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Estimation of spectral functionals for Levy-driven continuous-time linear models with tapered data

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Abstract: The paper is concerned with the nonparametric statistical estimation of linear spectral functionals for Lévy-driven continuous-time stationary linear models with tapered data. As an estimator for unknown functional we consider the averaged tapered periodogram. We analyze the bias of the estimator and obtain sufficient conditions assuring the proper rate of convergence of the bias to zero, necessary for asymptotic normality of the estimator. We prove a a central limit theorem for a suitable normalized stochastic process generated by a tapered Toeplitz type quadratic functional of the model. As a consequence of these results we obtain the asymptotic normality of our estimator.

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1. Introduction

In spectral analysis of time series the data are frequently tapered before calculating the statistics of interest. Instead of the original data $\{X(t),\, 0 \le t \le T\}$ the tapered data $\{h(t)X(t),\, 0 \le t \le T\}$ with the data taper $h(t)$ are used for all further calculations. Benefits of tapering the data have been widely reported in the literature. For example, data-tapers are introduced to reduce leakage effects, especially in the case when the spectrum of the model contains high peeks. Other application of data-tapers is in situations in which some of the data values are missing. Also, the use of tapers leads to the bias reduction, which is especially

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important when dealing with spatial data. In this case, the tapers can be used to fight the so-called "edge effects".

Much of statistical inferences about the spectrum based on tapered data are concerned with the discrete-time stationary models (see, e.g., Brillinger [7], R. Dahlhaus [9, 10], R. Dahlhaus and H. Künsch [11], Guyon [25], and references therein).

In this paper, we study the problem of nonparametric estimation of linear spectral functionals based on tapered data, in the case where the underlying model is a Lévy-driven continuous-time stationary linear process with possibly unbounded or vanishing spectral density function.

The model. Let $\{X(t),\ t\in\mathbb{R}\}$ be a Lévy-driven, real-valued, continuous-time stationary linear process defined by

$$X(t)=\int_{\mathbb{R}} a(t-s)\xi(ds), \tag{1.1}$$

where $a(\cdot)$ is a function from $L^2(\mathbb{R})$, and $\xi(t)$ is a Lévy process satisfying the conditions:

$$\mathbb{E}\xi(t)=0,\ \mathbb{E}\xi^2(1)=1 \text{ and } \mathbb{E}\xi^4(1)<\infty.$$

A Lévy process, $\{\xi(t),\ t\in\mathbb{R}\}$ is a process with independent and stationary increments, continuous in probability, with sample-paths which are right-continuous with left limits (càdlàg) and $\xi(0)=\xi(0-)=0$. The Wiener process $\{B(t),\ t\ge 0\}$ and the centered Poisson process $\{N(t)-\mathbb{E}N(t),\ t\ge 0\}$ are typical examples of centered Lévy processes. In the case where $\xi(t)=B(t)$, $X(t)$ is a Gaussian process.

Notice that the covariance function $r(t)$ of $X(t)$, which is an even function ($r(-t)=r(t)$), is given by

$$r(t)=\mathbb{E}X(t)X(0)=\int_{\mathbb{R}} a(t+x)a(x)dx, \tag{1.2}$$

and it possesses the spectral density

$$f(\lambda)=\frac{1}{2\pi}|\widehat{a}(\lambda)|^2=\frac{1}{2\pi}\left|\int_{\mathbb{R}} e^{-i\lambda t}a(t)dt\right|^2, \quad \lambda\in\mathbb{R}. \tag{1.3}$$

The functions $r(t)$ and $f(\lambda)$ are connected by the Fourier integral:

$$f(\lambda)=\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\lambda t}r(t)dt, \quad \lambda\in\mathbb{R}. \tag{1.4}$$

The function $a(\cdot)$ plays the role of a *time-invariant filter*.

Processes of the form (1.1) appear in many fields of science (economics, finance, physics, etc.), and cover a large class of popular models in continuous-time time series modeling. For instance, the so-called continuous-time autoregressive moving average (CARMA) models, which are the continuous-time analogs of the classical autoregressive moving average (ARMA) models in discrete-time case, are of the form (1.1) and play a central role in the representations of continuous-time stationary time series (see, e.g., Brockwell [8]).

The nonparametric estimation problem. Let $\{X(t),\, t \in \mathbb{R}\}$ be a centered stationary process with an *unknown* spectral density $f(\lambda)$, $\lambda \in \mathbb{R}$. We assume that $f(\lambda)$ belongs to a given (infinite-dimensional) class $\mathcal{F} \subset L^p := L^p(\mathbb{R})$ $(p \geq 1)$ of spectral densities possessing some specified smoothness properties. The problem is to estimate the value $J(f)$ of a given functional $J(\cdot)$ at an *unknown* "point" $f \in \mathcal{F}$ on the basis of the observed data $\{X(t),\, 0 \leq t \leq T\}$, and investigate the asymptotic (as $T \to \infty$) properties of the suggested estimators, depending on the dependence structure of the model $X(t)$ and smoothness structure of the "parametric" set $\mathcal{F}$.

This problem for discrete time stationary Gaussian processes has been considered in a number of papers. We cite merely the papers Dahlhaus and Wefelmeyer [12], Ginovyan [15, 19], and Ibragimov and Khas'minskii [26, 28], where can be found additional references.

For continuous time stationary Gaussian processes the problem was studied in Ginovyan [16, 17, 18, 20, 21], where efficient nonparametric estimators for linear and some nonlinear smooth spectral functionals were constructed and asymptotic bounds for minimax mean square risks of these estimators were obtained.

The problem of construction of consistent and asymptotically normal nonparametric estimators for linear and some nonlinear smooth spectral functionals in the case where the underlying model $X(t)$ is a Lévy-driven continuous-time stationary linear process defined by (1.1) with possibly unbounded or vanishing spectral density function has been studied in Ginovyan and Sahakyan [23].

In this paper we are interested in nonparametric estimation of spectral functionals $J(f)$ based on tapered data:

$$\{h_T(t)X(t),\ 0 \leq t \leq T\}, \tag{1.5}$$

where $h_T(t) := h(t/T)$ with a taper function $h(\cdot)$ satisfying assumption (T) below.

We assume that the estimand functional $J(f)$ is linear and continuous in $L^p(\mathbb{R})$, $p > 1$. Then $J(f)$ admits the representation

$$J = J(f) := \int_{\mathbb{R}} f(\lambda)g(\lambda)d\lambda, \tag{1.6}$$

where $g(\lambda) \in L^q(\mathbb{R}),\ 1/p + 1/q = 1$.

The estimator. As an estimator for functional $J(f)$, given by (1.6), we consider the averaged periodogram (or a simple "plug-in" statistic) based on the tapered data (1.5). To define the estimator, we first introduce some notation.

Denote by $H_{k,T}(\lambda)$ the continuous-time tapered Dirichlet type kernel, defined by

$$H_{k,T}(\lambda) := \int_{\mathbb{R}} h_T^k(t)e^{-i\lambda t}dt = \int_0^T h_T^k(t)e^{-i\lambda t}dt. \tag{1.7}$$

We set

$$H_k := \int_0^1 h^k(t)dt, \tag{1.8}$$

and assume that $H_2 \neq 0$.

Define the finite Fourier transform of the tapered data (1.5):

$$d_T^h(\lambda) := \int_0^T h_T(t)X(t)e^{-i\lambda t}dt, \tag{1.9}$$

and the tapered continuous periodogram $I_T^h(\lambda)$ of the process $X(t)$:

$$\begin{aligned} I_T^h(\lambda) : &= \frac{1}{C_T} d_T^h(\lambda) d_T^h(-\lambda) = \frac{1}{C_T} \left| \int_0^T h_T(t)X(t)e^{-i\lambda t}dt \right|^2 \\ &= \frac{1}{C_T} \int_0^T \int_0^T h_T(t)h_T(s)e^{-i\lambda(t-s)}X(t)X(s)dtds, \end{aligned} \tag{1.10}$$

where

$$C_T := 2\pi H_{2,T}(0) = 2\pi \int_0^T h_T^2(t)dt = 2\pi H_2\, T \neq 0. \tag{1.11}$$

Notice that for non-tapered case ($h(t) = 1$), we have $C_T = 2\pi T$.

An estimator J_T^h for functional (1.6) based on the tapered data (1.5) is defined to be the averaged tapered periodogram (or a simple "plug-in" statistic) defined by

$$\begin{aligned} J_T^h &= J(I_T^h) := \int_{\mathbb{R}} I_T^h(\lambda)g(\lambda)d\lambda \\ &= \frac{1}{C_T} \int_0^T \int_0^T h_T(t)h_T(s)b(t-s)X(t)X(s)\,dt\,ds, \end{aligned} \tag{1.12}$$

where C_T is as in (1.11), and $b(t)$ is the Fourier transform of function $g(\lambda)$:

$$b(t) := \widehat{g}(t) = \int_{\mathbb{R}} e^{i\lambda t}g(\lambda)d\lambda, \quad t \in \mathbb{R}. \tag{1.13}$$

We will refer to $g(\lambda)$ and to its Fourier transform $b(t) := \widehat{g}(t)$ as a *generating function* and *generating kernel* for the functional J_T^h, respectively.

Notation. Given numbers $p \geq 1$, $0 < \alpha < 1$, $r \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we set $\beta = \alpha + r$ and denote by $H_p(\beta)$ the L^p-Hölder class, that is, the class of those functions $\psi(\lambda) \in L^p(\mathbb{R})$, which have r-th derivatives in $L^p(\mathbb{R})$ and with some positive constant C satisfy

$$||\psi^{(r)}(\cdot + h) - \psi^{(r)}(\cdot)||_p \leq C|h|^\alpha.$$

Throughout the paper the letters C, c and M are used to denote positive constants, the values of which can vary from line to line. Also, by $\mathbb{I}_A(\cdot)$ we denote the indicator of a set $A \subset \mathbb{R}$.

The paper is structured as follows. In Section 2 we state the main results of the paper (Theorems 2.1 - 2.3). In Section 3 we give a number of preliminary results that are used in the proofs of main results, and also represent independent interest. In Section 4 we analyze the bias of the estimator J_T^h, and prove Theorem 2.1. In Section 5 we study the asymptotic distribution of a stochastic process generated by a tapered Toeplitz type quadratic functional of a Lévy-driven continuous-time linear process, and prove Theorems 2.2.

2. Main results

In this section we state the main results of this paper, involving bias rate convergence theorem, a central limit theorem and asymptotic normality of the estimator J_T^h. To this end, we first formulate conditions on model, generating function and taper function needed to state the results.

(A1) The filter $a(\cdot)$ and the generating kernel $b(\cdot)$ are such that

$$a(\cdot) \in L^p(\mathbb{R}) \cap L^2(\mathbb{R}), \quad b(\cdot) \in L^q(\mathbb{R}) \tag{2.1}$$

with

$$1 \leq p, q \leq 2, \quad \frac{2}{p} + \frac{1}{q} \geq \frac{5}{2}. \tag{2.2}$$

The spectral density f and the generating function g satisfy one of the following conditions.

(A2) $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and g is of bounded variation.
(A2$'$) $f \in H_p(\beta_1)$, $\beta_1 > 0$, $p \geq 1$ and $g(\lambda) \in H_q(\beta_2)$, $\beta_2 > 0$, $q \geq 1$ with $1/p + 1/q = 1$ and $\beta_1 + \beta_2 > 1/2$.
(T) The taper $h : \mathbb{R} \to \mathbb{R}$ is a continuous nonnegative function of bounded variation and of bounded support $[0, 1]$.

Our first theorem controls the bias $E(J_T^h) - J$ and provides sufficient conditions assuring the proper rate of convergence of bias to zero, necessary for asymptotic normality of the estimator J_T^h. Specifically, we have the following result, the proof of which is given in Section 4.

