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Grigor Gegham Gevorgyan

Central extensions of groups

SYNOPSIS

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Ատենախոսության թեման հաստատվել է ԵՊՀ մաթեմատիկայի և մեխանիկայի ֆակուլտետի խորհրդի կողմից:

Գիտական ղեկավար՝	ֆիզ-մաթ. գիտ. դոկտոր Վ. Աթաբեկյան
Պաշտոնական ընդդիմախոսներ՝	ֆիզ-մաթ. գիտ. դոկտոր Վ.Հ.Միքայելյան
	ֆիզ-մաթ. գիտ. թեկնածու Շ.Ա.Ստեփանյան
Առաջատար կազմակերպություն՝	Հայ - Ռուսական համալսարան

Պաշտպանությունը կկայանա 2025թ. հուլիսի 8–ին ժ. 15^{00}–ին Երևանի պետական համալսարանում գործող ԲՈՀ-ի 050 մասնագիտական խորհրդի նիստում (0025, Երևան, Ալեք Մանուկյան 1):

Ատենախոսությանը կարելի է ծանոթանալ ԵՊՀ-ի գրադարանում:
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Մասնագիտական խորհրդի գիտական քարտուղար,
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Dissertation topic was approved at a meeting of academic council of the faculty of Mathematics and Mechanics of the Yerevan State University.

Supervisor:	Doctor of phys-math. sciences V. Atabekyan
Official opponents:	Doctor of phys-math. sciences V.H.Mikaelian
	candidate of phys-math. sciences Sh.A.Stepanyan
Leading organization:	Russian - Armenian University

Defense of the thesis will be held at the meeting of the specialized council 050 of HAC of Armenia at Yerevan State University on July 8, 2025 at 15^{00} (0025, Yerevan, A.Manoogian str. 1).

The thesis can be found in the library of the YSU.
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Scientific secretary of the specialized council,
Doctor of mathematics, professor K.L.Avetisyan

General characteristics of the work

Relevance of the theme. If in a group G the equality $x^n = 1$ is a law, then we say that G is a periodic group of the exponent n, or G is a n-periodic group. All such groups compose the variety (that is the class of all groups satisfying each one of a given set of laws). A free group of rank m of this variety is denoted by $B(m, n)$. One of the most prominent problems in Group theory is the well-known Burnside problem: "Is any finitely generated group $B(m, n)$ finite?"

The negative solution to the Burnside problem was obtained in the classical series of works of S.I. Adian and P.S. Novikov [1]. Several years later, Adian in his monograph [2] modified and enforced the constructed theory and proved the following famous theorem: "for all odd $n \geq 665$ and finite $m > 1$ groups $B(m, n)$ are infinite."In monograph [2] it was constructed and investigated several other groups with new unusual properties. The first important series of groups constructed in [2] by means of generating and defining relations is denoted by $B(m, n, \alpha)$, where m is the number of generators of the group, $n \geq 665$ is an arbitrary odd number, and α is a natural parameter. In [2] it was proved that a free Burnside group $B(m, n)$ is the direct limit of groups $B(m, n, \alpha)$ over α.

The next important class is associated with the well-known problems of finite basis of the group theory, which was posed by B. Neumann in 1937. In [2] (see also [3]), it was proved that for any odd $n \geq 1003$ the following family of identities of two variables

$$\{(x^{pn}y^{pn}x^{-pn}y^{-pn})^n = 1\},$$

where the parameter p runs over all prime numbers, is nonreducible, that is, no one of these identities is a consequence of the others. So, for any odd $n \geq 1003$ there exists a continuum of different manifolds $\mathcal{A}_n(\Pi)$ corresponding to different sets of prime numbers Π. For each fixed value $m > 1$ there exists a continuum of non isomorphic groups $\Gamma(m, n, \Pi)$, where $\Gamma(m, n, \Pi)$ is a relatively free groups of rank m of the variety $\mathcal{A}_n(\Pi)$.

Afterwards, in work [4] (see also [5]) it was shown that, if a group G contains no involutions and is given by a finite number of relators of the form A^n, $i = 1, 2, ..., k$, where all exponents n_i are divisible by a fixed odd number $n \geq 665$, then the problems of equality and conjugacy of words are decidable in it.

