{|B||A|}+_B b_B\right)$$
$$+_A b_A=\phi_1^B\left(\phi_1^A x_{|B||A|}+_A\dots+_A\phi_{|A|}^A x_{|B|1}+_A b_A\right)+_B\dots+_B\phi_{|B|}^B(\phi_1^A x_{1|A|}+_A\dots$$
$$+_A\phi_{|A|}^A x_{11}+_A b_A)+_B b_B.$$

Fix $x_{ij}=0_{+_B}$, where $x_{ij}\neq x_{11}$ and $x_{ij}\neq x_{|B||A|}$, then we get

$$\phi_1^A\left(\phi_1^B x_{11}+_B b_B\right)+_A\phi_{|A|}^A\left(\phi_{|B|}^B x_{|B||A|}+_B b_B\right)+_A f_A=$$
$$\phi_1^B\left(\phi_1^A x_{|B||A|}+_A c_A\right)+_B\phi_{|B|}^B\left(\phi_{|A|}^A x_{11}+_A d_A\right)+_B f_B,$$

where c_A,d_a,f_A,f_B are elements from Q. From which we obtain

$$\alpha x_{11}+_A\beta x_{|B||A|}=\gamma x_{|B||A|}+_B\theta x_{11},$$

where $\alpha=\phi_1^A R_{b_B}^B\phi_1^B,\beta=R_{f_A}^A\phi_{|A|}^A R_{b_B}^B\phi_{|B|}^B,\gamma=\phi_1^B R_{c_A}^A\phi_1^A$ and $\theta=R_{f_B}^B\phi_{|B|}^B R_{d_A}^A\phi_{|A|}^A$ are surjections, where $R_{b_B}^B,R_{f_B}^B$ are the right translations of the group $(Q,+_B)$ and $R_{f_A}^A,R_{c_A}^A,R_{d_A}^A$ are the right translations of the group $(Q,+_A)$. From this we obtain

$$x_{11}+_A x_{|B||A|}=\theta h_\alpha x_{11}+_B\gamma h_\beta x_{|B||A|},$$

where h_α and h_β are the right inverses of the α and β. This means that the group $(Q,+_A)$ and the group $(Q,+_B)$ are principally homotopic and from Lemma 1 we get

$$x+_A y = x+_B y +_B g_{AB},$$
$$x+_B y = x+_A y +_A r_{AB},$$

where $g_{AB}, r_{AB} \in Q$.

Let us fix an operation $B \in \Sigma$, by this we will fix the group $(Q,+_B) = (Q,+)$ and for every operation $A \in \Sigma$ we obtain

$$A\left(x_1^{|A|}\right) = \phi_1^A x_1 +_A \ldots +_A \phi_{|A|}^A x_{|A|} +_A b_A = \phi_1^A x_1 + \cdots + \phi_{|A|}^A x_{|A|} + u_A, \tag{15}$$

where $u_A \in Q$, and for every $\phi_i^A, i = 1, \ldots, |A|$, we get

$$\phi_i^A(x+y) = \phi_i^A(x +_A y +_A r_{AB}) = \phi_i^A x +_A \phi_i^A y +_A \phi_i^A r_{AB} =$$
$$\phi_i^A x + \phi_i^A y + v = \phi_i^A x + \psi_i^A y,$$

where ψ_i^A is a surjection from Q to Q. It follows from Lemma 4 that ϕ_i^A, $i = 1, \ldots |A|$, are quasiendomorphisms of the group $(Q,+)$, and from Lemma 3 we have that $\phi_i^A = R_a \mu_i^A$, where μ_i^A is an endomorphism of the group $(Q,+)$ and R_a is the right translation of the group $(Q,+)$ by the element $a \in Q$. Hence we obtain

$$A(x_1^{|A|}) = \phi_1^A x_1 + \cdots + \phi_{|A|}^A x_{|A|} + u_A = \mu_1^A x_1 + \cdots + \mu_{|A|}^A x_{|A|} + v_A,$$

where μ_i^A, $i = 1, \ldots, |A|$, are sujective endomorphisms of the group $(Q,+)$ and $v_A \in Q$. Similar to the proof of the Theorem 3 we can show that $\mu_i^A \mu_j^A = \mu_{n+1-j}^A \mu_{n+1-i}^A$. □

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Դ. Ն. ՀԱՐՈՒԹՅՈՒՆՅԱՆ

ՌԵԳՈՒԼՅԱՐ ՊԱՐԱՄԵԴԻԱԼ ԲԱԺԱՆՈՒՄՈՎ ՀԱՆՐԱՀԱՇԻՎՆԵՐԻ ՄԱՍԻՆ

Այս հոդվածում դիտարկվում են n-տեղանի ռեգուլյար պարամեդիալ բաժանումով հանրահաշիվները և ցույց է տրվում, որ n-տեղանի ռեգուլյար պարամեդիալ բաժանումով հանրահաշվի յուրաքանչյուր գործողություն կարելի է գծայնորեն ներկայացնել նույն Աբելյան խմբի միջոցով: Ռեգուլյար պարամեդիալ բաժանումով հանրահաշիվների համար նմանատիպ արդյունքներ արդեն իսկ ստացվել են [1]-ում:

Д. Н. АРУТЮНЯН

ОБ АЛГЕБРАХ С РЕГУЛЯРНЫМИ ПАРАМЕДИАЛЬНЫМИ ДЕЛЕНИЯМИ

В этой статье изучаются n-арные регулярные алгебры с делением, удовлетворяющие гипертождеству парамедиальности. Показано, что каждая операция в n-арной регулярной парамедиальной алгебре с делением имеет линейное представление над одной и той же абелевой группой. Аналогичные результаты для регулярных медиальных алгебр с делением уже получены в [1].