Theorem 2.1 (Bias)**.** *Let the functionals $J := J(f)$ and $J_T^h := J(I_T^h)$ be defined by (1.6) and (1.12), respectively. Then under the conditions (A2) (or (A2$'$) and (T) the following asymptotic relation holds:*

$$T^{1/2} \left[\mathbb{E}(J_T^h) - J\right] \to 0 \quad \text{as} \quad T \to \infty. \tag{2.3}$$

The next theorem contains sufficient conditions for functional J_T^h to obey the central limit theorem (CLT), and is proved in Section 5.

Theorem 2.2 (CLT)**.** *Let $J := J(f)$ and $J_T^h := J(I_T^h)$ be defined by (1.6) and (1.12), respectively. Then under the conditions (A1) and (T) the functional J_T^h obeys the central limit theorem. More precisely, we have*

$$T^{1/2} \left[J_T^h - \mathbb{E}(J_T^h)\right] \xrightarrow{d} \eta \quad \text{as} \quad T \to \infty, \tag{2.4}$$

where the symbol $\xrightarrow{d}$ stands for convergence in distribution, and η is a normally distributed random variable with mean zero and variance $\sigma_h^2(J)$ given by

$$\sigma_h^2(J) = 4\pi e(h) \int_{\mathbb{R}} f^2(\lambda) g^2(\lambda) d\lambda + \kappa_4 e(h) \left[\int_{\mathbb{R}} f(\lambda) g(\lambda) d\lambda\right]^2. \tag{2.5}$$

Here κ_4 is the fourth cumulant of $\xi(1)$, and

$$e(h) := \frac{H_4}{H_2^2} = \int_0^1 h^4(t)dt \left(\int_0^1 h^2(t)dt\right)^{-2}. \tag{2.6}$$

Taking into account the equality

$$T^{1/2}\left[J_T^h - J\right] = T^{1/2}\left[\mathbb{E}(J_T^h) - J\right] + T^{1/2}\left[J_T^h - \mathbb{E}(J_T^h)\right], \tag{2.7}$$

as an immediate consequence of Theorems 2.1 and 2.2, we obtain the next result that contains sufficient conditions for a simple "plug-in" statistic $J(I_T^h)$ to be an asymptotically normal estimator for a linear spectral functional $J(f)$.

Theorem 2.3. *Let the functionals $J := J(f)$ and $J_T^h := J(I_T^h)$ be defined by (1.6) and (1.12), respectively. Then under the conditions (A1), (A2) (or (A2′) and (T) the statistic J_T^h is an asymptotically normal estimator for functional J. More precisely, we have*

$$T^{1/2}\left[J_T^h - J\right] \xrightarrow{d} \eta \quad \text{as} \quad T \to \infty, \tag{2.8}$$

where η is as in Theorem 2.2, that is, η is a normally distributed random variable with mean zero and variance $\sigma_h^2(J)$ given by (2.5) and (2.6).

Remark 2.1. Notice that if the underlying process $X(t)$ is Gaussian, then in formula (2.5) we have only the first term. Using the results from Ginovyan [17] and Ginovyan and Sahakyan [22], it can be shown that in this case the conditions (A2′) and (T) are sufficient for Theorem 2.3 to be true.

Remark 2.2. The result of Theorem 2.3 under different more restrictive conditions were stated in Avram et al. [2] (see also Sakhno [31]). For non-tapered case ($h(t) = \mathbb{I}_{(0,1)}(t)$), Theorems 2.1–2.3 were proved in Ginovyan [20, 21].

Remark 2.3. One of the common used approaches to derive central limit theorems for random quadratic functionals is the method-of-moments (see, e.g., R. Dahlhaus [9, 10], Avram et al. [2], and references therein). Taking into account the complexity of computing the moments of multiple integrals with respect to non-Gaussian Lévy noise (see Peccati and Taqqu [30], Chapter 7), it is not clear how this method can be carried out for our model. In this paper, similar to Bai et al. [3] and Ginovyan and Sahakyan [22], our proofs of the central limit theorems are based on a new approximation approach which reduces the quadratic integral form to a single integral form. This method can also be adapted to the discrete-time case.

Remark 2.4. Notice that linear and non-linear functionals of the periodogram play a key role in the parametric estimation of the spectrum of stationary processes, when using the minimum contrast estimation method with various contrast functions (see, e.g., Anh et al. [1], Ginovyan and Sahakyan [23], Leonenko and Sakhno [29], Sakhno [31], Taniguchi [32], and references therein). So, the results obtained in the present paper can be applied to prove consistency and asymptotic normality of minimum contrast estimators based on the Whittle and Ibragimov's contrast functionals for Lévy-driven continuous-time stationary linear models with tapered data. The details will be reported elsewhere.

3. Preliminaries

In this section we prove a number of auxiliary lemmas involving properties of continuous-time tapered Dirichlet and Fejér type kernels. Some of these properties for discrete-time tapered case were proved in Dahlhaus [9], and for continuous-time non-tapered case were established in Ginovyan and Sahakyan [22].

An important role in our analysis of the above mentioned properties is played by the function $L_T(\cdot) : \mathbb{R} \to \mathbb{R}$, $T \in \mathbb{R}^+$, defined by

$$L_T(t) = \begin{cases} T & \text{for} \quad |t| \le 1/T \\ 1/|t| & \text{for} \quad |t| > 1/T. \end{cases} \tag{3.1}$$

The function $L_T(t)$ possesses a number of interesting properties, allowing to estimate the cumulants of functional J_T^h defined (1.12) (see Dahlhaus [9] for discrete-time case, and Eichler [13] for continuous-time case). In this paper we will use the following property of function $L_T(t)$.

Lemma 3.1. *Let $m = 0, 1, \ldots,$ and*

$$L(t) := L_1(t) = \begin{cases} 1 & \text{for } |t| \le 1 \\ 1/|t| & \text{for } |t| > 1, \end{cases} \qquad t \in \mathbb{R}. \tag{3.2}$$

Then for $u \in \mathbb{R}$ we have

$$Q(u) := \int_{\mathbb{R}} L(t)L(t+u) \ln^m(2+|t|)dt \le M \ln^{m+1}(2+|u|)L(u), \tag{3.3}$$

where M is a constant depending on m.

Proof. For $|u| \le 2$ we have

$$\begin{aligned} Q(u) &\le \ln^m 3 \int_{|t|\le 3} dt + \int_{|t|>3} \frac{1}{|t(t+u)|} \ln^m(2+|t|)dt \\ &\le M + M \int_1^\infty \frac{1}{t^2} \ln^m(2+|t|)dt \le M \ln^{m+1}(2+|u|)L(u). \end{aligned} \tag{3.4}$$

In the case when $|u| > 2$, we can write

$$\begin{aligned} Q(u) &= \int_{|t|\le 1} \frac{\ln^m(2+|t|)}{|t+u|} dt + \int_{|t+u|\le 1} \frac{\ln^m(2+|t|)}{|t|} dt \\ &+ \int_{|t|>1,\ |t+u|>1} \frac{\ln^m(2+|t|)}{|t(t+u)|} dt =: Q_1(u) + Q_2(u) + Q_3(u). \end{aligned} \tag{3.5}$$

Now we estimate the functions $Q_i(u)$, $i = 1, 2, 3$. Assuming, without loss of generality, that $u > 2$, for $Q_1(u)$ we have

$$Q_1(u) = \int_{-1}^{1} \frac{\ln^m(2+|t|)}{t+u} dt \le \frac{M}{u-1} \le M \ln^{m+1}(2+|u|)L(u). \tag{3.6}$$

Similarly, for $Q_2(u)$ we get the estimate

$$Q_2(u) \le M \ln^{m+1}(2+|u|)L(u). \tag{3.7}$$

For $Q_3(u)$ we have

$$\begin{aligned} Q_3(u) &= \int_{-\infty}^{-u-1} + \int_{-u+1}^{-1} + \int_{1}^{\infty} \frac{\ln^m(2+|t|)}{|t(t+u)|} dt \\ &=: Q_{31}(u) + Q_{32}(u) + Q_{33}(u). \end{aligned} \tag{3.8}$$

We have

$$\begin{aligned} Q_{31}(u) &\le \int_{1}^{\infty} \frac{\ln^m(t+u+2)}{t(t+u)} dt \le \frac{\ln^m 3u}{u} \int_{1}^{u} \frac{1}{t} dt + M \int_{u}^{\infty} \frac{\ln^m(t+u)}{(t+u)^2} dt \\ &\le M \frac{\ln^{m+1} u}{u} + M \int_{2u}^{\infty} \frac{\ln^m t}{t^2} dt \ \le M \frac{\ln^{m+1} u}{u} \\ &\le M \ln^{m+1}(2+|u|)L(u). \end{aligned} \tag{3.9}$$

Similarly,

$$Q_{33}(u) \le M \ln^{m+1}(2+|u|)L(u). \tag{3.10}$$

Finally,

$$Q_{32}(u) \le \int_{1}^{u+1} \frac{\ln^m t}{t(t-u)} dt \le \frac{\ln^{m+1} u}{u} \le M \ln^{m+1}(2+|u|)L(u). \tag{3.11}$$

Combining the inequalities (3.4)– (3.11) we obtain (3.3).
Lemma 3.1 is proved. □

The next two lemmas contain some properties of the Dirichlet type kernel $H_{k,T}(\lambda)$ defined by (1.7). The next assertion is the continuous analog of formula (4) in Dahlhaus [9].

Lemma 3.2. *The function $H_{k,T}(\lambda)$ defined by (1.7) satisfies the following identity:*

$$\int_{\mathbb{R}} H_{k,T}(\lambda - u) H_{j,T}(u - \mu) du = 2\pi H_{k+j,T}(\lambda - \mu), \quad \lambda, \mu \in \mathbb{R}. \tag{3.12}$$

Proof. Without loss of generality we assume that $\mu = 0$, and observe that $H_{k,T} \in L^2(\mathbb{R})$. Hence by (1.7) we have $\widehat{H_{k,T}}(\zeta) = 2\pi h_T^k(-\zeta)$ and $\widehat{H_{k,T}(\lambda - \cdot)}(\zeta) = 2\pi h_T^k(\zeta) e^{-i\lambda\zeta}$, where $\hat{g}$ is the Fourier transform of g. Hence using Plansherel's identity, we get

$$\begin{aligned}\int_{\mathbb{R}} H_{k,T}(\lambda-u)H_{j,T}(u)du &= \frac{1}{2\pi}(2\pi)^2\int_{\mathbb{R}} h_T^k(\zeta)h_T^j(\zeta)e^{-i\lambda\zeta}d\zeta \\ &= 2\pi\int_{\mathbb{R}} h_T^{k+j}(\zeta)e^{-i\lambda\zeta}d\zeta = 2\pi H_{k+j,T}(\lambda),\end{aligned}$$

and the result follows.

Lemma 3.2 is proved. □

Lemma 3.3 (see Eichler [13]). *Let the functions $H_{k,T}(\cdot)$ and $L_T(\cdot)$ be defined by (1.7) and (3.1), respectively. Then for all $k \in \mathbb{N}$ and a constant C_k independent of T, the following inequality holds:*

$$|H_{k,T}(\lambda)| \le C_k\, L_T(\lambda). \tag{3.13}$$

Proof. Since the taper function h is assumed to be of bounded variation, then denoting by $V(h)$ the total variation of h, we can write

$$\int_{\mathbb{R}}\left|\prod_{i=1}^{k} h(t+u_i) - h^k(t)\right| dt \le ||h||_\infty^{k-1} V(h)\sum_{i=1}^{k}|u_i|. \tag{3.14}$$

Therefore

$$|H_{k,T}(\lambda)| \le \frac{1}{2}\int_{\mathbb{R}}\left|h^k(t) - h^k\left(t-\frac{\pi}{\lambda}\right)\right| dt \le \frac{1}{2}||h||_\infty^{k-1}V(h)k\pi|\lambda|^{-1}. \tag{3.15}$$

Taking into account that $|H_{k,T}(\lambda)| \le ||h||_\infty^k T$, we obtain (3.13).