Any group of the above given ones has the following properties: the group has a system of defining relations of the form $A^n = 1$ for some elements of A and each element a of the group with a finite order satisfies the relation $a^n = 1$. The following class of so called n-torsion groups was introduced in the joint work [6]. Suppose that G is a group defined by a system of generators X, $\mathcal{P}$ is the set of all its finite-order elements written in generators X, and $n > 1$ is a fixed natural number. A group G is called *n-torsion group*, if it can be defined as follows:

$$G = \langle X | R^n = 1, R \in \mathcal{P} \rangle.$$

In [7] n-torsion groups were named *partially Burnside groups.* A cyclic group of order n and any absolutely free group are n-torsion groups for an arbitrary natural n. A. Karrass, W. Magnus, and D. Solitar in [8] proved that in the group $G = \langle X | A^n = 1 \rangle$, where A is a simple word (that is, the word that is not the proper power of another word), the element A has order n in G and each finite-order element in G is conjugate to some power of A. In addition to [1]–[5], in his another work [9] Adian investigated the free groups of the variety satisfying the identity $[x,y]^n = 1$ and proved that the commutator subgroup of these groups are not periodic groups for odd $n > 1001$ (it is the solution to the McDonald problem). On the basis of work [9] it is easy to derive that these free groups also are n-torsion groups. In work [6] of 2019, it was proved that an n-periodic product of any family of n-torsion groups is an n-torsion group for any odd $n \geq 665$. It is easy to see that free groups of any variety of the form $\mathcal{B}_n\mathcal{U}$, where $\mathcal{B}_n$ is the Burnside variety and $\mathcal{U}$ is the variety whose free groups are torsion-free, also are n-torsion groups. Some groups which, in fact, are n-torsion were also constructed and studied in the works by A.Yu. Ol'shanskii, S.V. Ivanov, I.G. Lysenok, and other authors (see, e.g., [10]–[11]). The authors of work [6] showed that for odd $n \geq 665$ for each n-torsion group we can construct a theory analogous to the theory constructed in monograph [2], which allows studying the n-torsion groups by the methods developed in [2] and investigating their key properties. In particular, in [6] it was proved that any n-torsion group may be given by a certain independent system of defining relations of the form $A^n = 1$ for certain words A. We call such representation of the n-torsion group the Adian representation.

The first main results in th Chapter 1 is about central extensions of n-torsion group and ther subgroups.

The next results are about central extensions of free groups of infinitely based varieties of S.I. Adian by free Abelian groups, about their some verbal subgroups, Schur multipliers and some applications.

In the Chapter 3 we consider the laws from the point of view of probability. Consider the following general problem. Let us choose an arbitrary word $w(x_1, ..., x_t)$ in the free group F_t of rank t and some finitely generated group G. Let $d_r(w)$ denote the number of all ordered rows $(g_1, ..., g_t)$ of length t of elements of the group G from the ball $B_{G,S}(r)$ for which $w(g_1, ..., g_t) = 1$ in G. Then the number $\frac{d_r(w)}{(\gamma_G(r))^t}$ shows the probability of the identity $w = 1$ being satisfied in the ball $B_{G,S}(r)$ of the group G. The question arises: if for $r \to \infty$ we have $\frac{d_r(w)}{(\gamma_G(r))^t} \to 1$, then is $w = 1$ an identity in G (see[12], Problem 1.3)?

The aim of the thesis:

• to prove that any Abelian group D can be embedded as a center into some group A_D so that the quotient group A_D/D is isomorphic to the given n-torsion group,

• to prove that every finite subgroup of each n-torsion group is a cyclic group,

• to prove that for any countable abelian group $\mathcal{D}$ and $n \geq 1003$ there exists a group $A_{\mathcal{D}}(G)$ such that verbal subgroup of the group $A_{\mathcal{D}}(G)$ corresponding to the word system $\{[x^{pn}, y^{pn}]^n\}$, $p \in \Pi$, coincides with the Abelian group $\mathcal{D}$,

• to prove that the group $A_{\mathcal{D}}(G)$ is a free group of rank m in the variety of groups defined by the system of identities $[[x^{pn}, y^{pn}]^n, z] = 1$, $p \in \Pi$,

• to prove that the Schur multiplier of the free group of the variety generated by the system of identities $\{[x^{pn}, y^{pn}]^n = 1\}$, $p \in \Pi$, is a free Abelian group of countable rank,

• to prove that In the variety of groups defined by the law $[x^n, y] = 1$ there exist two generator infinite torsion groups,

• to prove that For any sufficiently large odd number n there is a 4-generated group G, in which the identity $x_1^n = 1$ does not hold and we have $\frac{d_r(x_1^n)}{\gamma(r)} \to 1$ as $r \to \infty$.

The methods of investigations. In the thesis we apply methods and use the results obtained on the basis of the theory of Novikov-Adian, created for the study of periodic groups, as well as other known methods of Combinatorial Group Theory.

Scientific innovation. All results are new.

Practical and theoretical value. The results of the work have theoretical character. The results of the thesis can be used in the study of n-torsion groups, periodic groups, free groups of some varieties, central extensions, almost identities in some n-periodic products of groups.

Approbation of the results. The obtained results were presented at the scientific research seminar of Group theory at Yerevan State University, 2021-2025; at the International Conference Mal'tsev Meeting on the , 20??, Novosibirsk, Russia.

