ԵՐԵՎԱՆԻ ՊԵՏԱԿԱՆ ՀԱՄԱԼՍԱՐԱՆ

Արման Արթուրի Բայրամյան

Պարբերական խմբեր, դրանց ավտոմորֆիզմները և հավանականային նույնություններ խմբերում

Ա.01.06 «Հանրահաշիվ և թվերի տեսություն» մասնագիտությամբ
ֆիզիկամաթեմատիկական գիտությունների թեկնածուի
գիտական աստիճանի հայցման ատենախոսության

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Arman Bayramyan

Periodic Groups, Their Automorphisms and Probabilistic Identities in Groups

SYNOPSIS

of dissertation for the degree of candidate of
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Ատենախոսության թեման հաստատվել է Երևանի պետական համալսարանում։

Գիտական ղեկավար՝ ֆիզ.-մաթ. գիտ. դոկտոր Վ. Ս. Աթաբեկյան

Պաշտոնական ընդդիմախոսներ՝ ֆիզ.-մաթ. գիտ. դոկտոր Յու. Մ. Մովսիսյան
ֆիզ.-մաթ. գիտ. դոկտոր Ա. Է. Գուտերման

Առաջատար կազմակերպություն՝ ՀՀ ԳԱԱ Մաթեմատիկայի ինստիտուտ

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Մասնագիտական խորհրդի գիտական քարտուղար,
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The topic of the dissertation was approved at Yerevan State University.

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Leading organization: Institute of Mathematics NAS RA

The defense of the dissertation will take place on July 2, 2026 at 15^{00} at the meeting of the Specialized Council of Mathematics 050 of the HESC of RA acting at Yerevan State University (address: Yerevan, 0025, 1 Alex Manoogian).

The dissertation is available at the library of YSU.

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Overview

Relevance of the topic. The Burnside problem, posed by William Burnside in 1902 [1], is one of the most influential problems in twentieth-century group theory. Its general form asks whether a finitely generated group in which every element has finite order must be finite; the bounded form asks the same question for groups of finite exponent n, that is, groups satisfying $g^n = 1$ for every element g. The bounded problem is equivalent to asking whether the free Burnside group of rank m and exponent n,

$$B(m, n) = \langle a_1, \dots, a_m \mid w^n = 1 \text{ for all } w \in F_m \rangle,$$

is finite. Burnside himself proved that $B(m, n)$ is finite for $n \le 3$ [1]; the cases $n = 4$ and $n = 6$ were settled by I. Sanov [2] and M. Hall [3], respectively. A negative answer to the general problem was given by E. Golod and I. Shafarevich [4], who constructed a finitely generated infinite p-group. The bounded problem was resolved in 1968 by P. S. Novikov and S. I. Adian [5], who proved that $B(m, n)$ is infinite for $m \ge 2$ and odd $n \ge 4381$; Adian later refined this bound to odd $n \ge 665$ in his monograph [6]. The publication of the monograph [6] was followed by new interpretations and further development of the theory. We particularly emphasize a number of remarkable works by A. Yu. Olshanskii (see [7] and the references therein), as well as the works of S. Ivanov [8] and I. Lysenok [9, 10].

The combinatorial techniques developed by Novikov and Adian have had a profound impact well beyond the original Burnside problem. The structural theory of free Burnside groups has given rise to several new constructions, in particular the n-periodic product of groups, introduced by Adian [11], and n-torsion (or partial Burnside) groups [12, 13], defined as groups in which every element either has infinite order or satisfies $x^n = 1$. Both notions play a central role in the present dissertation.

A new direction in the study of infinite groups concerns probabilistic identities. The history of probabilistic questions in group theory goes back to Gustafson's classical observation [14]: in a compact group, the commuting probability is either equal to 1 (the group is abelian) or at most $5/8$. Similar gap phenomena have been established for finite solvable groups: Guralnick and Wilson [15] showed that if the probability that two random elements generate a solvable subgroup of a finite group exceeds $11/30$, then the group itself is solvable, and analogous gap results for the Burnside-type identities $x^p = 1$ and various commutator-like words are due to Laffey [16, 17] and to Delizia, Jezernik, Moravec and Nicotera [18].

For a finitely generated infinite group there is no uniform probability measure, and a sequence of measures has to be specified instead. Following Antolín, Martino and Ventura [19] and Amir, Blachar, Gerasimova and Kozma [20], one fixes a sequence of probability measures $\{\mu_k\}$ on G and says that a word w is an M-probabilistic identity in G if

$$\limsup_{k \to \infty} \mu_k\big(\{(g_1, \dots, g_m) : w(g_1, \dots, g_m) = 1\}\big) = 1.$$

The central question in this framework was posed in [20] (Question 13.3): does the probabilistic satisfaction of an identity imply its universal satisfaction in G? In essence, this question asks whether every almost-identity is a genuine identity.