Lemma 3.3 is proved. □

For a number k ($k = 2, 3, \ldots$) and a taper function h satisfying assumption (T) consider the following Fejér type kernel function:

$$\Phi_{k,T}^h(\mathbf{u}) := \frac{H_T(\mathbf{u})}{(2\pi)^{k-1}H_{k,T}(0)}, \qquad \mathbf{u} = (u_1, \ldots, u_{k-1}) \in \mathbb{R}^{k-1}, \tag{3.16}$$

where

$$H_T(\mathbf{u}) := H_{1,T}(u_1)\cdots H_{1,T}(u_{k-1})H_{1,T}\left(-\sum_{j=1}^{k-1}u_j\right), \tag{3.17}$$

and the function $H_{k,T}$ is defined by (1.7) with $H_{k,T}(0) = T\cdot H_k \neq 0$ (see (1.8)).

The next lemma shows that the kernel $\Phi_{k,T}^h$ is an approximation identity.

Lemma 3.4. *For any $k = 2, 3, \ldots$ and a taper function h satisfying assumption (T) the kernel $\Phi_{k,T}^h(\mathbf{u})$, $\mathbf{u} = (u_1, \ldots, u_{k-1}) \in \mathbb{R}^{k-1}$, possesses the following properties:*

a) $\sup_{T>0}\int_{\mathbb{R}^{k-1}}\left|\Phi_{k,T}^h(\mathbf{u})\right| d\mathbf{u} = C_1 < \infty$*;*

b) $\int_{\mathbb{R}^{k-1}}\Phi_{k,T}^h(\mathbf{u})\,d\mathbf{u} = 1$*;*

c) $\lim_{T\to\infty}\int_{\mathbb{E}_\delta^c}\left|\Phi_{k,T}^h(\mathbf{u})\right| d\mathbf{u} = 0$ *for any $\delta > 0$;*

d) If $k>2$ for any $\delta>0$ there exists a constant $M_\delta>0$ such that for $T>0$

$$\left\|\Phi^h_{k,T}\right\|_{L^{p_k}(\mathbb{E}^c_\delta)} \le M_\delta, \tag{3.18}$$

where $p_k=\frac{k-2}{k-3}$ for $k>3$, $p_3=\infty$ and

$$\mathbb{E}^c_\delta = \mathbb{R}^{k-1}\setminus \mathbb{E}_\delta,$$
$$\mathbb{E}_\delta = \{\mathbf{u}=(u_1,\dots,u_{k-1})\in\mathbb{R}^{k-1} : |u_i|\le\delta,\ i=1,\dots,k-1\}.$$

e) If the function $\Psi\in L^1(\mathbb{R}^{k-1})\bigcap L^{k-2}(\mathbb{R}^{k-1})$ is continuous at $\mathbf{v}=(v_1,\dots,v_{k-1})$ (L^0 is the space of measurable functions), then

$$\lim_{T\to\infty}\int_{\mathbb{R}^{k-1}}\Psi(\mathbf{u}+\mathbf{v})\Phi^h_{k,T}(\mathbf{u})d\mathbf{u}=\Psi(\mathbf{v}). \tag{3.19}$$

Proof. We start with a). Observe that by (1.7),

$$H_{1,T}(\lambda):=\int_{\mathbb{R}} h\left(\frac{t}{T}\right)e^{-i\lambda t}dt = T\int_{\mathbb{R}} h(t)e^{-iT\lambda t}dt = T\hat{h}(T\lambda), \tag{3.20}$$

where $\hat{h}$ is the Fourier transform of h. Hence, in view of (3.17), we can write

$$\begin{aligned}
&\int_{\mathbb{R}^{k-1}}|H_T(\mathbf{u})|\,d\mathbf{u}\\
&=T^k\int_{\mathbb{R}^{k-1}}\left|\hat{h}(Tu_1)\cdots\hat{h}(Tu_{k-1})\hat{h}(Tu_1+\cdots+Tu_{k-1})\right|du_1\cdots du_{k-1}\\
&=T\int_{\mathbb{R}^{k-1}}\left|\hat{h}(u_1)\cdots\hat{h}(u_{k-1})\hat{h}(u_1+\cdots+u_{k-1})\right|du_1\cdots du_{k-1}.
\end{aligned} \tag{3.21}$$

Since h is a function of bounded variation with support on $[0,1]$, we have

$$\left|\hat{h}(u)\right|\le M\cdot L(u) \quad \text{for} \quad u\in\mathbb{R},$$

where L is defined in (3.2) and M is a constant depending on h. Taking into account that $H_{k,T}(0)=T\cdot H_k$, from (3.3), (3.16) and (3.21) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{k-1}}\left|\Phi^h_{k,T}(\mathbf{u})\right|d\mathbf{u}\\
&\le M\int_{\mathbb{R}^{k-1}}|L(u_1)\cdots L(u_{k-1})L(u_1+\cdots+u_{k-1})|\,du_1\cdots du_{k-1}\\
&\le M\int_{\mathbb{R}^{k-2}}|L(u_1)\cdots L(u_{k-2})L(u_1+\cdots u_{k-2})|\\
&\times\ln(2+|u_1+\cdots+u_{k-2}|)du_1\cdots du_{k-2}\\
&\le M\int_{\mathbb{R}^{k-3}}|L(u_1)\cdots L(u_{k-3})L(u_1+\cdots u_{k-3})|\\
&\times\ln^2(2+|u_1+\cdots+u_{k-3}|)du_1\cdots du_{k-3}
\end{aligned}$$

$$\leq \cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\leq M \int_{\mathbb{R}} \left|L^2(u_1)\right| \ln^{k-2}(2+|u_1|)du_1 < \infty, \tag{3.22}$$

and the assertion a) is proved.

The assertion b) follows immediately from (3.12), (3.16) and (3.17).

To prove assertion c) we write

$$\int_{\mathbb{E}_\delta^c} |\Phi_{k,T}^h(\mathbf{u})| \leq \int\limits_{|u_1|>\delta} + \int\limits_{|u_2|>\delta} + \cdots + \int\limits_{|u_{k-1}|>\delta} |\Phi_{k,T}^h(\mathbf{u})| d\mathbf{u} =: J_1 + J_2 + \cdots + J_{k-1}.$$

Using the same arguments as in (3.22), for $s = 1, 2 \ldots, k-1$ we get

$$J_s \leq M \int_{|u_s|>T\delta} \left|L^2(u_s)\right| \ln^{k-2}(2+|u_s|)du_s \to 0 \quad \text{as} \quad T \to \infty,$$

and the result follows.

To prove assertion d), we first observe that by (3.13),

$$\int_{\mathbb{R}} H_{1,T}^{p_k}(u)du \leq C \cdot T^{p_k-1} \quad \text{and} \quad |H_{1,T}(u)| \leq C_\delta \quad \text{for} \quad |u| > \delta, \quad T > 0. \tag{3.23}$$

For $k = 3$ by (3.13) we have

$$\left|\Phi_{3,T}^h(u_1, u_2)\right| = \left|\frac{H_{1,T}(u_1)H_{1,T}(u_2)H_{1,T}(u_1+u_2)}{(2\pi)^2 T H_3}\right| \leq M_\delta, \quad (u_1, u_2) \in \mathbb{E}_\delta^c,$$

since for any $(u_1, u_2) \in \mathbb{E}_\delta^c$ at least two of three numbers $|u_1|,\ |u_2|,\ |u_1+u_2|$ are greater then $\delta/3$.

In the case where $k \geq 4$ we have

$$\begin{aligned}\int_{\mathbb{E}_\delta^c} \left|\Phi_{k,T}^h(\mathbf{u})\right|^{p_k} d\mathbf{u} \quad &\leq \int\limits_{|u_1|>\delta} \left|\Phi_{k,T}^h(\mathbf{u})\right|^{p_k} d\mathbf{u} + \cdots \\ &\quad + \int\limits_{|u_{k-1}|>\delta} \left|\Phi_{k,T}^h(\mathbf{u})\right|^{p_k} d\mathbf{u} =: I_1 + \cdots + I_{k-1}.\end{aligned} \tag{3.24}$$

We estimate I_1, the integrals $I_2, \ldots, I_{k-1}$ can be estimated similarly. We have

$$\begin{aligned} I_1 \quad &\leq \int\limits_{|u_1|>\delta,\ |u_2|>\delta/k} \left|\Phi_{k,T}^h(\mathbf{u})\right|^{p_k} d\mathbf{u} + \cdots + \int\limits_{|u_1|>\delta,\ |u_{k-1}|>\delta/k} \left|\Phi_{k,T}^h(\mathbf{u})\right|^{p_k} d\mathbf{u} \\ &+ \int\limits_{|u_1|>\delta,\ |u_2|\leq\delta/k,\ldots,|u_{k-1}|\leq\delta/k} \left|\Phi_{k,T}^h(\mathbf{u})\right|^{p_k} d\mathbf{u} \\ &=: \quad I_{12} + I_{13} + \cdots + I_{1k}. \end{aligned} \tag{3.25}$$

According to (3.16) and (3.23), we have

$$
\begin{aligned}
I_{12} &\le C_\delta \cdot \frac{1}{T^{p_k}} \\
&\quad \times \int\limits_{|u_2|>\delta/k} H_{1,T}^{p_k}(u_2)\cdots H_{1,T}^{p_k}(u_{k-1}) H_{1,T}^{p_k}(u_1+\cdots+u_{k-1})du_1du_3\cdots du_{k-1}du_2 \\
&\le C_\delta T^{(k-2)(p_k-1)-p_k} \int\limits_{|u_2|>\delta/k} \frac{1}{u_2^2}\,du_2 \le M_\delta. \qquad (3.26)
\end{aligned}
$$

Likewise, we get

$$I_{1s} \le M_\delta, \qquad s = 3,4,\dots,k-1. \qquad (3.27)$$

Note that in the integral I_{1k}, we have $|u_1+\cdots+u_{k-1}| > \delta - \delta(k-2)/k > \delta/k$, and hence by (3.23) we obtain

$$
\begin{aligned}
I_{1k} &\le C_{\delta/k} \cdot \frac{1}{T^{p_k}} \int\limits_{|u_1|>\delta} H_{1,T}^{p_k}(u_1)H_{1,T}^{p_k}(u_2)\cdots H_{1,T}^{p_k}(u_{k-1})du_2\cdots du_{k-1}du_1 \\
&\le M_\delta \int\limits_{|u_1|>\delta} \frac{1}{u_1^2}\,du_1 \le M_\delta. \qquad (3.28)
\end{aligned}
$$

From (3.24) – (3.28) we obtain (3.18), and thus the assertion d) is proved.

The assertion e) we prove for $\mathbf{v} = \mathbf{0} := (0,\dots,0)$, in the general case one can consider the function $\overline{\Psi}(\mathbf{u}) = \Psi(\mathbf{u}+\mathbf{v})$ with $\overline{\Psi}(0) = \Psi(\mathbf{v})$. By assertion b) of the lemma we have

$$R_{k,T}^h := \int_{\mathbb{R}^3} \Psi(\mathbf{u})\Phi_{k,T}^h(\mathbf{u})d\mathbf{u} - \Psi(\mathbf{0}) = \int_{\mathbb{R}^{k-1}} [\Psi(\mathbf{u}) - \Psi(\mathbf{0})]\Phi_{k,T}^h(\mathbf{u})d\mathbf{u}. \quad (3.29)$$

Next, for any $\varepsilon > 0$ we can find $\delta > 0$ to satisfy

$$|\Psi(\mathbf{u}) - \Psi(\mathbf{0})| < \frac{\varepsilon}{C_1} \quad \text{for} \quad \mathbf{u} \in \mathbb{E}_\delta, \qquad (3.30)$$

where C_1 is the constant from assertion a).