The main results of the thesis

1 Chapter 1.

Let us consider an alphabet of letters $\{x_1, x_2, \cdots, x_m, \cdots\}$. We denote by F_∞ the free group freely generated by $x_1, x_2, \cdots, x_m, \cdots$ and denote by F_m the free group generated by $x_1, x_2, \cdots, x_m$ for $m \geq 1$. If G is a group, $\alpha : \{x_1, x_2, \cdots, x_m, \cdots\} \to G$ is a mapping of the free generators of F_∞ into G, then the image of the some word $w(x_1, x_2, \cdots, x_m)$ under the corresponding homomorphism $\alpha : F_\infty \to G$ is called a value of the word w in G. In this situation the letters x_i are described as variables, and $w = w(x_1, x_2, \cdots, x_m)$ is referred to as a word in n variables.

Definition 1.1. *(see [13]) The word w is called a law or identity in the group G if the only possible value of w in G is 1 for all possible homomorphisms $\alpha : F_n \to G$.*

Definition 1.2. *The word u is a consequence of the set of words W if u is a law in a group G whenever every word from W is a law in G. The sets W_1 and W_2 of words are equivalent if every word in W_1 is a consequence of W_2 and vice versa.*

Definition 1.3. *A variety of groups is the class of all groups satisfying each one of a given set of laws.*

Definition 1.4. *(see [13]) A group G is called relatively free if it possesses a set of generators such that every relator of these generators is a law in G.*

Let X be an arbitrary group alphabet, $\mathcal{R}$ be some set of words, written in this alphabet, $n > 1$ be a fixed natural number, and

$$G = \langle X | R^n = 1, R \in \mathcal{R} \rangle \tag{1}$$

be the presentation of some group G.

Definition 1.5. *The group* (1) *is said to be an n-torsion group if for every element $Y \in G$, either $Y^n = 1$, or Y is of infinite order.*

Proposition 1.1. *([6, Proposition 3.2]) The system of defining relations $\{A^n = 1, A \in \bigcup_{\alpha=1}^{\infty} E_\alpha\}$ of the group*

$$G = \left\langle X \mid A^n = 1, A \in \bigcup_{\alpha=1}^{\infty} E_\alpha \right\rangle \tag{2}$$

is an independent system of relations, that is, any of these relations does not follow from the others.

A cyclic group of order n and infinite cyclic group are n-torsion groups for an arbitrary natural n. A. Karrass, W. Magnus, and D. Solitar in [8] proved that in the group $G = \langle X | A^n = 1 \rangle$, where A is a simple word (that is, the word that is not the proper power of another word), the element A has order n in G and each finite-order element in G is conjugate to some power of A.

In addition to [1]–[5], in his another work [9] Adian investigated the free groups of the variety satisfying the identity $[x, y]^n = 1$ and proved that the commutator (or derived) subgroup of these groups are not periodic groups for odd $n > 1001$ (it is the solution to the McDonald problem). On the basis of work [9] it is easy to derive that these free groups also are n-torsion groups.

In the work [6] of 2019, it was proved that an n-periodic product of any family of n-torsion groups is an n-torsion group for any odd $n \geq 665$. Recall that n-periodic products of groups were introduced by Adian in 1976 in [14].

The class of n-torsion groups is sufficiently broad. For instance, it is easy to see, that the free groups of arbitrary rank m as well as the free Burnside groups $B(m, n)$ are n-torsion groups for any natural n. By definition, the free Burnside group $B(m, n)$ of period n and rank m has the following presentation:

$$B(m, n) = \langle a_1, a_2, ..., a_m | w^n = 1 \rangle,$$

where w runs through the set of all words in the alphabet $\{a_1, a_2, ..., a_m\}$. The group $B(m, n)$ is the quotient group of the free group F_m of rank m by the normal (verbal) subgroup F_m^n generated by all possible nth powers of the elements of F_m.

It immediately follows from the definition of the n-torsion group G that the mapping from G to $B(X, n)$ is identical on the generators X is continued to the surjective homomorphism, where $B(X, n)$ is a free Burnside group of period n with the same system of generators X as G. In particular, any non cyclic n-torsion group is infinite if n has an odd divisor $k \geq 665$ or a divisor of the form $k = 16m \geq 8000$ due to the above-mentioned Adian theorem and due to the Lysenok theorem [15] (see also [10]).

As it was mentioned in [6], the groups $B(m, n, \alpha)$, $\Gamma(m, n, \Pi)$, $m \geq 1$ (the set of which is continual), the free groups of a variety satisfying the identity $[x, y]^n = 1$ are also n-torsion groups (see [3]–[9]). Some other n-torsion groups and their properties have been studied in the works (see, e.g., [16]–[17]). The following proposition allows us a way to construct a new series of n-torsion groups.

Proposition 1.2. *If F is an absolutely free group and N is a normal subgroup of F such that the quotient group F/N is a torsion-free group, then the group F/N^n is an n-torsion group for any $n \geq 1$, where N^n is the subgroup generated by the n-th powers of all elements from N .*

It is easy to see that free groups of any variety of the form $\mathcal{B}_n\mathcal{U}$, where $\mathcal{B}_n$ is the Burnside variety and $\mathcal{U}$ is the variety whose free groups are torsion-free, also are n-torsion groups. Some groups which, in fact, are n-torsion were also constructed and studied in the works by A.Yu. Ol'shanskii, S.V. Ivanov, I.G. Lysenok, and other authors (see, e.g., [10]–[11]).