For some words the answer is positive: Tointon [21] proved that a positive commuting probability $P([x, y] = 1)$ forces G to be virtually abelian, and Amir et al. [20] obtained the analogous result for the identity $x^2 = 1$. The case of the Burnside identity $x^n = 1$ is fundamentally different. The transience of the simple symmetric random walk on free Burnside groups, proved by

Adian [22], suggests that an n-periodic product can satisfy $x^n = 1$ "with high probability" even when it contains elements of infinite order. This intuition was confirmed by the author together with V. S. Atabekyan [23], whose construction uses only 3 generators and produces an n-torsion group with no free subgroup of rank 2. The construction of [24] sharpens the latter result further. Question 13.3 of [20] is thus answered in the negative for $x^n = 1$, while remaining an open problem for other identities. Another solution to this question was obtained by Goffer, Greenfeld and Olshanskii [25], who constructed examples resolving this and several related problems in the same direction. In contrast to our approach, the construction in [25] requires both large rank and large exponent and yields a group containing a non-abelian free subgroup.

A separate line of research, also active and important, concerns the automorphisms of free Burnside groups and their fixed points. An interesting problem in this area was formulated by V. D. Mazurov in the Kourovka Notebook (Problem 17.70 in [26]): does an automorphism of prime order q of $B(q, p)$ that cyclically permutes the free generators have a non-trivial fixed point? The case $q = 2$ was settled affirmatively by Atabekyan and Aslanyan [27] by a direct construction, while the case of odd primes q remains open.

The aim and objectives of the thesis. The main objectives of the dissertation are the following.

1. To construct a finitely generated group with a probabilistic identity which is not an identity of the group, thus answering Question 13.3 of Amir, Blachar, Gerasimova and Kozma [20].

2. To obtain explicit lower bounds for the growth function of the free Burnside group $B(m, n)$ for arbitrary rank $m \geq 2$, needed as an important tool for objective (1).

3. To study the automorphisms of free Burnside groups and, in particular, Mazurov's fixed-point problem [26].

Scientific novelty. All the main results presented in the dissertation are new. The principal novel contributions are:

- Adian's growth estimate for $B(2, n)$ [6] is extended to all ranks $m \geq 2$ (Theorem 1.1). This bound is the key technical ingredient in the proofs of Theorems 1.2 and 1.3.

- Question 13.3 of Amir, Blachar, Gerasimova and Kozma [20] is answered in the negative. A continuum of pairwise non-isomorphic 3-generator groups is constructed, in each of which $x^n = 1$ holds probabilistically but not universally (Theorems 1.2 and 1.3).

- Mazurov's problem (Kourovka Notebook 17.70 [26]) is reformulated in the language of non-abelian cohomology. It is shown (Theorem 1.4) that an automorphism α of prime order q has a non-trivial fixed point in $B(q, p)$ if and only if the first non-abelian cohomology set $H^1(\langle\alpha\rangle, N)$ is non-trivial, where N is the normal closure of all p-th powers in the free group F_q.

Theoretical and practical significance. The dissertation is of a theoretical nature. The combinatorial and geometric methods developed in the dissertation may be useful in further structural studies of infinite groups and in related questions of geometric and asymptotic group theory.

Method of research. The work combines methods of combinatorial group theory, geometric group theory and homological algebra. The main tools are the Novikov–Adian theory [5, 6], Olshanskii's graded small cancellation theory [7], van Kampen diagrams, Adian's theory of n-periodic products of groups [11], and the framework of n-torsion groups [12, 13]. For the part concerning automorphisms and fixed points, non-abelian cohomology and Bass–Serre theory of group actions on trees [28] are used.

Approbation of results. The main results of the dissertation were reported at several international conferences, including:

- International Conference on Algebra, Logic, and Their Applications, Yerevan, Armenia, 2024.
- Young Geometric Group Theory XIII, Copenhagen, Denmark, 2025.
- Postgraduate Group Theory Conference, Bristol, UK, 2025.
- International Scientific Conference "Lomonosov", Moscow, Russia, 2026.

Publications. The main results of the dissertation are published in 3 papers in peer-reviewed mathematical journals [23, 24, 29]; a complete list is given at the end of this synopsis.

Structure and content of the thesis. The dissertation is written in English. It consists of an Introduction, five main chapters, a Conclusion and a bibliography of 75 titles. The total volume is 73 pages.

The Main Content of the Dissertation

Chapter 1. Introduction. The chapter begins with a survey of the history of the Burnside problem and its resolution by Novikov and Adian [5, 6]. Then we introduce the algebraic constructions on which the dissertation builds. For odd $n \geq 665$, the n-periodic product of a family of groups $\{G_i\}_{i\in I}$, due to Adian [11], is a quotient $\prod_{i\in I}^n G_i$ of the free product $*_{i\in I} G_i$ by a system of relations of the form $A^n = 1$ chosen according to a specific inductive procedure analogous to the Novikov–Adian construction for free Burnside groups. A key structural property of the n-periodic product is the following: for every element $x \in \prod_{i\in I}^n G_i$, either $x^n = 1$, or x is conjugate to an element of one of the factors G_i. The class of n-torsion groups (also called partial Burnside groups) [12, 13] consists of groups in which every element either has infinite order or satisfies $x^n = 1$. These two notions are tightly linked: an n-periodic product of n-torsion groups is itself an n-torsion group.