Now, if $k = 2$ we have

$$\left|\Phi_{2,T}^h(u)\right| = \left|\frac{H_{1,T}(u)H_{1,T}(-u)}{(2\pi)TH_2}\right| \le \frac{M_\delta}{T}, \quad |u| > \delta,$$

and hence, by assertions a) and c) of the lemma and (3.29), for sufficiently large T we obtain

$$
\begin{aligned}
|R_{2,T}^h| &\le \int_{|u|\le\delta} + \int_{|u|>\delta} |\Psi(u) - \Psi(0)|\left|\Phi_{2,T}^h(u)\right| du \\
&\le \frac{\varepsilon}{C_1}\int_{\mathbb{R}} \left|\Phi_{2,T}^h(u)\right| du + \frac{M_\delta}{T}\int_{\mathbb{R}} |\Psi(u)|du + |\Psi(0)|\int_{|u|>\delta} \left|\Phi_{2,T}^h(u)\right| du \\
&\le 3\varepsilon.
\end{aligned}
$$

If $k > 2$ we consider the decomposition $\Psi = \Psi_1 + \Psi_2$ such that

$$\|\Psi_1\|_{k-2} \leq \frac{\varepsilon}{M_\delta} \quad \text{and} \quad \|\Psi_2\|_\infty < \infty, \tag{3.31}$$

where M_δ is as in assertion d). Applying assertions a) - d) of the lemma and formulas (3.29) - (3.31) for sufficiently large T we obtain

$$\begin{aligned} |R^h_{k,T}| &\leq \int_{\mathbb{E}_\delta} |\Psi(\mathbf{u}) - \Psi(\mathbf{0})| \left|\Phi^h_{k,T}(\mathbf{u})\right| d\mathbf{u} + \int_{\mathbb{E}^c_\delta} |\Psi_1(\mathbf{u})| \left|\Phi^h_{k,T}(\mathbf{u})\right| d\mathbf{u} \\ &+ \int_{\mathbb{E}^c_\delta} |\Psi_2(\mathbf{u}) - \Psi(\mathbf{0})| \left|\Phi^h_{k,T}(\mathbf{u})\right| d\mathbf{u} \leq \frac{\varepsilon}{C_1} \int_{\mathbb{E}_\delta} \left|\Phi^h_{k,T}(\mathbf{u})\right| d\mathbf{u} \\ &+ \|\Psi_1\|_{k-2} \left\|\Phi^h_{k,T}\right\|_{L^{p_k}(\mathbb{E}^c_\delta)} + \left(\|\Psi_2\|_\infty + |\Psi(0)|\right) \int_{\mathbb{E}^c_\delta} \left|\Phi^h_{k,T}(\mathbf{u})\right| d\mathbf{u} \leq 3\,\varepsilon. \end{aligned}$$

This combined with (3.29) yields (3.19) for $\mathbf{v} = \mathbf{0}$. Lemma 3.4 is proved. ☐

Remark 3.1. For nontapered case ($h(t) = \mathbb{I}_{(0,1)}(t)$), Lemma 3.4 was stated in Ginovyan and Sahakyan [22] (see also Bentkus [4]). For discrete time tapered case the assertions a)-c) and e) of Lemma 3.4 were proved in Dahlhaus [9].

4. The Bias. Proof of Theorem 2.1

In this section we prove Theorem 2.1, that is, we show that under the conditions (A2) (or (A2′) and (T) the bias $E(J^h_T) - J$ of estimator J^h_T satisfies the asymptotic relation (2.3).

We first prove two lemmas.

Lemma 4.1. *Assume that the conditions (A2) and (T) are satisfied. Then the following asymptotic relation holds as $T \to \infty$:*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(\lambda) g(\lambda + \mu) \Phi^h_{2,T}(\mu) d\lambda d\mu = \int_{\mathbb{R}} f(\lambda) g(\lambda) d\lambda + o\left(T^{-1/2}\right), \tag{4.1}$$

where $\Phi^h_{2,T}(\mu)$ is given by (3.16).

Proof. We have

$$\begin{aligned} I_T &:= \int_{\mathbb{R}^2} f(\lambda) g(\lambda + \mu) \Phi^h_{2,T}(\mu) d\lambda d\mu \\ &= \int_{\mathbb{R}} \psi(\mu) \Phi^h_{2,T}(\mu) d\mu = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\psi}(\zeta) \hat{\Phi}^h_{2,T}(\zeta) d\zeta, \end{aligned} \tag{4.2}$$

where

$$\psi(\mu) = \int_{\mathbb{R}} f(\lambda) g(\lambda + \mu) d\lambda, \quad \hat{\psi}(\zeta) = \hat{f}(\zeta) \hat{g}(\zeta), \quad \lambda, \zeta \in \mathbb{R}. \tag{4.3}$$

According to (3.16) and (3.20) we can write

$$\Phi^h_{2,T}(\mu) = \frac{H_{1,T}(\mu) H_{1,T}(-\mu)}{2\pi T H_{2,T}(0)} = \frac{T}{2\pi H_2} \hat{h}(T\mu) \hat{h}(-T\mu) = \frac{T}{2\pi H_2} \varphi(T\mu), \tag{4.4}$$

where $H_2 = \int_0^1 h^2(t) dt$ and $\varphi(\zeta) := \hat{h}(\zeta) \hat{h}(-\zeta)$.

Observe that for function $q(\lambda) := \int_{\mathbb{R}} h(t)h(t-\lambda)dt$ we have $\hat{q} = \varphi \in L^1(\mathbb{R})$. Hence

$$q(\lambda) = \frac{1}{2\pi}\int_{\mathbb{R}} \varphi(\zeta)e^{i\lambda\zeta}d\zeta = \frac{1}{2\pi}\hat{\varphi}(-\lambda)$$

and

$$\hat{\varphi}(\zeta) = 2\pi q(-\zeta) = 2\pi \int_{\mathbb{R}} h(t)h(t+\zeta)dt = 2\pi \int_0^1 h(t)h(t+\zeta)dt.$$

Now, by (4.4) we have

$$\hat{\Phi}^h_{2,T}(\zeta) = \frac{1}{H_2}\int_0^1 h(t)h\left(t+\frac{\zeta}{T}\right)dt,$$

and hence, by (4.2) and (4.3), we obtain

$$I_T = \frac{1}{2\pi H_2}\int_{\mathbb{R}} \hat{f}(\zeta)\hat{g}(\zeta)\int_0^1 h(t)h\left(t+\frac{\zeta}{T}\right)dtd\zeta.$$

Therefore, we can write

$$\begin{aligned} I_T - \int_{\mathbb{R}} f(\lambda)g(\lambda)d\lambda &= I_T - \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\zeta)\hat{g}(\zeta)d\zeta \\ &= \frac{1}{2\pi H_2}\int_{\mathbb{R}} \hat{f}(\zeta)\hat{g}(\zeta)\int_0^1 \left[h(t)h\left(t+\frac{\zeta}{T}\right) - h^2(t)\right]dtd\zeta \\ &= \frac{1}{2\pi H_2}\int_{|\zeta|\le T} \hat{f}(\zeta)\hat{g}(\zeta)\int_0^1 h(t)\left[h\left(t+\frac{\zeta}{T}\right) - h(t)\right]dtd\zeta \\ &- \frac{1}{2\pi}\int_{|\zeta|>T} \hat{f}(\zeta)\hat{g}(\zeta)d\zeta =: R_1 - R_2. \end{aligned} \tag{4.5}$$

Since by assumption g and h are of bounded variation, we have

$$|\hat{g}(\zeta)| \le K\frac{1}{|\zeta|}, \qquad |h(\zeta)| < K, \qquad \int_0^1 |h(t+\zeta) - h(t)|\,dt \le K|\zeta|, \qquad \zeta \in \mathbb{R},$$

where the constant K does not depend on ζ. Hence, we have

$$\begin{aligned} |R_1| &\le K^3 \int_{|\zeta|\le T}\left|\hat{f}(\zeta)\cdot\frac{1}{\zeta}\cdot\frac{\zeta}{T}\right|d\zeta \\ &\le \frac{K^3}{T}\left[\int_{|\zeta|\le\sqrt{T}} |\hat{f}(\zeta)|d\zeta + \int_{\sqrt{T}<|\zeta|\le T} |\hat{f}(\zeta)|d\zeta\right] \\ &\le \frac{K^3}{T}\left[\|\hat{f}\|_2(2\sqrt{T})^{1/2} + \left\{\int_{|\zeta|>\sqrt{T}} |\hat{f}(\zeta)|^2d\zeta\right\}^{1/2} T^{1/2}\right] \qquad (4.6) \\ &= o\left(T^{-1/2}\right), \end{aligned}$$

and

$$\begin{aligned} |R_2| &\leq K\int_{|\zeta|>T}\left|\hat{f}(\zeta)\frac{1}{\zeta}\right|d\zeta \\ &\leq K\left\{\int_{|\zeta|>T}\left|\hat{f}(\zeta)\right|^2 d\zeta\right\}^{1/2}\left\{\int_{|\zeta|>T}\frac{1}{\zeta^2}d\zeta\right\}^{1/2} = o\left(T^{-1/2}\right) \quad (4.7)\end{aligned}$$

as $T \to \infty$. Combining (4.5) – (4.7), we obtain (4.1).
Lemma 4.1 is proved. □

Denote

$$\Delta_{2,T}^h := \left|\int_{\mathbb{R}^2} f(\lambda)g(\lambda+\mu)\Phi_{2,T}^h(\mu)d\lambda d\mu - \int_{\mathbb{R}} f(\lambda)g(\lambda)d\lambda\right|. \quad (4.8)$$

Lemma 4.2. *Assume that the conditions (A2′) and (T) are satisfied. Then the following inequality holds:*

$$\Delta_{2,T}^h \leq C_h \begin{cases} T^{-(\beta_1+\beta_2)}, & \text{if } \beta_1+\beta_2<1 \\ T^{-1}\ln T, & \text{if } \beta_1+\beta_2=1 \\ T^{-1}, & \text{if } \beta_1+\beta_2>1, \end{cases} \qquad T>0, \quad (4.9)$$

where C_h is a constant depending on h.