For any odd $n \geq 1003$, by $\Gamma(m, n, \Pi)$ is denoted the free group of rank m of the variety of groups, determined by the following family of identities of two variables:

$$\{(x^{pn}y^{pn}x^{-pn}y^{-pn})^n = 1\}, \tag{3}$$

where the parameter p runs over an arbitrary set of prime numbers Π. In [29], it was proved that this groups are n-torsion and that any finite subgroup of each free group $\Gamma(m, n, \Pi)$ of an arbitrary rank $m \geq 1$ is a cyclic group. A similar assertion for free Burnside groups $B(m, n)$ of an odd period $n \geq 665$ and of any rank has been proved earlier by Adian in [2] (see Ch. VII of [2]). Notice that for absolute free groups this assertion has a simple proof.

In Chapter 1 we will prove a generalization of a result about finite subgroups of free Burnside groups. In [29] we proved that every finite subgroup of each relatively free n-torsion group is a cyclic group for any odd $n \geq 1003$. Later in [31] we proved the following a more general result.

The aim of Chapter 1 is to prove the following results.

Theorem 1.1. *(see [29], [31]) Every finite subgroup of each n-torsion group is a cyclic group of exponent n for any odd $n \geq 1003$.*

Theorem 1.2. *(see [31]) Any m-generated Abelian group D can be embedded as a center into some group A_D so that the quotient group A_D/D is isomorphic to the given n-torsion* *group with representation* (2)*, and in this group there are at least* m *defining relations.*

For the proof of Theorem 1.2 we propose the following construction of a group $A_{\mathcal{D}}(G)$ by fixing an arbitrary not more than countable Abelian group $\mathcal{D}$ defined by the generators and defining relations (see also [18]):

$$\mathcal{D} = \langle d_1, d_2, \ldots, d_i, \ldots | r = 1, r \in \mathcal{R} \rangle, \tag{4}$$

where $\mathcal{R}$ is some set of words in the group alphabet $d_1, d_2, \ldots, d_i, \ldots$. By condition of Theorem 1.2 the inequality $|J| \geq |\{d_1, d_2, \ldots, d_i, \ldots\}|$ is valid.

By $A_{\mathcal{D}}(G)$ we denote the group defined by the system of generators of two types

$$a_1, a_2, \ldots, a_m \tag{5}$$

and

$$d_1, d_2, \ldots, d_i, \ldots, \tag{6}$$

and the system of defining relations of three kinds

$$r = 1 \quad \text{for all} \quad r \in \mathcal{R}, \tag{7}$$

$$a_i d_j = d_j a_i, \tag{8}$$

$$A_j^n = d_j \tag{9}$$

for all $i = 1, 2, \ldots, m$, $j \in J$ and $A_j \in \mathcal{E}$. In this case, if $|J| > |\{d_1, d_2, \ldots, d_i, \ldots\}| = k$, then for all $j > k$ we define

$$A_j^n = d_k. \tag{10}$$

For the groups $A_{\mathcal{D}}(G)$ the following main theorem is true.

Theorem 1.3. *For any* $m > 1$*, odd* $n \geq 1003$*, and any Abelian group* $\mathcal{D}$ *with representation* (4) *the following statements are true:*

1. *The center of the group* $A_{\mathcal{D}}(G)$ *coincides with* $\mathcal{D}$*.*
2. *The quotient group of the group* $A_{\mathcal{D}}(G)$ *by the subgroup* $\mathcal{D}$ *is the given* n*-torsion group* G*.*

2 Chapter 2.

The aim of this chapter is to construct and study central extensions of free groups of infinitely based varieties of S.I. Adian by free Abelian groups. Using these extensions, we will obtain the description of the Schur multiplier of the free groups of Adian varieties.

As it was mentioned in Chapter **??**, these varieties are given by the following system of laws (identities) in two variables