Another powerful result from Adian's work that plays an important role in this thesis is his construction of infinite independent systems of group identities. In [30], Adian showed that for any odd $n \geq 1003$, the following family of two-variable identities is irreducible:

$$\{[x^{pn}, y^{pn}]^n = 1 : p \text{ prime}\}.$$

For each subset $\mathcal{P}$ of the set of primes, the relatively free group of rank 2 in the variety defined by the identities

$$\{[x^{pn}, y^{pn}]^n = 1 : p \in \mathcal{P}\},$$

is denoted $\mathcal{F}(n, \mathcal{P})$. By the irreducibility of the system, distinct subsets $\mathcal{P}$ yield non-isomorphic groups $\mathcal{F}(n, \mathcal{P})$, producing a continuum of pairwise non-isomorphic 2-generator groups.

The chapter concludes by formulating Mazurov's problem [26] on the fixed points of automorphisms of $B(q, p)$ that cyclically permute the free generators, and by stating the main theorems of the dissertation.

The numbering of theorems and lemmas in this synopsis follows that of the dissertation. The four main theorems stated in Section 1.7 are:

Theorem 1.1 ([29]). *Let S be a free generating set for the free Burnside group $B(m,n)$, and let $\gamma = \gamma_{B(m,n),S}$ be the growth function of $B(m,n)$ with respect to S. Then for all $m \geq 2$, odd $n \geq 665$, and any natural number r,*

$$\gamma(r) > \frac{m}{m-1}\left(2m - 1 - 2 \cdot 10^{-3}\right)^r - 1.$$

Theorem 1.2 ([23]). *Let n be a sufficiently large fixed odd number. In the n-periodic product $G = B(2,n) *^n \mathbb{Z}$, the identity $x^n = 1$ is an M-probabilistic identity, but does not hold universally.*

Theorem 1.3 ([24]). *For any sufficiently large fixed odd n, in the n-periodic product $\mathbb{A}_{\mathcal{P}} = \mathcal{F}(n,\mathcal{P}) *^n \mathbb{Z}_n$ the identity $x^n = 1$ is an M-probabilistic identity but does not hold universally. Moreover, the groups $\{\mathbb{A}_{\mathcal{P}}\}$ form a continuum family of pairwise non-isomorphic 3-generator groups, indexed by the subsets $\mathcal{P}$ of the set of primes.*

Theorem 1.4. *Let $q \geq 2$ be prime, $p \geq 2$, and let α be the automorphism of $B(q,p)$ of order q induced by the cyclic permutation of the free generators. Then:*

1. *$H^1(\langle\alpha\rangle, F_q) = \{1\}$ for all $q \geq 2$;*
2. *there is a bijection of pointed sets $C_{B(q,p)}(\alpha) \cong H^1(\langle\alpha\rangle, N)$,*

where $C_{B(q,p)}(\alpha)$ is the fixed-point subgroup of α and $N = \langle\langle w^p : w \in F_q\rangle\rangle$. In particular, α has a non-trivial fixed point in $B(q,p)$ if and only if $H^1(\langle\alpha\rangle, N) \neq \{1\}$.

Together with these theorems, Section 1.7 also states the following corollary on n-torsion groups.

Corollary 1.1 ([29]). *Let G be an m-generated group with $m \geq 2$. If G admits a surjective homomorphism onto $B(m,n)$ for some odd $n \geq 665$, then there exists an m-element generating set S for G such that*

$$\gamma_{G,S}(r) > \frac{m}{m-1}\left(2m - 1 - 2 \cdot 10^{-3}\right)^r - 1.$$

In particular, all finitely generated non-cyclic n-torsion groups for odd $n \geq 665$ satisfy this bound.

Chapter 2. Periodic groups and related constructions. This chapter collects the algebraic constructions developed by Adian and his collaborators that form the foundation for the present work: free Burnside groups, n-periodic products and n-torsion groups.

Definition 2.2. *For integers $m \geq 1$ and $n \geq 2$, the free Burnside group of rank m and exponent n is the quotient of the free group F_m by the normal subgroup generated by all n-th powers. Equivalently,*

$$B(m,n) = \langle a_1, \dots, a_m \mid w^n = 1 \text{ for all } w \in F_m\rangle.$$

A summary of properties of $B(m,n)$ established by the Novikov–Adian theory [5, 6] is given (Theorem 2.1, Proposition 2.1). For odd $n \geq 665$ the free Burnside groups of rank $m \geq 2$ and exponent n are infinite groups; word and conjugacy problems in $B(m,n)$ are decidable; every abelian subgroup is cyclic of order dividing n; every finite subgroup is cyclic; the centre is trivial; the group has exponential growth and is not amenable.

We then turn to n-periodic products of groups, introduced by Adian in [11]. This construction provides a unified framework in which the Novikov–Adian classification of periodic words extends from the free group to the free product of an arbitrary family of groups. The n-periodic product is an exact, associative and hereditary operation; these are properties shared by both the free product and the direct product. The n-periodic product thus resolves Maltsev's problem on the existence of group products with these three properties beyond the free product and the direct product [11, 31]. For the constructions in the dissertation, the central property of n-periodic products is the following dichotomy.