Proof. According to (3.16) and (3.17) we have

$$\Phi_{2,T}^h(\mu) = \frac{H_{1,T}(\mu)H_{1,T}(-\mu)}{2\pi T H_2(0)}, \quad (4.10)$$

where $H_2 = \int_0^1 h^2(t)dt > 0$. The following properties of the kernel $\Phi_{2,T}^h$ follow from Lemma 3.4, and formulas (3.13) and (4.10):

$$\begin{aligned} &\int_{\mathbb{R}} \Phi_{2,T}^h(\mu)(\mu)\,d\mu = 1, \qquad \int_{|\mu|\geq 1}\left|\Phi_{2,T}^h(\mu)\right|d\mu \leq C_h\,T^{-1}, \\ &\int_{|\mu|\leq 1}\left|\Phi_{2,T}^h(\mu)\mu^{\alpha}\right|d\mu \leq C_h \begin{cases} T^{-\alpha}, & \text{if } 0<\alpha<1 \\ T^{-1}\ln T, & \text{if } \alpha=1 \\ T^{-1}, & \text{if } \alpha>1. \end{cases} \end{aligned} \quad (4.11)$$

Since the function $\Phi_{2,T}^h(\mu)$ is even, we have

$$\int_{\mathbb{R}^2} f(\lambda)g(\lambda+\mu)\Phi_{2,T}^h(\mu)d\lambda d\mu = \int_{\mathbb{R}^2} f(\lambda+\mu)g(\lambda)\Phi_{2,T}^h(\mu)d\lambda d\mu,$$

and hence, using the equality

$$\int_{\mathbb{R}} f(\lambda)g(\lambda)dt = \int_{\mathbb{R}} f(\lambda+\mu)g(\lambda+\mu)dt, \qquad u \in \mathbb{R},$$

we get

$$2\Delta_{2,T}^{h}=\left|\int_{\mathbb{R}}\Phi_{2,T}^{h}(\mu)\int_{\mathbb{R}}\left[f(\lambda+\mu)-f(\lambda)\right]\left[g(\lambda+\mu)-g(\lambda)\right]d\lambda d\mu\right|. \tag{4.12}$$

Using Hölder's inequality from (4.12) we get

$$2\Delta_T\le\int_{\mathbb{R}}\left|\Phi_{2,T}^{h}(\mu)\right|\cdot\|f(\mu+\cdot)-f(\cdot)\|_{L^p}\|g(\mu+\cdot)-g(\cdot)\|_{L^q}d\mu \tag{4.13}$$

and hence by the conditions of the lemma,

$$\Delta_T\le C\int_{|\mu|\le1}\left|\Phi_{2,T}^{h}(\mu)\mu^{\beta_1+\beta_2}\right|d\mu+C\|f\|_{L^p}\|g\|_{L^q}\int_{|\mu|>1}\left|\Phi_{2,T}^{h}(\mu)\right|d\mu. \tag{4.14}$$

From (4.11) and (4.14) follows (4.9).

Lemma 4.2 is proved. □

Remark 4.1. For nontapered case ($h(t)=\mathbb{I}_{(0,1)}(t)$), Lemma 4.2 was proved in Ginovyan [17] (Lemma 4), (see also, Ginovyan et al. [24], Theorem 2.4). For discrete-time tapered case it was proved in Dahlhaus [9] (Lemma 4).

Proof of Theorem 2.1. We first show that the expected value of the estimator J_T^h is given by formula:

$$\mathbb{E}(J_T^h)=\int_{\mathbb{R}}\int_{\mathbb{R}}f(\lambda)g(\lambda+\mu)\Phi_{2,T}^{h}(\mu)d\lambda d\mu, \tag{4.15}$$

where $\Phi_{2,T}^{h}(\cdot)$ is defined by (3.16). Indeed, using (1.4), (1.12) and (3.16), we can write

$$\begin{aligned}
\mathbb{E}(J_T^h) &= \frac{1}{C_T}\int_0^T\int_0^T h_T(t)h_T(s)\widehat{g}(t-s)r(t-s)\,dt\,ds\\
&= \frac{1}{C_T}\int_0^T\int_0^T h_T(t)h_T(s)\int_{\mathbb{R}}e^{i\lambda(s-t)}g(\lambda)d\lambda\int_{\mathbb{R}}e^{i\mu(t-s)}f(\mu)d\mu\,dt\,ds\\
&= \frac{1}{C_T}\int_{\mathbb{R}}\int_{\mathbb{R}}g(\lambda)f(\mu)\int_0^T\int_0^T h_T(t)h_T(s)e^{i(\lambda-\mu)(s-t)}\,dt\,dsd\lambda d\mu\\
&= \frac{1}{C_T}\int_{\mathbb{R}}\int_{\mathbb{R}}g(\lambda)f(\mu)\left|\int_0^T h_T(t)e^{-i(\lambda-\mu)t}dt\right|^2 d\lambda d\mu\\
&= \frac{1}{C_T}\int_{\mathbb{R}}\int_{\mathbb{R}}f(\lambda)g(\mu)\left|H_{1,T}(\lambda-\mu)\right|^2 d\lambda d\mu\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}f(\lambda)g(\mu)\Phi_{2,T}^{h}(\lambda-\mu)d\lambda d\mu,
\end{aligned}$$

and (4.15) follows. Thus, for the bias $\mathbb{E}(J_T^h)-J$ of estimator J_T^h we have

$$\mathbb{E}(J_T^h)-J=\int_{\mathbb{R}}\int_{\mathbb{R}}f(\lambda)g(\lambda+\mu)\Phi_{2,T}^{h}(\mu)d\lambda d\mu-\int_{\mathbb{R}}f(\lambda)g(\lambda)d\lambda. \tag{4.16}$$

Now, the asymptotic relation (2.3) under conditions (A2) and (T) follows from (4.16) and Lemma 4.1, while under the conditions (A2′) and (T) it follows from (4.16) and Lemma 4.2.

Theorem 2.1 is proved. □

5. A central limit theorem for tapered Toeplitz type quadratic functionals. Proof of Theorem 2.2

In this section we study the asymptotic distribution of a suitable normalized stochastic process $\{Q_T^h(t),\, t \in [0,1]\}$, generated by a tapered Toeplitz type quadratic functional of a Lévy-driven continuous-time linear process $X(t)$ given by (1.1). We show that under conditions (A1) and (T) the process $Q_T(t)$ obeys a central limit theorem, that is, the finite-dimensional distributions of the standard $\sqrt{T}$ normalized process $Q_T(t)$ tend to those of a normalized standard Brownian motion. Then, using this result we prove Theorem 2.2.

We will use the following notation. The symbol $*$ will stand for the convolution:

$$(\varphi_1 * \varphi_2)(u) = \int_{\mathbb{R}} \varphi_1(u-x)\varphi_2(x)dx,$$

while the symbol $\bar{*}$ will be used to denote the reversed convolution:

$$(\varphi^{\bar{*}2})(u) = (\varphi \,\bar{*}\, \varphi)(u) = \int_{\mathbb{R}} \varphi(u+x)\varphi(x)dx.$$

Also, we will use the following well-known identities:

$$\mathcal{F}(\varphi_1 * \varphi_2) = \mathcal{F}(\varphi_1)\cdot\mathcal{F}(\varphi_2) \quad \text{and} \quad \mathcal{F}(\varphi_1 \bar{*} \varphi_2) = \mathcal{F}(\varphi_1)\cdot\overline{\mathcal{F}(\varphi_2)}, \tag{5.1}$$

where $\mathcal{F}(h) := \widehat{h}$ denotes the Fourier transform of a function h.

Denote by Q_T^h the tapered Toeplitz type quadratic functional of the process $X(t)$ from formula (1.12), that is,

$$Q_T^h := \int_0^T \int_0^T h_T(u)h_T(v)b(u-v)X(u)X(v)\,du\,dv. \tag{5.2}$$

We are interested in the asymptotic distribution (as $T \to \infty$) of the stochastic process $\{Q_T^h(t),\, t \in [0,1]\}$, generated by the functional Q_T^h:

$$Q_T^h(t) := \int_0^{Tt} \int_0^{Tt} h_T(u)h_T(v)X(u)X(v)dudv, \quad t \in [0,1]. \tag{5.3}$$

The next theorem, which is the tapered version of Theorem 2.1 of Bai et al. [3], contains sufficient conditions for the process $\{Q_T^h(t),\, t \in [0,1]\}$ to obey central limit theorem.

Theorem 5.1. *Under the conditions (A1) and (T) the process $Q_T^h(t)$ defined in (5.3) obeys central limit theorem. More precisely, we have*

$$\widetilde{Q}_T^h(t) := T^{-1/2}\left(Q_T^h(t) - \mathbb{E}Q_T^h(t)\right) \xrightarrow{f.d.d.} \sigma_h(Q)B(t), \quad \text{as} \quad T \to \infty, \tag{5.4}$$

where the symbol $\xrightarrow{f.d.d.}$ *stands for convergence of finite-dimensional distributions,* $B(t)$ *is a standard Brownian motion, and*

$$\sigma_h^2(Q) = H_4 \int_{\mathbb{R}} [2K_A(v) + \kappa_4 K_B(v)] \, dv, \tag{5.5}$$

where H_4 *is as in (1.8),* κ_4 *is the fourth cumulant of* $\xi(1)$*, and*

$$K_A(v) = \Big((a * b)^{\bar{*}2} \cdot a^{\bar{*}2}\Big)(v), \qquad K_B(v) = \Big((a*b) \cdot a\Big)^{\bar{*}2}(v). \tag{5.6}$$

We first introduce the notions of multiple *off-diagonal* (Itô-type) and *with-diagonal* (Stratonovich-type) stochastic integrals with respect to Lévy noise, and briefly discuss their properties (see, e.g., Bai et al. [3], Farré et al. [14], Peccati and Taqqu [30]). Let f be a function in $L^2(\mathbb{R}^k)$, then the following off-diagonal multiple stochastic integral, called Itô-Lévy integral, is well-defined:

$$I_k^\xi(f) = \int_{\mathbb{R}^k}' f(x_1, \dots, x_k) \xi(dx_1) \dots \xi(dx_k), \tag{5.7}$$

where $\xi(t)$ is a Lévy process with $\mathbb{E}\xi(t) = 0$ and $\mathrm{Var}[\xi(t)] = \sigma_\xi^2 t$, and the prime $'$ indicates that we do not integrate on the diagonals $x_i = x_j$, $i \neq j$.

The multiple integral $I_k^\xi(\cdot)$ satisfies the following inequality:

$$\|I_k^\xi(f)\|_{L^2(\Omega)}^2 \leq k! \sigma_\xi^{2k} \|f\|_{L^2(\mathbb{R}^k)}^2, \tag{5.8}$$

and the inequality in (5.8) becomes equality if f is symmetric:

$$\|I_k^\xi(f)\|_{L^2(\Omega)}^2 = k! \sigma_\xi^{2k} \|f\|_{L^2(\mathbb{R}^k)}^2. \tag{5.9}$$

We will need a stochastic Fubini's theorem (see Bai et al. [3], Lemma 3.1, or Peccati and Taqqu [30], Theorem 5.13.1).

Lemma 5.1. *Let* (S, μ) *be a measure space with* $\mu(S) < \infty$*, and let* $f(s, x_1, \dots, x_k)$ *be a function on* $S \times \mathbb{R}^k$ *such that*

$$\int_S \int_{\mathbb{R}^k} f^2(s, x_1, \dots, x_k) dx_1 \dots dx_k \mu(ds) < \infty,$$

then we can change the order of the multiple stochastic integration $I_k^\xi(\cdot)$ *and the deterministic integration* $\int_S f(s, \cdot)\mu(ds)$*:*

$$\int_S I_k^\xi\big(f(s, \cdot)\big) \mu(ds) = I_k^\xi \left(\int_S f(s, \cdot) \mu(ds) \right).$$

The *with-diagonal* counterpart of the integral $I_k^\xi(f)$, called a Stratonovich-type stochastic integral, is defined by

$$\mathring{I}_k^\xi(f) := \int_{\mathbb{R}^k} f(x_1, \dots, x_k) \xi(dx_1) \dots \xi(dx_k), \tag{5.10}$$

which includes all the diagonals. We refer to Farré et al. [14] for a comprehensive treatment of Stratonovich-type integrals $\mathring{I}_k^{\xi}(f)$. Observe that for the with-diagonal integral $\mathring{I}_k^{\xi}(f)$ to be well-defined, the integrand f needs also to be square-integrable on all the diagonals of $\mathbb{R}^k$ (see Bai et al. [3], Farré et al. [14]).