$$\{(x^{pn}y^{pn}x^{-pn}y^{-pn})^n = 1\}, \tag{11}$$

where the parameter p runs through all prime numbers and $n \geq 1003$ is an arbitrary fixed odd number. The system (11) of laws is independent, that is, none of these identities is a consequence of the others (the question of the existence of such systems was posed by B. Neumann in 1937). This implies that for any odd $n \geq 1003$ there exists a continuum of different varieties $\mathcal{A}_n(\Pi)$ corresponding to different sets of primes Π, if we require that $p \in \Pi$. At the same time, for fixed $m > 1$ there exists a continuum of non-isomorphic groups $\Gamma(m, n, \Pi)$, where $\Gamma(m, n, \Pi)$ is a relatively free group of rank m of the variety $\mathcal{A}_n(\Pi)$. These varieties were first constructed by S.I. Adian in [19, 3]. Their detailed description is also contained in the monograph [2]. A number of new interesting properties of free groups $\Gamma(m, n, \Pi)$ of $\mathcal{A}_n(\Pi)$ varieties were obtained recently in [17, 20]. In fact, in [17] it was proved that the centralizer of any non-identity element in a relatively free group $\Gamma(m, n, \Pi)$ in any of the varieties under consideration is cyclic, and for every $m > 1$ the set of all non-isomorphic free groups of rank m in these varieties is of the cardinality of the continuum. Moreover, all these groups have trivial centre, all their abelian subgroups are cyclic, and all their non-trivial normal subgroups are infinite. For any free group $\Gamma(m, n, \Pi)$ in any of these varieties was also obtainad a description of the automorphisms of the semigroup $End(\Gamma(m, n, \Pi))$, answering a question posed by Plotkin in 2000. In particular, it was proved that the automorphism group of any semigroup $End(\Gamma(m, n, \Pi))$ is canonically embedded in the automorphisms group $Aut(End(\Gamma(m, n, \Pi)))$ of the semigroup $End(\Gamma(m, n, \Pi))$. In [20] it was proved that every normal automorphism (i.e., an automorphism that stabilizes any normal subgroup) of noncyclic free groups $\Gamma(m, n, \Pi)$ is an inner automorphism. Recall that the first result about normal automorphisms of some n-torsion groups was obtained in [21], [22]. In particular, in [21] it was proved that for an arbitrary odd $n \geq 1003$ and $m > 1$ every automorphism of the free Burnside group $B(m, n)$ that stabilizes every maximal normal subgroup N of $B(m, n)$ of infinite index is an inner automorphism. For the same values of m and n it was established that the subgroup of inner automorphisms of $Aut(B(m, n))$ is maximal among the subgroups in which the orders of the elements are bounded by n. Some other results about normal automorphisms of n-torsion groups can be found in [23], [24].

Let us instead of an Abelian group $\mathcal{D}$ (4) chose the *free Abelian group* $\mathcal{D}$ of countable rank given by the following generators and defining relations:

$$\mathcal{D} = \langle d_1, d_2, \ldots, d_i, \ldots \mid \forall i, j\ [d_i, d_j] = 1 \rangle. \tag{12}$$

Let us also instead of a group G take the group $G = \Gamma(m, n, \Pi)$.

The next result is a more precise modification of Proposition 1.3. When instead of n-torsion group G we take the free group $\Gamma = \Gamma(m, n, \Pi)$ of the variety defined by the law $[x^{pn}, y^{pn}]^n = 1$, then we get an opportunity to consider and to describe the verbal subgroup of $A_{\mathcal{D}}(\Gamma)$ generated by the word $[x^{pn}, y^{pn}]^n$. For the obtained group $A_{\mathcal{D}}(\Gamma)$ we get the following theorems.

Theorem 2.1. *The group $A_{\mathcal{D}}(\Gamma)$ is a free group of rank m in the variety of groups $\mathfrak{D}$ defined by the system of identities $[[x^{pn}, y^{pn}]^n, z] = 1$, $p \in \Pi$.*

Theorem 2.2. *For any $m > 1$ and odd $n \geq 1003$:*

1) *the center of the group $A_{\mathcal{D}}(\Gamma)$ coincides with $\mathcal{D}$;*

2) *the quotient group of the group $A_{\mathcal{D}}(\Gamma)$ by the subgroup $\mathcal{D}$ is the group $G = \Gamma(m, n, \Pi)$;*

3) *verbal subgroup of the group $A_{\mathcal{D}}(\Gamma)$ corresponding to the word system $\{[x^{pn}, y^{pn}]^n\}$, $p \in \Pi$, coincides with the Abelian group $\mathcal{D}$ (12).*

From point 3) of Theorem 2.2 it immediately follows

Corollary 2.1. *The center of the group $A_{\mathcal{D}}(\Gamma)$ is a free Abelian group of countable rank. Moreover, the elements $\{d_j | j \in \mathbb{N}\}$ are free generators of the center $A_{\mathcal{D}}(\Gamma)$.*

Theorems 2.1, 2.2 are also allow to investigate the Schur multipliers of the free groups $A_{\mathcal{D}}(G) = F_m/[F_m, N]$ for different ranks $m > 1$. It follows from Theorems 2.1, 2.2 that

Theorem 2.3. *The Schur multiplier of the free group of the variety generated by the system of identities $\{[x^{pn}, y^{pn}]^n = 1\}$, $p \in \Pi$, is a free Abelian group of countable rank for all finite ranks $m > 1$ and odd periods $n \geq 1003$.*

The next theorem gives a positive answer to Macedonska's question: *Does there exist a finitely generated torsion group of unbounded exponent generating a proper variety?.*

Theorem 2.4. *In the variety of groups defined by the law $[x^n, y] = 1$ there exist two generator infinite periodic groups with the unbounded exponent groups.*

The above mentioned Theorems 2.1, 2.2 and Corollary 2.1 were published in [31]. The proof of Theorem 2.3 was published in the paper [32].