Theorem 2.7 (Adian, [11]). *Let $n \geq 665$ be odd. For every element $x \in \prod_{i \in I}^{n} G_i$, either $x^n = 1$, or x is conjugate to an element of one of the factors G_i.*

The chapter then introduces n-torsion groups (also called partial Burnside groups), studied by Adian–Atabekyan [12] and by Boatman [13].

Definition 2.4 ([12]). *A group G with presentation*

$$G = \langle S \mid R^n = 1,\ R \in \mathcal{R} \rangle$$

for some set of words $\mathcal{R}$ over S is called an n-torsion group if for every element $y \in G$, either $y^n = 1$ or y has infinite order.

Cyclic groups, free groups and free Burnside groups are n-torsion groups. The following four results, stated in Section 2.4 of the dissertation, are used throughout the subsequent chapters.

Theorem 2.8 ([12]). *For odd $n \geq 665$, the n-periodic product of any family of n-torsion groups is itself an n-torsion group.*

Corollary 2.1. *For odd $n \geq 665$, the groups $B(m,n) *^n \mathbb{Z}$ are n-torsion groups for any $m \geq 1$.*

Proposition 2.2([12]). *Every m-generated n-torsion group G admits a canonical surjective homomorphism $G \twoheadrightarrow B(m,n)$.*

Lemma 2.6 ([13]). *Let G be an n-torsion group with generating set S and let W be a word over $S^{\pm 1}$. If U is the shortest word such that W is conjugate to a power of U in G, then every letter $s \in S^{\pm 1}$ occurring in U also occurs in W.*

The chapter ends with Adian's construction [30] of infinite independent systems of group identities, which gives access to a continuum of distinct group varieties.

Theorem 2.9 ([30]). *For any odd $n \geq 1003$, the family of two-variable identities*

$$\{[x^{pn}, y^{pn}]^n = 1 : p \text{ prime}\}$$

is irreducible.

For each subset $\mathcal{P}$ of the set of primes, let $\mathcal{F}(n,\mathcal{P})$ denote the relatively free group of rank 2 in the variety defined by the identities $\{[x^{pn}, y^{pn}]^n = 1 : p \in \mathcal{P}\}$. The following two propositions, established by Adian and Atabekyan in [32], are then recorded.

Proposition 2.3([32]). *For distinct subsets $\mathcal{P}$ and $\mathcal{P}'$ of the set of primes, the groups $\mathcal{F}(n,\mathcal{P})$ and $\mathcal{F}(n,\mathcal{P}')$ are non-isomorphic.*

Proposition 2.4 ([32]). *For odd $n \geq 1003$, the groups $\mathcal{F}(n,\mathcal{P})$ are n-torsion groups.*

Since the set of all subsets of primes is uncountable, Proposition 2.3 produces a continuum of pairwise non-isomorphic 2-generator n-torsion groups. Combining Proposition 2.4 with Theorem 2.8 shows that the 3-generator n-periodic products $\mathbb{A}_{\mathcal{P}} = \mathcal{F}(n, \mathcal{P}) *^n \mathbb{Z}_n$ are themselves n-torsion.

Proposition 2.5. *For odd $n \geq 1003$, the groups $\mathbb{A}_{\mathcal{P}}$ are n-torsion groups.*

Chapter 3. Preliminaries on geometric group theory. This chapter introduces the geometric tools used in the proofs of the main results: van Kampen diagrams, graded small cancellation theory, A-maps, contiguity submaps and the theory of minimal partial Burnside presentations developed by Boatman [13] after Olshanskii [7].

Definition 3.1. *A van Kampen diagram over a presentation $\langle S \mid \mathcal{R} \rangle$ is a finite, simply connected, labelled planar map Δ in which each oriented edge is labelled by a letter of $S \cup S^{-1}$, with inverse edges receiving inverse labels, and the contour label of each face is a cyclic permutation of some relator in $\mathcal{R}$ or its inverse.*

By van Kampen's lemma (Theorem 3.1), a word W represents the identity in the group $\langle S \mid \mathcal{R} \rangle$ if and only if there exists a van Kampen diagram with contour label W. The conjugacy analogue uses annular diagrams (Theorem 3.2): two words represent conjugate elements if and only if there is an annular diagram with the given labels on its two boundary components.

The classical small cancellation theory, originating in Dehn's work in the 1910s and developed systematically by Greendlinger and by Lyndon and Schupp [33, 34], treats group presentations in which the relators have uniformly bounded mutual overlaps. This theory is, however, ill-suited for free Burnside groups, where the relators are n-th powers of unbounded length. To overcome this obstacle, Olshanskii developed in the 1980s the graded small cancellation theory [7], in which relators are stratified by rank and small cancellation conditions are imposed only between relators of the same rank.

The central object of the graded theory is the A-map: a graded planar map whose cells satisfy inductive metric conditions on the contour lengths and on the overlaps between cells. The overlap between two cells, or between a cell and a section of the contour, is formalized through the notion of a contiguity submap. By Theorem 3.3, every reduced diagram over a minimal partial Burnside presentation is an A-map.