The with-diagonal integral $\mathring{I}_k^{\xi}(f)$ can be expressed by off-diagonal integrals of lower orders using the Hu-Meyer formula (see Farré et al. [14], Theorem 5.9). We will only use the special case when $k = 2$, in which case we have

$$\mathring{I}_2^{\xi}(f) = \int_{\mathbb{R}^2}' f(x_1, x_2)\xi(dx_1)\xi(dx_2) + \int_{\mathbb{R}} f(x, x)\xi_c^{(2)}(dx) + \int_{\mathbb{R}} f(x, x)dx, \quad (5.11)$$

where

$$\xi_c^{(2)}(t) = \xi^{(2)}(t) - \mathbb{E}\xi^{(2)}(t) = \xi^{(2)}(t) - |t|, \quad (5.12)$$

and $\xi^{(2)}(t)$ is the quadratic variation of $\xi(t)$, which is non-deterministic if $\xi(t)$ is non-Gaussian (see Farré et al. [14], equation (10)). The centered process $\xi_c^{(2)}(t)$ is called a second order *Teugels martingale*, which is a Lévy process with the same filtration as $\xi(t)$, whose quadratic variation is deterministic:

$$[\xi_c^{(2)}(t), \xi_c^{(2)}(t)] = \kappa_4 t,$$

where κ_4 is the fourth cumulant of $\xi(1)$. For any $f, g \in L^2(\mathbb{R})$, one has (see Farré et al. [14], page 2153),

$$\mathbb{E}\left[\int_{\mathbb{R}} f(x)\xi_c^{(2)}(dx) \int_{\mathbb{R}} g(x)\xi_c^{(2)}(dx)\right] = \kappa_4 \int_{\mathbb{R}} f(x)g(x)dx. \quad (5.13)$$

The decomposition (5.11) implies that

$$\mathbb{E}\mathring{I}_k^{\xi}(f) = \int_{\mathbb{R}} f(x, x)dx.$$

Consider now the following integrals, the first of which is an off-diagonal double integral and the second is a single integral with respect to Teugels martingale $\xi_c^{(2)}(t)$:

$$\int_{\mathbb{R}^2}' f(x_1, x_2)\xi(dx_1)\xi(dx_2) \quad \text{and} \quad \int_{\mathbb{R}} g(x)\xi_c^{(2)}(dx). \quad (5.14)$$

Notice that for any $f \in L^2(\mathbb{R}^2)$ and $g \in L^2(\mathbb{R})$ the integrals in (5.14) are uncorrelated (see Bai et al. [3]).

To prove Theorem 5.1, we first establish two lemmas. We set

$$R_T^h(x_1, x_2) = \frac{1}{\sqrt{T}} \int_0^T \int_0^T h_T(u)h_T(v)b(u - v)a(u - x_1)a(v - x_2)dudv, \quad (5.15)$$

and

$$S_T^h(x_1, x_2) = \frac{1}{\sqrt{T}} \int_0^T |h_T(v)|^2 \left[(a * b)(v - x_1)\right] \left[a(v - x_2)\right] dv. \quad (5.16)$$

Lemma 5.2. *Let $R_T^h(x_1,x_2)$ and $S_T^h(x_1,x_2)$ be as in (5.15) and (5.16) with $x_1 \neq x_2$. Then under conditions (A1) and (T) the following assertions hold.*

(a) We have

$$\lim_{T\to\infty} \|S_T^h\|^2_{L^2(\mathbb{R}^2)} = H_4 \int_{\mathbb{R}} K_A(u)du, \tag{5.17}$$

where $K_A(\cdot)$ is as in (5.6).

(b) We have

$$\lim_{T\to\infty} \|R_T^h - S_T^h\|_{L^2(\mathbb{R}^2)} = 0. \tag{5.18}$$

(c) For any $M > 0$, there exists a function $c_M^h(\cdot,\cdot)$ supported on $[-2M,2M]^2$, so that the function

$$S_{T,M}^h(x_1,x_2) = \frac{1}{\sqrt{T}} \int_0^T c_M^h(v-x_1, v-x_2)dv,$$

satisfies the relation:

$$\lim_{M\to\infty} \limsup_{T\to\infty} \|R_T^h - S_{T,M}^h\|_{L^2(\mathbb{R}^2)} = 0. \tag{5.19}$$

Proof. In the proof we use Young's inequality for convolution (see, e.g., Bogachev [6], Theorem 3.9.4), stating that for any numbers p, q, r satisfying $1 \leq p, q, r \leq \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, and for any functions $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$ the function $f * g$ is defined almost everywhere, $f * g \in L^r(\mathbb{R})$, and one has

$$\|f * g\|_r \leq \|f\|_p \|g\|_q. \tag{5.20}$$

In view of Riesz-Thorin theorem, without loss of generality, we can assume that

$$a(\cdot) \in L^p(\mathbb{R}), \quad b(\cdot) \in L^q(\mathbb{R}), \quad \frac{2}{p} + \frac{1}{q} = \frac{5}{2}. \tag{5.21}$$

Let p and q be as in (5.21). Define the numbers q_1, q_1^*, q_2 to satisfy the following equations:

$$\frac{1}{q_1} + \frac{1}{q_1^*} = 1, \quad 1 + \frac{1}{q_1^*} = \frac{2}{p}, \quad 1 + \frac{1}{q_1} = \frac{2}{q_2}, \quad 1 + \frac{1}{q_2} = \frac{1}{p} + \frac{1}{q}. \tag{5.22}$$

(Going from the last to the first equality in (5.22), one can solve successively for q_2, q_1^*, q_1 and then verify using (5.21) that the first equality in (5.22) holds.)

To prove (5.17) observe first that

$$\begin{aligned} &\|S_T^h\|^2_{L^2(\mathbb{R}^2)} \\ &= \frac{1}{T} \int_0^T \int_0^T |h_T(v_1)|^2 |h_T(v_2)|^2 \int_{\mathbb{R}} [(a*b)(v_1 - x_1)] \, [(a*b)(v_2 - x_1)] \, dx_1 \\ &\times \int_{\mathbb{R}} [a(v_1 - x_2)] \, . \, [a(v_2 - x_2)] \, dx_2 dv_1 dv_2 \end{aligned}$$

$$= \frac{1}{T}\int_{\mathbb{R}}\int_{\mathbb{R}} |h_T(v_1)|^2|h_T(v_2)|^2\Big((a*b)^{\bar{*}2}\cdot a^{\bar{*}2}\Big)(v_1-v_2)dv_1dv_2$$
$$= \frac{1}{T}\int_{\mathbb{R}} |h_T(v_2)|^2\int_{\mathbb{R}} |h_T(v+v_2)|^2\Big((a*b)^{\bar{*}2}\cdot a^{\bar{*}2}\Big)(v)dvdv_2$$
$$= \frac{1}{T}\int_{\mathbb{R}} \Big((a*b)^{\bar{*}2}\cdot a^{\bar{*}2}\Big)(v)\int_{\mathbb{R}} |h_T(u)|^2|h_T(v+v_2)|^2dv_2dv$$
$$= \int_{\mathbb{R}} \Big((a*b)^{\bar{*}2}\cdot a^{\bar{*}2}\Big)(v)\int_0^1 |h(u)|^2|h(\frac{v}{T}+u)|^2dudv. \tag{5.23}$$

Since h is a bounded and continuous function, we have by the dominated convergence theorem, that

$$\lim_{T\to\infty}\int_0^1 |h(u)|^2|h(\frac{v}{T}+u)|^2du = H_4, \qquad v\in\mathbb{R}. \tag{5.24}$$

On the other hand, by using Hölder's inequality and Young's inequality for convolution (see (5.20)), we can write (with $M_h = \|h\|_\infty < \infty$)

$$\begin{aligned}
\|S_T^h\|^2_{L^2(\mathbb{R}^2)} &\leq M_h^4\int_{\mathbb{R}}\Big((|a|*|b|)^{\bar{*}2}\cdot|a|^{\bar{*}2}\Big)(v)\ dv\\
&\underset{\text{Hölder}}{\leq} M_h^4\|(|a|*|b|)^{\bar{*}2}\|_{q_1}\||a|^{\bar{*}2}\|_{q_1^*}\\
&\underset{\text{Young}}{\leq} M_h^4\|(|a|*|b|)^{\bar{*}2}\|_{q_1}\|a\|_p^2 \underset{\text{Young}}{\leq} M_h^4\||a|*|b|\|^2_{q_2}\|a\|_p^2\\
&\underset{\text{Young}}{\leq} M_h^4\|a\|_p^4\|b\|_q^2<\infty.
\end{aligned} \tag{5.25}$$

Hence, by dominated convergence theorem, from (5.23) and (5.24) we obtain (5.17).

Observe that similar to (5.25) we can prove that

$$\|R_T^h\|^2_{L^2(\mathbb{R}^2)} \leq M_h^4\|a\|_p^4\|b\|_q^2. \tag{5.26}$$

Next, we have

$$\begin{aligned}
&R_T^h(x_1,x_2) - S_T^h(x_1,x_2)\\
&= \frac{1}{\sqrt{T}}\int_0^T\int_0^T h_T(u)h_t(v)b(u-v)a(u-x_1)a(v-x_2)dudv\\
&-\frac{1}{\sqrt{T}}\int_0^T\int_{\mathbb{R}} |h_t(v)|^2\, b(u-v)a(u-x_1)a(v-x_2)dudv\\
&= \frac{1}{\sqrt{T}}\int_0^T\int_0^T [h_T(u)-h_T(v)]\, h_T(v)b(u-v)a(u-x_1)a(v-x_2)dudv\\
&-\frac{1}{\sqrt{T}}\int_0^T\int_{\mathbb{R}\setminus[0,T]} |h_T(v)|^2\, b(u-v)a(u-x_1)a(v-x_2)dudv\\
&=: I_T^1(x_1,x_2) - I_T^2(x_1,x_2).
\end{aligned} \tag{5.27}$$

Consider first the case when the functions $a(\cdot)$ and $b(\cdot)$ have compact support. Assuming that $\operatorname{supp} b \subset [-M, M]$ we have

$$
\begin{aligned}
I_T^1(x_1,x_2) &\leq \omega\left(\frac{2M}{T},h\right)\frac{M_h}{\sqrt{T}}\int_0^T\int_{\mathbb{R}}|b(u-v)a(u-x_1)a(v-x_2)|\,dudv \\
&= \omega\left(\frac{2M}{T},h\right)\frac{M_h}{\sqrt{T}}\int_0^T\left(|a|*|b|\right)(v-x_1)|a(v-x_2)|dv,
\end{aligned}
$$

where $\omega(\delta,h)$ is the modulus of continuity of function h. As in (5.25) we can show that

$$
\|I_T^1\|_{L^2(\mathbb{R}^2)} \leq \omega\left(\frac{2M}{T},h\right)M_h\|a\|_p^4\|b\|_q^2 \to 0 \quad \text{as} \quad T\to\infty. \tag{5.28}
$$

For $I_T^2(x_1,x_2)$ we have

$$
\begin{aligned}
I_T^2(x_1,x_2) &\leq \frac{M_h^2}{\sqrt{T}}\int_0^T\int_T^\infty|b(u-v)a(u-x_1)a(v-x_2)|\,dudv \\
&+ \frac{M_h^2}{\sqrt{T}}\int_0^T\int_{-\infty}^0|b(u-v)a(u-x_1)a(v-x_2)|\,dudv \\
&= \frac{M_h^2}{\sqrt{T}}\int_{T-M}^T\int_T^{T+M}|b(u-v)a(u-x_1)a(v-x_2)|\,dudv \\
&+ \frac{M_h^2}{\sqrt{T}}\int_0^M dv\int_{-M}^0|b(u-v)a(u-x_1)a(v-x_2)|\,dudv \\
&=: I_{T,1}^2(x_1,x_2)+I_{T,2}^2(x_1,x_2).
\end{aligned}
$$

We have

$$
\begin{aligned}
I_{T,1}^2(x_1,x_2) &\leq \frac{M_h^2}{\sqrt{T}}\int_{T-M}^T\int_{\mathbb{R}}|b(u-v)a(u-x_1)du|\,|a(v-x_2)|\,dv \\
&= \frac{M_h^2}{\sqrt{T}}\int_{T-M}^T(|a|*|b|)(v-x_1)\,|a(v-x_2)|\,dv.
\end{aligned}
$$

Similar to (5.23) we obtain

$$
\begin{aligned}
\|I_{T,1}^2\|_{L^2(\mathbb{R}^2)}^2 &\leq \frac{M_h^2}{T}\int_{T-M}^T\int_{T-M}^T\Big((|a|*|b|)^{\bar{*}2}\cdot|a|^{\bar{*}2}\Big)(v_1-v_2)dv_1dv_2 \\
&\leq \frac{M_h^2M}{T}\int_{\mathbb{R}}\Big((|a|*|b|)^{\bar{*}2}\cdot|a|^{\bar{*}2}\Big)(v)dv \leq \frac{M_h^2M}{T}\|a\|_p^4\|b\|_q^2\to 0
\end{aligned}
$$

as $T\to\infty$. Similarly, we get

$$
\|I_{T,2}^2\|_{L^2(\mathbb{R}^2)}\to 0 \quad \text{as} \quad T\to\infty,
$$

and hence

$$
\|I_T^2\|_{L^2(\mathbb{R}^2)}\to 0 \quad \text{as} \quad T\to\infty.
$$

This, combined with (5.27) and (5.28) proves (5.18).