3 Chapter 3.

Consider an arbitrary finitely generated group G with a set of generators S. Let $B_{G,S}(r)$ denote the ball of radius r centered at the unit element of the Cayley graph of this group (S-ball). We denote the number of elements in this ball by $\gamma_G(r)$. It is well known that for a free group of rank m the equality

$$\gamma_{F_m}(r) = \frac{m}{m-1}((2m-1)^r - 1)$$

holds.

Now consider the following general problem. Let us choose an arbitrary word $w(x_1, ..., x_t)$ in the free group F_t of rank t and some finitely generated group G. Let $d_r(w)$ denote the number of all ordered rows $(g_1, ..., g_t)$ of length t of elements of the group G from the ball $B_{G,S}(r)$ for which $w(g_1, ..., g_t) = 1$ in G. Then the number

$$\frac{d_r(w)}{(\gamma_G(r))^t}$$

shows the probability of the identity $w = 1$ being satisfied in the ball $B_{G,S}(r)$ of the group G. The question arises: if for $r \to \infty$ we have

$$\frac{d_r(w)}{(\gamma_G(r))^t} \to 1,$$

then is $w = 1$ an identity in G (see [12], Problem 1.3)?

A negative answer to this question was proved in [26]. Moreover, it was established that for every odd $n > 10^{10}$ and large enough $k \gg 1$ there exists a group G, generated by a finite set $S = \{s_1, ..., s_k\}$ such that the $\dfrac{d_r(x_1^n)}{\gamma_G(r)} \to 1$ for the word $w = x_1^n$, but $x_1^n = 1$ isn't an identity in G ($t = 1$). In the work [27] it was considered an n-periodic product of a free periodic group of rank 2 and an infinite cyclic group. It was proved that in this product, the probability of satisfying the identity $x^n = 1$ is 1, but it does not hold on the entire product. In [28] considered n-periodic products of cyclic groups of order n and two-generator relatively free groups satisfying identities of the form $[x^{pn}, y^{pn}]^n = 1$ and proved that in each of these products, the probability of satisfying $x^n = 1$ is equal to 1, despite the fact that the identity does not hold throughout any of these groups.

In this chapter we will strengthen the result of [26] by showing that we can take $k = 4$ and provide a simpler direct proof of the following main result.

Theorem 3.1. *For any sufficiently large odd number n there is a 4-generated group G, in which the identity $x_1^n = 1$ does not hold and we have*

$$\frac{d_r(x_1^n)}{\gamma(r)} \to 1 \quad as \quad r \to \infty.$$

The main result of this chapter was published in [33].

References

Список литературы

[1] P.S. Novikov, S.I. Adian, Infinite periodic groups. I, II, III., *Izv. Akad. Nauk SSSR Ser. Mat.*, 1968, 32:1, 212–244, 251–524, 709–731. transl. *Math. USSR-Izv.*, 1968, **2**:1, 212–244, 251–524, 709–731.

[2] S.I. Adian, The Burnside problem and identities in groups, *Springer-Verlag. Berlin-New York, Results in Mathematics and Related Areas*, 1979, **95**.

[3] S. I. Adian, Infinite irreducible systems of group identities, Math. USSR-Izv. **4**:4, 721–739 (1970). https://doi.org/10.1070/IM1970v004n04ABEH000928

[4] S. I. Adian, On the word problem for groups defined by periodic relations, in *Burnside Groups*, Lecture Notes in Mathematics, vol. 806 (Springer, Berlin, 1980), pp. 41–46.

[5] S. I. Adian, Groups with periodic defining relations, Math. Notes **83**, 293–300 (2008). https://doi.org/10.1134/S0001434608030012

[6] S. I. Adian and V. S. Atabekyan, n-torsion groups, J. Contemp. Math. Anal. **54**, 319–327 (2019). https://doi.org/10.3103/S1068362319060013

[7] N. S. Boatman, Partial-Burnside groups (Dissertation), 2012,

https://etd.library.vanderbilt.edu/etd-11302012-113318

http://hdl.handle.net/1803/14921

[8] A. Karrass, W. Magnus, and D. Solitar, Elements of finite order in groups with a single defining relation, Commun. Pure Appl. Math. **13**, 57–66 (1960). https://doi.org/10.1002/cpa.3160130107

[9] S. I. Adyan, Groups with periodic commutators, Dokl. Math. **62**, 174–176 (2000).

[10] S. V. Ivanov, The free Burnside groups of sufficiently large exponents, Int. J. of Algebra and Computation, 4 , 1-307, 1994.

[11] Atabekyan V.S. Uniform Nonamenability of Subgroups of Free Burnside Groups of Odd Period. *Math. Notes* **85** (2009), 496–502. https://doi.org/10.1134/S0001434609030213

[12] Amir G., Blachar G., et al. *Probabilistic Laws on Infinite Groups* (2023). https://doi.org/10.48550/arXiv.2304.09144

[13] H. Neumann, Varieties of groups, *Springer-Verlag, Ergebnisse der Mathemati und ihrer Grenzgebiete*, Berlin-Heidelberg-New York, Band 37, 1967.