The chapter also recalls the framework of partial Burnside presentations [13], including the following result.

Lemma 3.9. *Let G be a partial Burnside group with generating set S, $b \in S$ a generator of infinite order, $k \neq 0$, and let w be a geodesic word in G conjugate to b^k. Then $w = cb^k c^{-1}$ in G for some geodesic word c, with $|w|_G = |k| + 2|c|$.*

Chapter 4. Growth and probability on groups. In this chapter we study the growth of free Burnside groups and describe the framework of probabilistic identities. Let G be a group with finite symmetric generating set S.

Definition 4.1. *The Cayley graph of G with respect to S, denoted* $\mathrm{Cay}(G, S)$*, is the graph whose vertex set is G and whose edge set consists of all pairs (g, gs) for $g \in G$ and $s \in S \cup S^{-1}$.*

The Cayley graph carries a natural metric.

Definition 4.2. *The word metric on G with respect to S is defined by*

$$d_S(g, h) = \ell_S(g^{-1}h),$$

where $\ell_S(x)$ denotes the word length of x, i.e., the smallest non-negative integer r such that x can be written as a product of r elements from S. Equivalently, $d_S(g,h)$ is the length of a shortest path from g to h in $\mathrm{Cay}(G,S)$.

Definition 4.3. *The ball of radius r centered at the identity in the Cayley graph is the set*

$$B_{G,S}(r) = \{g \in G : \ell_S(g) \leq r\}.$$

The growth function of G with respect to S is

$$\gamma_{G,S}(r) = |B_{G,S}(r)| = |\{g \in G : \ell_S(g) \leq r\}|.$$

Growth functions can be classified into three classes up to the standard equivalence relation: polynomial, intermediate and exponential.

Later in this chapter we define the natural sequence of uniform measures on Cayley balls, and introduce the notion of an M-probabilistic identity. For the word $w = x^n$, showing that $x^n = 1$ is a probabilistic identity amounts to proving that

$$\frac{|B_{G,S}(r) \cap D|}{|B_{G,S}(r)|} \longrightarrow 0 \qquad \text{as } r \to \infty,$$

where $D = \{g \in G : g^n \neq 1\}$ is the set of elements failing the identity. The proofs in Chapter 5 thus reduce to bounding $|B_{G,S}(r) \cap D|$ from above and $|B_{G,S}(r)|$ from below.

A group has exponential growth if $\gamma_{G,S}(r) \sim e^r$. In his monograph [6], Adian proved that $B(m,n)$ has exponential growth for all $m \geq 2$ and odd $n \geq 665$ and gave the explicit estimate

$$\gamma_{B(2,n)}(r) > 4 \cdot (2.9)^{r-1}$$

for any free generating set with 2 generators. For the probabilistic constructions in this dissertation, explicit growth estimates are needed for higher ranks. By generalizing Adian's combinatorial argument [6], which counts reduced words avoiding any subword of the form A^8, one obtains Theorem 1.1. The case $m = 3$ is recorded as a separate corollary, since this is the bound used in Chapter 5.

Corollary 4.1 ([29]). *For $m = 3$, odd $n \geq 665$ and any natural number r,*

$$\gamma_{B(3,n)}(r) > \tfrac{3}{2} \cdot (4.998)^r - 1.$$

Since every m-generated n-torsion group admits a surjective homomorphism onto $B(m,n)$ (Proposition 2.2), this lower bound transfers to all such groups (Corollary 1.1). In particular, the 3-generator groups $\mathbb{A}_{\mathcal{P}}$ and $B(2,n) *^n \mathbb{Z}$ studied in Chapter 5 satisfy the same lower bound.

As a further application, Theorem 1.1 strengthens the estimate of Atabekyan and Mikaelian [35] on the size of product sets in $B(m,n)$.

Corollary 4.2 ([29]). *For any finite symmetric subset S of the free Burnside group $B(m,n)$ of odd exponent $n \geq 1003$ that generates a non-cyclic subgroup,*

$$|S^s| \geq 4 \cdot 2.998^{s/(400d)^3 - 1},$$

where d is the least odd divisor of n satisfying $d \geq 1003$.

Chapter 5. Proofs of the main results. This chapter contains the proofs of two main results of this dissertation (Theorems 1.2 and 1.3). Fix a subset $\mathcal{P}$ of the set of prime numbers and consider the n-periodic product

$$\mathbb{A}_{\mathcal{P}} = \mathcal{F}(n, \mathcal{P}) *^{n} \mathbb{Z}_n,$$

where $\mathcal{F}(n, \mathcal{P})$ is the relatively free group of rank 2 defined above, with free generators b_1, b_2, and $\mathbb{Z}_n = \langle a \rangle$ is the cyclic group of order n. The group $\mathbb{A}_{\mathcal{P}}$ is generated by the set $S = \{a, b_1, b_2\}$, and we write $S' = \{b_1, b_2\}$ for the generating set of $\mathcal{F}$. Thus the group $\mathbb{A}_{\mathcal{P}}$ is 3-generated and, by Proposition 2.5, is an n-torsion group. By the exactness of the n-periodic product, $\mathcal{F}(n, \mathcal{P})$ embeds into $\mathbb{A}_{\mathcal{P}}$. Since $\mathcal{F}(n, \mathcal{P})$ contains elements of infinite order, the identity $x^n = 1$ does not hold universally in $\mathbb{A}_{\mathcal{P}}$.