In the case when the supports of functions $a(\cdot)$ and $b(\cdot)$ are not bounded, for $M > 0$ we set

$$\begin{aligned} a_M(x) &= a(x)1_{[-M,M]}(x), \qquad \bar{a}_M(x) = a(x) - a_M(x), \\ b_M(x) &= b(x)1_{[-M,M]}(x), \qquad \bar{b}_M(x) = b(x) - b_M(x) \end{aligned}$$

and define

$$R^h_{M,T}(x_1, x_2) = \frac{1}{\sqrt{T}} \int_0^T \int_0^T h_T(u) h_T(v) b_M(u-v) a(u-x_1) a(v-x_2) du dv,$$
$$S^h_{M,T}(x_1, x_2) = \frac{1}{\sqrt{T}} \int_0^T |h_T(v)|^2 \left[(a_M * b_M)(v-x_1)\right] \left[a_M(v-x_2)\right] dv. \quad (5.29)$$

In view of (5.15), (5.29) and the identity

$$\begin{aligned} baa - b_M a_M a_M &= (baa - b_M aa) + (b_M aa - b_M a_M a) + (b_M a_M a - b_M a_M a_M) \\ &= \bar{b}_M aa + b_M \bar{a}_M a + b_M a_M \bar{a}_M, \end{aligned}$$

we have

$$\begin{aligned} &S^h_T(x_1, x_2) - S^h_{M,T}(x_1, x_2) \\ &= \frac{1}{\sqrt{T}} \int_0^T \int_{\mathbb{R}} |h_T(u)|^2 \quad \big[\bar{b}_M(u-v) a(u-x_1) a(v-x_2) \\ &+ b_M(u-v) \bar{a}_M(u-x_1) a(v-x_2) + b_M(u-v) a_M(u-x_1) \bar{a}_M(v-x_2)\big] du dv, \end{aligned}$$

and similar to (5.25), we get

$$\|S^h_T - S^h_{M,T}\|^2_{L^2(\mathbb{R}^2)} \le M_h^4 \big(\|b^-_M\|_q^2 \|a\|_p^4 + \|b_M\|_q^2 \|a^-_M\|_p^2 \|a\|_p^2 + \|b_M\|_q^2 \|a_M\|_p^2 \|a^-_M\|_p^2\big),$$

where the right-hand side does not depend on T.

Since $\|\bar{a}_M\|_p \to 0$ and $\|\bar{b}_M\|_q \to 0$ as $M \to \infty$, one obtains

$$\lim_{M\to\infty} \sup_{T>0} \|S^h_T - S^h_{M,T}\|_{L^2(\mathbb{R}^2)} = 0. \quad (5.30)$$

Similarly, we get

$$\lim_{M\to\infty} \sup_{T>0} \|R^h_T - R^h_{M,T}\|_{L^2(\mathbb{R}^2)} = 0 \quad (5.31)$$

which proves (5.18) in the general case.

To prove assertion (c) it is enough to set

$$c_M(x_1, x_2) = (a_M * b_M)(x_1) a_M(x_2), \quad (5.32)$$

and use (5.18), (5.30) and the inequality

$$\|R^h_T - S^h_{T,M}\|_{L^2(\mathbb{R}^2)} \le \|R^h_T - S^h_T\|_{L^2(\mathbb{R}^2)} + \|S^h_T - S^h_{T,M}\|_{L^2(\mathbb{R}^2)}.$$

Lemma 5.2 is proved. □

The next result is similar to Lemma 5.2, where $\mathbb{R}^2$ is replaced by $\mathbb{R}$. We set

$$R_T^h(x) = R_T^h(x,x) = \frac{1}{\sqrt{T}} \int_0^T \int_0^T h_T(u)h_T(v)b(u-v)a(u-x)a(v-x)dudv$$

and

$$S_T^h(x) = S_T^h(x,x) = \frac{1}{\sqrt{T}} \int_0^T |h_T(v)|^2 (a*b)(v-x)a(v-x)dv,$$

where $R_T^h(\cdot,\cdot)$ and $S_T^h(\cdot,\cdot)$ are as in (5.15) and (5.16) with $x_1 = x_2 = x$.

Lemma 5.3. *Let $a(\cdot)$ and $b(\cdot)$ be as in (2.1), with p and q satisfying*

$$1 \le p, q \le 2, \quad \frac{2}{p} + \frac{1}{q} \ge 2. \tag{5.33}$$

Then under condition (T) the following assertions hold.

(a) We have

$$\lim_{T\to\infty} \|S_T^h\|^2_{L^2(\mathbb{R})} = H_4 \int_{\mathbb{R}} K_B(u)du, \tag{5.34}$$

where $K_B(\cdot)$ is as in (5.6).

(b) We have

$$\lim_{T\to\infty} \|R_T^h - S_T^h\|_{L^2(\mathbb{R})} = 0. \tag{5.35}$$

(c) For any $M > 0$, there exists a function $d_M^h(\cdot)$ supported on $[-2M, 2M]$, so that the function

$$S_{T,M}^h(x) = \frac{1}{\sqrt{T}} \int_0^T d_M^h(v-x)dv,$$

satisfies the relation:

$$\lim_{M\to\infty} \limsup_{T\to\infty} \|R_T^h - S_{T,M}^h\|_{L^2(\mathbb{R})} = 0. \tag{5.36}$$

Remark 5.1. Obviously the condition (5.33) is implied by condition (2.2).

Proof of Lemma 5.3. The proof is similar to that of Lemma 5.2. We thus outline the key steps of the proof omitting the details.

As in the proof of Lemma 5.2, in view of Riesz-Thorin theorem, without loss of generality, we can assume that

$$\frac{2}{p} + \frac{1}{q} = 2.$$

Define the number p^* to satisfy the following equations:

$$\frac{1}{p} + \frac{1}{p^*} = 1, \quad 1 + \frac{1}{p^*} = \frac{1}{p} + \frac{1}{q}. \tag{5.37}$$

To prove (5.34) observe that

$$\begin{aligned}\|S_T^h\|_{L^2(\mathbb{R}^2)}^2 &= \frac{1}{T}\int_0^T\int_0^T |h_T(v_1)|^2|h_T(v_2)|^2 \\ &\times \int_{\mathbb{R}} [(a*b)(v_1-x)]\,[(a*b)(v_2-x)]\,[a(v_1-x)]\,.\,[a(v_2-x)]\,dx dv_1 dv_2 \\ &= \frac{1}{T}\int_{\mathbb{R}}\int_{\mathbb{R}} |h_T(v_1)|^2|h_T(v_2)|^2\Big((a*b)\cdot a\Big)^{\bar{*}2}(v_1-v_2)dv_1dv_2 \\ &= \frac{1}{T}\int_{\mathbb{R}} |h_T(v_2)|^2\int_{\mathbb{R}} |h_T(v+v_2)|^2\Big((a*b)\cdot a\Big)^{\bar{*}2}(v)dvdv_2 \\ &= \frac{1}{T}\int_{\mathbb{R}} \Big((a*b)\cdot a\Big)^{\bar{*}2}(v)\int_{\mathbb{R}} |h_T(u)|^2|h_T(v+v_2)|^2dv_2dv \\ &= \int_{\mathbb{R}} \Big((a*b)\cdot a\Big)^{\bar{*}2}(v)\int_0^1 |h(u)|^2|h(\frac{v}{T}+u)|^2dudv. \end{aligned} \tag{5.38}$$

Using Hölder's inequality and Young's inequality for convolution (see (5.20)) and (5.37), we can write ($M_h = \|h\|_\infty < \infty$)

$$\begin{aligned}\|S_T^h\|_{L^2(\mathbb{R}^2)}^2 \quad &\le \quad M_h^4\int_{\mathbb{R}}\Big((a*b)\cdot a\Big)^{\bar{*}2}(v)\ dv \underset{\text{Young}}{\le} M_h^4\|(|a|*|b|)\cdot a\|_1^2 \\ &\underset{\text{Hölder}}{\le} \quad M_h^4\|(|a|*|b|)\|_{p^*}^2\|a\|_p^2 \underset{\text{Young}}{\le} M_h^4\|a\|_p^4\|b\|_q^2 < \infty.\end{aligned}$$

Using (5.24) and the dominated convergence theorem, from (5.38) we obtain (5.34).

The proof of items (b) and (c) is similar to that of Lemma 5.2, and so is omitted.