[14] Adian S.I. Periodic Products of Groups. *Proc. Steklov Inst. Math.* **142** (1976), 3–21.

[15] I. G. Lysenok, Infinite Burnside groups of even exponent,Izv.Math.,60 (3), 453–654, 1996.

[16] A.Yu. Ol'shanskii, Geometry of defining relations in groups, *Math. Appl. (Soviet Ser.)*, Kluwer, Dordrecht, ISBN 0-7923-1394-1, 1991, 70, 505.

[17] S. I. Adian, V. S. Atabekyan, On free groups in the infinitely based varieties of S. I. Adian, Izv. RAN. Ser. Mat., 81:5 (2017), 3–14.

[18] S. I. Adian and V. S. Atabekyan, Central extensions of free periodic groups, Sb. Math. **209**, 1677–1689 (2018). https://doi.org/10.1070/SM9007

[19] S. I. Adyan, Infinite irreducible systems of group identities, Soviet Math. Dokl., 11 (1970), 113–115.

[20] S. I. Adian, V. S. Atabekyan, Normal Automorphisms of Free Groups of Infinitely Based Varieties, Math. Notes, 108:2 (2020), 149–154.

[21] V. S. Atabekyan, Normal automorphisms of free Burnside groups, Izv. Math., 75:2 (2011), 223–237.

[22]] E. A. Cherepanov, Normal automorphisms of free Burnside groups of large odd exponents, Internat. J. Algebra Comput. 16:5 (2006), 839–847.

[23] A. L. Gevorgyan, On automorphisms of periodic products of groups, Proceedings of the YSU, Physical and Mathematical Sciences, 2012, no. 2, 3–9.

[24] Atabekyan V.S., Aslanyan H.T., Grigoryan A.E., Normal Automorphisms of Free Burnside Groups of Period 3, Armen. J. Math., 9:2 (2017), 60–67.

[25] Amir, G., Blachar, G., Gerasimova, M., Kozma, G. (2023). Probabilistic Laws on Infinite Groups (arXiv:2304.09144). arXiv. http://arxiv.org/abs/2304.09144

[26] Goffer, G., Greenfeld, B. (2023). Probabilistic Burnside groups (arXiv:2306.11204). arXiv. http://arxiv.org/abs/2306.11204

[27] V. S. Atabekyan, A. A. Bayramyan, "Probabilistic Identities in n-Torsion Groups", Journal of Contemporary Mathematical Analysis, 59:6 (2024), 455-459

[28] V. S. Atabekyan, A. A. Bayramyan, V. H. Mikaelian, A continuum of non-isomorphic 3-generator groups with probabilistic law. (2025) arXiv:2504.20591. https://doi.org/10.48550/arXiv.2504.20591

List of publications of the author

[29] A. Gevorgyan, G.Gevorgyan. Finite Subgroups of the Relatively Free n-Torsion Groups, *Journal of Contemporary Mathematical Analysis* **55** (2020), 1–4. https://doi.org/10.3103/S1068362320010045

[30] G.G. Gevorgyan, A.G.Gevorgyan. Finite Subgroups of the Relatively Free n-Torsion Groups. International Conference MAL'TSEV MEETING November 16–20, 2020 Collection of Abstracts. p.143.

[31] V. Atabekyan, G.Gevorgyan. Central Extensions of n-Torsion Groups, *Journal of Contemporary Mathematical Analysis* **57** (2022), 26–34. https://doi.org/10.3103/S1068362322010034

[32] G. Gevorgyan. On Schur Multipliers Of Some Relatively Free Groups, *Proceedings of the YSU A: Physical and Mathematical Sciences* **Vol. 56 No. 1 (257)** (2022), 1–6. https://doi.org/10.46991/PYSU:A/2022.56.1.001

[33] G. Gevorkyan, V. Dilanyan. Almost identities in groups, *Proceedings of the YSU A: Physical and Mathematical Sciences* **Vol. 58 No. 1 (263)** (2024), 8–12. https://doi.org/10.46991/PYSU:A.2024.58.1.008

ԱՄՓՈՓՈՒՄ

Գրիգոր Գեղամի Գևորգյան

Խմբերի կենտրոնական ընդլայնումներ

Ատենախոսությունը նվիրված է n-ոլորումով խմբերին, դրանց կենտրոնական ընդլայնումներին և որոշ կիրառությունների: Ենթադրենք, որ G-ն X ծնորդների համակարգով խումբ է, $\mathcal{P}$-ն նրա բոլոր վերջավոր կարգի փարրերի բազմությունն է՝ արտահայտված X ծնորդներով, և $n > 1$-ը ֆիքսված բնական թիվ է: G խումբը կոչվում է *n-ոլորումով խումբ*, եթե այն կարող է սահմանվել հետևյալ կերպ. $G = \langle X | R^n = 1, R \in \mathcal{P} \rangle$:

Աշխատանքում ուսումնասիրվել են բավականաչափ մեծ կենտ n թվերին համապատասխանող n-ոլորումով խմբեր, հատուկ ալգորիթմով կառուցվել են դրանց կենտրոնական ընդլայնումներ, ուսումնասիրվել դրանց հատկությունները և կիրառություններ:

Ատենախոսությունում ստացվել են հետևյալ հիմնական արդյունքները.