Let $D = \{g \in \mathbb{A}_{\mathcal{P}} : g^n \neq 1\}$ denote the set of elements for which the identity $x^n = 1$ fails. By the characteristic property of n-periodic products (Theorem 2.7) and the structure of the factors, every element of D is conjugate to an element of $\mathcal{F}(n, \mathcal{P})$ of infinite order. Such conjugates are described by the following chain of structural lemmas, built on Lemma 2.6 and on the geometric tools of Chapter 3. Here the parameters $\beta = 1 - \bar{\beta}$ and γ are chosen according to the least parameter principle.

Lemma 5.1. *Let V be a word over $S^{\pm 1}$ in $\mathbb{A}_{\mathcal{P}}$ that is conjugate to some word W over $S'^{\pm 1} = \{b_1^{\pm 1}, b_2^{\pm 1}\}$, and let U be a shortest word such that V is conjugate to a power of U. Then U is a word over $S'^{\pm 1}$.*

Lemma 5.2. *There exists a cyclic shift V_1 of V such that $V_1 = TU^kT^{-1}$ in $\mathbb{A}_{\mathcal{P}}$, with $\bar{\beta}\,|U^k| < |V|$ and $|T| < \gamma(|V| + |U^k|)$.*

Lemma 5.3. *There exists a decomposition $V = XTU^kT^{-1}X^{-1}$ in $\mathbb{A}_{\mathcal{P}}$, where U is a word over $S'^{\pm 1}$, X is a prefix of V, and*

$$|U^k| < \bar{\beta}^{-1}\,|V|, \qquad |T| < \gamma(1 + \bar{\beta}^{-1})\,|V|.$$

Lemma 5.4. *Let d_r denote the number of elements $g \in B_{\mathbb{A}_{\mathcal{P}}}(r)$ that are conjugate to some element of $\mathcal{F}$ in $\mathbb{A}_{\mathcal{P}}$. Then*

$$d_r < r \cdot \gamma_{\mathcal{F}}\left(\bar{\beta}^{-1} r\right) \cdot \gamma_{\mathbb{A}_{\mathcal{P}}}(3\gamma r).$$

Taking into account that $|D \cap B_{\mathbb{A}_{\mathcal{P}}}(r)| \leq d_r$, and combining Lemma 5.4 with the growth bound $|B_{\mathbb{A}_{\mathcal{P}}}(r)| > \frac{3}{2} \cdot (4.998)^r - 1$ from Theorem 1.1 and Corollary 4.1 one can obtain

$$\frac{|D \cap B_{\mathbb{A}_{\mathcal{P}}}(r)|}{|B_{\mathbb{A}_{\mathcal{P}}}(r)|} \longrightarrow 0 \qquad \text{as } r \to \infty,$$

which proves Statement (1) of Theorem 1.3. Statement (2) follows from the existence of elements of infinite order in $\mathcal{F}(n, \mathcal{P})$. Statement (3) is obtained from the non-isomorphism of the groups $\mathcal{F}(n, \mathcal{P})$ (Proposition 2.3) together with a countability argument applied to the set of pairs of generators in $\mathbb{A}_{\mathcal{P}}$. This gives a negative answer to Question 13.3 of [20].

Theorem 1.2 is proved by a similar scheme applied to $G = B(2, n) *^{n} \mathbb{Z}$, using Theorem 2.7 to characterize the set of elements not satisfying $x^n = 1$, the geometric Lemma 3.9, and the growth bound from Theorem 1.1 and Corollary 4.1.

Chapter 6. Automorphisms of free Burnside groups. The last chapter discusses Mazurov's problem (Kourovka Notebook, 17.70 in [26]): is it true that an automorphism of prime order q of $B(q,p)$ that cyclically permutes the free generators has a non-trivial fixed point? The case $q=2$ is settled by the following theorem of Atabekyan and Aslanyan [27], which provides an explicit fixed point.

Theorem 6.1 ([27]). *Let φ be an arbitrary automorphism of order 2 of a periodic group G of odd period n. If $\varphi(h) \neq h^{-1}$ for some $h \in G$, then the element $s = \varphi(h) \cdot (h^{-1}\varphi(h))^{(n-1)/2}$ is a non-trivial fixed point of φ.*

For $q \geq 3$ the problem remains open. The direct algebraic approach of Theorem 6.1 does not generalize to higher-order automorphisms. The dissertation develops a cohomological approach to the problem. Let α be the automorphism of F_q defined by $\alpha(a_i) = a_{i+1}$, indices taken modulo q; since the subgroup $N = \langle\langle w^p : w \in F_q \rangle\rangle$ is fully invariant in F_q, α induces an automorphism of $B(q,p) \cong F_q/N$.