Lemma 5.3 is proved. □

Proof of Theorem 5.1. By (5.11) and Lemma 5.1 one can write

$$\widetilde{Q}_T^h = A_T^h(t) + B_T^h(t),$$

where

$$A_T^h(t) = \int_{\mathbb{R}^2}' \frac{1}{\sqrt{T}}\int_0^{Tt}\int_0^{Tt} h_T(u)h_T(v)b(u-v)a(u-x_1)a(v-x_2)dudv\ \xi(dx_1)\xi(dx_2),$$

and

$$B_T^h(t) = \int_{\mathbb{R}} \frac{1}{\sqrt{T}}\int_0^{Tt}\int_0^{Tt} h_T(u)h_T(v)b(u-v)a(u-x)a(v-x)dudv\xi_c^{(2)}(dx). \tag{5.39}$$

Choosing $c_M^h(x_1,x_2)$ as in Lemma 5.2 and setting

$$A_{T,M}^h(t) = \int_{\mathbb{R}^2}' \frac{1}{\sqrt{T}}\int_0^{Tt} c_M^h(u-x_1,u-x_2)du\ \xi(dx_1)\xi(dx_2), \tag{5.40}$$

one has by (5.9) and relation (5.19) of Lemma 5.2 that

$$\lim_{M\to\infty}\limsup_{T\to\infty}\mathbb{E}|A_T^h(t)-A_{T,M}^h(t)|^2=0,\quad \forall t>0. \tag{5.41}$$

Choosing $d_M^h(x)$ as in Lemma 5.3 and setting

$$B_{T,M}^h(t)=\int_{\mathbb{R}}\frac{1}{\sqrt{T}}\int_0^{Tt}d_M^h(u-x)du\;\xi_c^{(2)}(dx), \tag{5.42}$$

one has by (5.9) and relation (5.36) of Lemma 5.3 that

$$\lim_{M\to\infty}\limsup_{T\to\infty}\mathbb{E}|B_T^h(t)-B_{T,M}^h(t)|^2=0,\quad \forall t>0. \tag{5.43}$$

To complete the proof of the theorem, in view of (5.41) and (5.43), it is enough to show that

$$\widetilde{Q}_{T,M}^h(t):=A_{T,M}^h(t)+B_{T,M}^h(t)\xrightarrow{f.d.d.}\sigma_{h,M}(Q)B(t)\quad\text{as}\quad T\to\infty \tag{5.44}$$

with $\sigma_{h,M}(Q)\geq 0$ satisfying

$$\lim_{M\to\infty}\sigma_{h,M}^2(Q)=\lim_{T\to\infty}\mathrm{Var}[A_T^h(1)+B_T^h(1)]=\sigma_h^2(Q). \tag{5.45}$$

To this end, observe first that by the stochastic Fubini Lemma 5.1, one has

$$\widetilde{Q}_{T,M}^h(t)=\frac{1}{\sqrt{T}}\int_0^{Tt}Y_M^h(u)du,$$

where

$$Y_M^h(u)=\int_{\mathbb{R}^2}' c_M^h(u-x_1,u-x_2)\xi(dx_1)\xi(dx_2)+\int_{\mathbb{R}}d_M^h(u-x)\;\xi_c^{(2)}(dx),$$

and $\xi_c^{(2)}(\cdot)$ is the Teugel martingale defined in (5.12). Note that $Y_M^h(u)$ is independent of the σ-field generated by $\{\xi(s):\ s<u-2M,\ s>u+2M\}$ since $c_M^h(\cdot,\cdot)$ vanishes outside $[-2M,2M]^2$ and $d_M^h(\cdot)$ vanishes outside $[-2M,2M]$, implying that $Y_M^h(u)$ is a stationary $4M$-dependent process. Then the convergence in (5.44) can be deduced from a classical central limit theorem for M-dependent processes by combining the discretization argument in the proof of Theorem 18.7.1 of Ibragimov and Linnik [27] and Theorem 5.2 of Billingsley [5].

To show (5.45), we first note that the random variables $A_T(1)$ and $B_T(1)$ are uncorrelated. Hence by (5.9) and (5.17) with $k=2$, we have

$$\mathrm{Var}[A_T^h]\to 2H_4\int_{\mathbb{R}}K_A(u)du,$$

and by (5.9), (5.13) and (5.34) we obtain

$$\mathrm{Var}[B_T^h]\to H_4\kappa_4\int_{\mathbb{R}}K_B(u)du.$$

This completes the proof of Theorem 5.1. □

Proof of Theorem 2.2. We have to prove that

$$T^{1/2}\left[J_T^h - E(J_T^h)\right] \xrightarrow{d} \eta \sim N\left(0, \sigma_h^2(J)\right) \quad \text{as} \quad T \to \infty, \tag{5.46}$$

where $\sigma_h^2(J)$ is given by (2.5) and (2.6).

Now we show that the relation (5.46) follows from Theorem 5.1. Indeed, observe first that in view of (1.8), (1.11), (1.12), (5.2) and (5.4) we can write

$$\begin{aligned} T^{1/2}\left[J_T^h - E(J_T^h)\right] &= T^{1/2}C_T^{-1}\left[Q_T^h - E(Q_T^h)\right] \\ &= \frac{T^{1/2}}{2\pi H_2 T}\left[Q_T^h - E(Q_T^h)\right] = \frac{1}{2\pi H_2}\widetilde{Q}_T^h(1). \end{aligned} \tag{5.47}$$

It follows from Theorem 5.1 that

$$\widetilde{Q}_T^h(1) := T^{-1/2}\left(Q_T^h(1) - EQ_T^h(1)\right) \xrightarrow{d} \eta_1 \sim N\left(0, \sigma_h^2(Q)\right) \quad \text{as} \quad T \to \infty, \tag{5.48}$$

where $\sigma_h^2(Q)$ is given by

$$\sigma_h^2(Q) = H_4 \int_{\mathbb{R}} [2K_A(u) + \kappa_4 K_B(u)]\, du. \tag{5.49}$$

Next, using (5.1) and Parseval-Plansherel theorem, we can write (see Bai et al. [3]):

$$\int_{\mathbb{R}} K_A(u)du = \int_{\mathbb{R}} \left((a*b)^{\bar{*}2} \cdot a^{\bar{*}2}\right)(u)du = 8\pi^3 \int_{\mathbb{R}} f^2(\lambda)g^2(\lambda)d\lambda \tag{5.50}$$

and

$$\int_{\mathbb{R}} K_B(u)du = \int_{\mathbb{R}} \left((a*b) \cdot a\right)^{\bar{*}2}(u)du = 4\pi^2 \left[\int_{\mathbb{R}} f(\lambda)g(\lambda)d\lambda\right]^2. \tag{5.51}$$

Therefore, in view of (5.47) and (5.49)-(5.51), we have

$$\begin{aligned} \sigma_h^2(J) &= \frac{1}{4\pi^2 H_2^2}\sigma_h^2(Q) \\ &= 4\pi e(h)\int_{\mathbb{R}} f^2(\lambda)g^2(\lambda)d\lambda + \kappa_4 e(h)\left[\int_{\mathbb{R}} f(\lambda)g(\lambda)d\lambda\right]^2, \end{aligned} \tag{5.52}$$

where $e(h)$ is as in (2.6). Now the relation (5.46) with $\sigma_h^2(J)$ given by (2.5) and (2.6) follows from (5.47), (5.48) and (5.52).

Theorem 2.2 is proved. □

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References

[1] V. V. Anh, N. N. Leonenko, and L. Sakhno, *Minimum contrast estimation of random processes based on information of second and third orders*, J. Statist. Planning Inference **137** (2007), 1302–1331. MR2301481

[2] F. Avram, N. N. Leonenko, and L. Sakhno, *Harmonic analysis tools for statistical inference in the spectral domain*, In: Dependence in Probability and Statistics, P. Doukhan et al. (eds.), Lecture Notes in Statistics, vol. 200, Springer, 2010, 59–70. MR2731826

[3] S. Bai, M. S. Ginovyan, and M. S. Taqqu, *Limit theorems for quadratic forms of Levy-driven continuous-time linear processes*. Stochast. Process. Appl. **126** (2016), 1036–1065. MR3461190

[4] R. Bentkus, *On the error of the estimate of the spectral function of a stationary process*, Litovskii Mat. Sb. **12** (1972), 55–71. MR0319332

[5] P. Billingsley, *Convergence of Probability Measures*, John Wiley & Sons, New York, 1999. MR1700749

[6] V. Bogachev, *Measure Theory*, vol. I, Springer-Verlag, Berlin, 2007 MR2267655

[7] D. R. Brillinger, *Time Series: Data Analysis and Theory,* Holden Day, San Francisco, 1981. MR0595684

[8] P. J. Brockwell, *Recent results in the theory and applications of CARMA processes,* Annals of the Institute of Statistical Mathematics **66**(4) (2014), 647–685. MR3224604

[9] R. Dahlhaus, *Spectral analysis with tapered data,* J. Time Ser. Anal. **4** (1983), 163–174. MR0732895

[10] R. Dahlhaus, *A functional limit theorem for tapered empirical spectral functions,* Stoch. Process. Appl. **19** (1985), 135–149. MR0780726

[11] R. Dahlhaus and H. Künsch, *Edge effects and efficient parameter estimation for stationary random fields,* Biometrika **74**(4) (1987), 877–882. MR0919857

[12] R. Dahlhaus and W. Wefelmeyer, *Asymptotically optimal estimation in misspecified time series models,* Ann. Statist. **24** (1996), 952–974. MR1401832

[13] M. Eichler, *Empirical spectral processes and their applications to stationary point processes,* Ann. Appl. Probab. **5**(4) (1995), 1161–1176. MR1384370

[14] M. Farré, M. Jolis, and F. Utzet, *Multiple Stratonovich integral and Hu–Meyer formula for Lévy processes*, The Annals of Probability **38**(6) (2010), 2136–2169. MR2683627

[15] M. S. Ginovyan, *Asymptotically efficient nonparametric estimation of functionals of a spectral density having zeros,* Theory Probab. Appl. **33**(2) (1988), 296–303. MR0954578

[16] M. S. Ginovyan, *On estimating the value of a linear functional of the spectral density of a Gaussian stationary process,* Theory Probab. Appl. **33**(4) (1988), 722–726. MR0979749

[17] M. S. Ginovyan, *On Toeplitz type quadratic functionals in Gaussian stationary process,* Probab. Theory Relat. Fields **100** (1994), 395–406. MR1305588

[18] M. S. Ginovyan, *Asymptotic properties of spectrum estimate of stationary Gaussian processes,* J. Cont. Math. Anal. **30**(1) (1995), 1–16. MR1643528

[19] M. S. Ginovyan, *Asymptotically efficient nonparametric estimation of nonlinear spectral functionals,* Acta Appl. Math. **78** (2003), 145–154. MR2024019

[20] M. S. Ginovyan, *Efficient Estimation of Spectral Functionals for Gaussian Stationary Models,* Comm. Stochast. Anal. **5**(1) (2011), 211–232. MR2808543

[21] M. S. Ginovyan, *Efficient Estimation of Spectral Functionals for Continuous-time Stationary Models,* Acta Appl. Math. **115**(2) (2011), 233–254. MR2818916

[22] M. S. Ginovyan and A. A. Sahakyan, *Limit Theorems for Toeplitz quadratic functionals of continuous-time stationary process,* Probab. Theory Relat. Fields **138** (2007), 551–579. MR2299719

[23] M. S. Ginovyan and A. A. Sahakyan, *Robust estimation for continuous-time linear models with memory,* Probability Theory and Mathematical Statistics, **95** (2016), 75–91. MR3631645

[24] M. S. Ginovyan, A. A. Sahakyan, and M. S. Taqqu, *The trace problem for Toeplitz matrices and operators and its impact in probability,* Probability Surveys **11** (2014), 393–440. MR3290440

[25] X. Guyon, *Random Fields on a Network: Modelling, Statistics and Applications,* Springer, New York, 1995. MR1344683

[26] R. Z. Has'minskii and I. A. Ibragimov, *Asymptotically efficient nonparametric estimation of functionals of a spectral density function,* Probab. Theory Related Fields **73** (1986), 447–461. MR0859842

[27] I. A. Ibragimov and Yu. V. Linnik, *Independent and stationary sequences of random variables*, Wolters-Noordhoff, 1971. MR0322926

[28] I. A. Ibragimov and R. Z. Khasminskii, *Asymptotically normal families of distributions and efficient estimation,* Ann. Statist. **19** (1991), 1681–1724. MR1135145

[29] N. N. Leonenko and L. Sakhno, *On the Whittle estimators for some classes of continuous-parameter random processes and fields,* Stat & Probab. Letters **76** (2006), 781–795. MR2266092

[30] G. Peccati and M. S. Taqqu, Moments, *Cumulants and Diagrams: a Survey With Computer Implementation*, Springer Verlag, 2011. MR2791919

[31] L. Sakhno, *Minimum Contrast Method for Parameter Estimation in the Spectral Domain,* In: Modern Stochastics and Applications (V. Korolyuk et al. eds.), Springer Optimization and Its Applications, vol. 90, Springer, 2014, 319–336. MR3236082

[32] M. Taniguchi, *Minimum contrast estimation for spectral densities of stationary processes*. J. R. Stat. Soc. Ser. B-Stat. Methodol. **49** (1987), 315–325. MR0928940