- Ապացուցվել է, որ ցանկացած Աբելյան D խումբ կարելի է ներդրել որպես կենտրոն ինչ-որ A_D խմբի մեջ այնպես, որ A_D/D քանորդ խումբն իզոմորֆ լինի տրված n-ոլորումով խմբին,
- ապացուցվել է, որ յուրաքանչյուր n-ոլորումով խմբի ցանկացած վերջավոր ենթախումբ ցիկլիկ խումբ է,
- ապացուցվել է, որ ցանկացած հաշվելի աբելյան $\mathcal{D}$ խմբի և $n \geq 1003$ կենտ թվի համար գոյություն ունի $A_{\mathcal{D}}(G)$ խումբ, որի $[x^{pn}, y^{pn}]^n$ $(p \in \Pi)$ բառերի համակարգին համապատասխանող վերբալ ենթախումբը համընկնում է $\mathcal{D}$ Աբելյան խմբի հետ,
- ապացուցվել է, որ $A_{\mathcal{D}}(G)$ խումբը $[[x^{pn}, y^{pn}]^n, z] = 1$ $(p \in \Pi)$ նույնությունների համակարգով որոշված խմբերի բազմաձևության m ռանգի հարաբերկան ազատ խումբ է,
- ապացուցվել է, որ $[x^{pn}, y^{pn}]^n = 1$ $(p \in \Pi)$ նույնությունների համակարգով որոշված վերջավոր ռանգի հարաբերական ազատ խմբի Շուրի բազմապատկիչը հաշվելի ռանգի ազատ աբելյան խումբ է,
- ապացուցվել է, որ $[x^n, y] = 1$ նույնությունով սահմանված խմբերի բազմաձևությունում գոյություն ունեն երկու ծնորդով անսահմանափակ էքսպոնենտով անվերջ պարբերական խմբեր,
- ապացուցվել է, որ ցանկացած բավականաչափ մեծ կենտ n թվի համար գոյություն ունի 4 ծնորդով G խումբ, որում $x_1^n = 1$ նույնությունը տեղի չունի, սակայն $\dfrac{d_r(x_1^n)}{\gamma(r)} \to 1$ երբ $r \to \infty$:

Заключение

Геворгян Григор Гегамович

О центральных расширениях групп

Диссертация посвящена изучению групп с n-кручением, их центральным расширениям и некоторым приложениям. Предположим, что G группа с системой образующих X, $\mathcal{P}$ множество всех её элементов конечного порядка, выраженных через образующие X, и $n > 1$ фиксированное натуральное число. Группа G называется *группой с n-кручением*, если она может быть задана следующим образом: $G = \langle X \mid R^n = 1, R \in \mathcal{P} \rangle$.

В работе исследованы группы с n-кручением, соответствующие достаточно большим нечётным числам n, с помощью специального алгоритма построены их центральные расширения, изучены их свойства и приложения.

В диссертации получены следующие основные результаты:

• Доказано, что любую абелеву группу D можно вложить в качестве центра некоторой группы A_D так, что факторгруппа A_D/D изоморфна заданной группе с n-порождением.

• Доказано, что любая конечная подгруппа произвольной группы с n-кручением является циклической группой.

• Доказано, что для любой счётной абелевой группы $\mathcal{D}$ и нечетного числа $n \geq 1003$ существует группа $A_{\mathcal{D}}(G)$, у которой вербальная подгруппа, соответствующая системе слов $[x^{pn}, y^{pn}]^n$ $(p \in \Pi)$, совпадает с абелевой группой $\mathcal{D}$.

• Доказано, что группа $A_{\mathcal{D}}(G)$ является свободной группой ранга m в многообразии групп, заданном системой тождеств $[[x^{pn}, y^{pn}]^n, z] = 1$ $(p \in \Pi)$.

• Доказано, что множитель Шура свободной группы конечного ранга многообразия групп, порожденной системой тождеств $[x^{pn}, y^{pn}]^n = 1$ $(p \in \Pi)$ является свободной абелевой группой счётного ранга.

• Доказано, что в многообразии групп, заданном тождеством $[x^n, y] = 1$, существуют бесконечные периодические группы с неограниченной экспонентой с двумя порождающими.

• Доказано, что для любого достаточно большого нечётного числа n существует группа G с четырьмя образующими, в которой тождество $x_1^n = 1$ не выполняется, однако $\dfrac{d_r(x_1^n)}{\gamma(r)} \to 1$ при $r \to \infty$.