The first statement of Theorem 1.4 is the following.

Proposition 6.1. *For any prime $q \geq 2$, $H^1(\langle\alpha\rangle, F_q) = \{1\}$.*

The proof uses Serre's fixed-point theorem for finite groups acting without inversion on trees ([28]), applied to the natural action of the semidirect product $F_q \rtimes_\alpha \langle t \rangle$ on the Cayley tree of F_q.

The second statement is obtained by applying an exact sequence of pointed sets in non-abelian cohomology [36] to the short exact sequence $1 \to N \to F_q \to B(q,p) \to 1$. Two auxiliary facts are needed: Proposition 6.1 above, and the following lemma, in which $C_G(\alpha)$ denotes the fixed-point subset of a group G under the action of the automorphism α.

Lemma 6.1. *$C_{F_q}(\alpha) = \{1\}$ and $C_N(\alpha) = \{1\}$.*

Together these results produce a bijection of pointed sets between the fixed-point set of α in $B(q,p)$ and a non-abelian first cohomology set:

Proposition 6.2. *For any prime $q \geq 2$ and any $p \geq 2$, $C_{B(q,p)}(\alpha) \cong H^1(\langle\alpha\rangle, N)$.*

Combining Propositions 6.1 and 6.2 gives Theorem 1.4. Mazurov's problem is thus reduced to a cohomological question on the action of $\langle\alpha\rangle$ on the normal subgroup $N \leq F_q$.

Conclusion

The dissertation contains the following main results.

- An explicit lower bound for the growth function of the free Burnside group $B(m,n)$ is obtained for all $m \geq 2$ and odd $n \geq 665$ (Theorem 1.1). This bound extends Adian's estimate [6] for $B(2,n)$ to arbitrary rank and give, as a corollary, the same lower bound for all finitely generated non-cyclic n-torsion groups (Corollary 1.1).

- A negative answer is given to Question 13.3 of Amir, Blachar, Gerasimova and Kozma [20]: probabilistic satisfaction of an identity does not, in general, imply its universal satisfaction (Theorems 1.2 and 1.3). A continuum of pairwise non-isomorphic 3-generator groups is constructed, in each of which $x^n = 1$ holds probabilistically but not universally.

- Mazurov's problem is reformulated in cohomological terms (Theorem 1.4). It is shown that the automorphism α of $B(q,p)$ has a non-trivial fixed point if and only if the first non-abelian cohomology set $H^1(\langle\alpha\rangle, N)$ is non-trivial, where N is the normal closure of all p-th powers in F_q.

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ԱՄՓՈՓՈՒՄ

Պարբերական խմբեր, դրանց ավտոմորֆիզմները և հավանականային նույնություններ խմբերում

Ատենախոսությունը նվիրված է ազատ բեռնսայդյան խմբերի հետ կապված մի շարք հարցերի ուսումնասիրությանը: Մասնավորապես անդրադարձ է կատարվում ազատ բեռնսայդյան խմբերի աճի ֆունկցիային, Ադյանի տեսության հիման վրա կառուցված խմբերում հավանականային նույնություններին, ինչպես նաև ազատ բեռնսայդյան խմբերի ավտոմորֆիզմներին:

Ատենախոսության հիմնական նպատակն է պատասխանել Ամիրի, Բլաշարի, Գերասիմովայի և Կոզմայի կողմից առաջադրված բաց խնդրին խմբերում հավանականային նույնությունների վերաբերյալ, ստանալ ազատ բեռնսայդյան խմբերի աճի ստորին գնահատականներ, և ոչ աբելյան կոհոմոլոգիաների միջոցով վերաձևակերպել ազատ պարբերական խմբերի ավտոմորֆիզմների վերաբերյալ Մազուրովի խնդիրը (Կոուրովյան տետր, խնդիր 17.70):

Աշխատանքում ստացվել են հետևյալ հիմնական արդյունքները.

- Ընդհանրացվել են Ս. Ադյանի կողմից 2 ռանգի համար ստացված աճի գնահատականները: Ապացուցվել է (թեորեմ 1.1), որ կամայական $m \geq 2$ ռանգի և կենտ $n \geq 665$ էքսպոնենտի համար ազատ բեռնսայդյան $B(m,n)$ խմբի աճի ֆունկցիան բավարարում է

$$\gamma_{B(m,n)}(r) > \frac{m}{m-1}(2m-1-2\cdot 10^{-3})^r - 1$$

  անհավասարությանը: Այս գնահատականը էական դեր է խաղում հավանականային նույնությունով խմբերի կառուցման մեջ:

- Հավանականային նույնությունների մասին Ամիր-Բլաշար-Գերասիմովա-Կոզմայի հարցին տրվել է բացասական պատասխան: Ապացուցվել է (թեորեմ 1.2), որ $B(2,n)*^n\mathbb{Z}$ n-պարբերական արտադրյալում $x^n = 1$ պայմանը բավարարվում է 1 հավանականությամբ, սակայն այն չի հանդիսանում խմբի նույնություն: Այս արդյունքը ուժեղացվել է՝ կառուցելով երեք ծնորդով n-ոլորումով խմբերի անհաշվելի ընտանիք՝ հիմնված Ադյանի անվերջ անկախ նույնությունների համակարգի վրա, որոնցում $x^n = 1$ առնչությունը կրկին հավանականային նույնություն է, բայց խմբային նույնություն չէ (թեորեմ 1.3): Ստացված ընտանիքի խմբերը զույգ առ զույգ իզոմորֆ չեն:

- q պարզ ռանգի և p պարզ էքսպոնենտի $B(q,p)$ ազատ բեռնսայդյան խմբերի ավտոմորֆիզմների մասին Մազուրովի խնդիրը վերաձևակերպվել է ոչ աբելյան կոհոմոլոգիաների տեսության լեզվով: Կիրառելով Բասս-Սեռի տեսությունը՝ նախ ապացուցվել է, որ q ռանգի F_q ազատ խմբի համար $H^1(\langle\alpha\rangle, F_q) = \{1\}$: Ապա, կիրառելով կոհոմոլոգիական ճշգրիտ հաջորդականությունները, ապացուցվել է (թեորեմ 1.4), որ ավտոմորֆիզմի անշարժ կետերի բազմությունը փոխմիարժեք համապատասխանության մեջ է $H^1(\langle\alpha\rangle, N)$ կոհոմոլոգիական բազմության հետ, որտեղ N-ը $F_q \twoheadrightarrow B(q,p)$ բնական էպիմորֆիզմի միջուկն է: Արդյունքում Մազուրովի խնդրի լուծումը հանգեցվել է համապատասխան կոհոմոլոգիական բազմության ոչ տրիվիալության ստուգմանը:

Առենախոսության մեջ ստացված արդյունքները և մշակված կոմբինատորական և երկրաչափական մեթոդներն ունեն տեսական նշանակություն: Դրանք կարող են օգտակար լինել անվերջ խմբերի հետագա ուսումնասիրությունների, ինչպես նաև խմբերի երկրաչափական և ասիմպտոտական տեսություններում հարակից հարցե-րի ուսումնասիրության համար:

ЗАКЛЮЧЕНИЕ

Периодические группы, их автоморфизмы и вероятностные тождества в группах

Диссертация посвящена исследованию ряда вопросов, связанных со свободными бернсайдовыми группами. В частности, в работе рассматриваются функция роста свободных бернсайдовых групп, вероятностные тождества в группах, построенных на основе теории Адяна, а также автоморфизмы свободных бернсайдовых групп.

Основная цель диссертации — ответить на открытый вопрос, поставленный Амиром, Блашаром, Герасимовой и Козмой о вероятностных тождествах в группах, получить нижние оценки роста свободных бернсайдовых групп и переформулировать проблему Мазурова об автоморфизмах свободных периодических групп на языке неабелевых когомологий (Коуровская тетрадь, проблема 17.70).

В работе получены следующие основные результаты:

- Обобщены оценки роста, полученные С. Адяном для ранга 2. Доказано (Теорема 1.1), что для произвольного ранга $m \geq 2$ и нечетного показателя $n \geq 665$ функция роста свободной бернсайдовой группы $B(m,n)$ удовлетворяет неравенству

$$\gamma_{B(m,n)}(r) > \frac{m}{m-1}(2m-1-2\cdot 10^{-3})^r - 1.$$

  Эта оценка играет существенную роль в построении групп с вероятностным тождеством.

- Дан отрицательный ответ на вопрос Амира-Блашара-Герасимовой-Козмы о вероятностных тождествах. Доказано (Теорема 1.2), что в n-периодическом произведении $B(2,n) *^n \mathbb{Z}$ условие $x^n = 1$ выполняется с вероятностью 1, однако оно не является тождеством группы. Этот результат был усилен построением несчетного семейства n-крученых групп с тремя образующими, основанного на системе бесконечных независимых тождеств Адяна, в которых соотношение $x^n = 1$ вновь является вероятностным тождеством, но не групповым тождеством (Теорема 1.3). Группы полученного семейства попарно неизоморфны.

- Проблема Мазурова об автоморфизмах свободных бернсайдовых групп $B(q,p)$ простого ранга q и простого показателя p переформулирована на языке теории неабелевых когомологий. Применяя теорию Басса-Серра, сначала доказано, что для свободной группы F_q ранга q выполнено $H^1(\langle\alpha\rangle, F_q) = \{1\}$. Затем, используя точные когомологические последовательности, доказано (Теорема 1.4), что множество неподвижных точек автоморфизма находится во взаимно однозначном соответствии с когомологическим множеством $H^1(\langle\alpha\rangle, N)$, где N — ядро естественного эпиморфизма $F_q \twoheadrightarrow B(q,p)$. В результате решение проблемы Мазурова сведено к проверке нетривиальности соответствующего когомологического множества.

Результаты, полученные в диссертации, и разработанные комбинаторные и геометрические методы имеют теоретическое значение. Они могут быть полезны для дальнейших исследований бесконечных групп, а также для изучения смежных вопросов в геометрической и асимптотической теориях групп.